Analysis of a M/G/1/K Queue with Vacations

Systems with Exhaustive Service, Multiple or Single Vacations

We consider here the finite capacity M/G/1/K queue with the variation that the server goes for vacations when it is idle. This service model is referred to as one providing *exhaustive service,* as the server cannot go for a vacation until all the jobs presently in the system have been served. This is the service model being considered here as this leads to a simple analytical model. Note that it is also possible to have a *gated service* model where the server only serves those customers that it finds in the system when it first starts service following its vacation. It then leaves for vacation again.

The vacation model itself may be of different types. In Section 4.1, we had considered a *multiple vacation* model. Here a server, on returning from a vacation, goes for another vacation if it finds the system still empty. In this case, the server resumes normal service if it finds one or more jobs waiting when it returns from a vacation. Note that multiple vacations, one after the other, will be possible in this model. An alternative vacation model, discussed in Section 4.2, assumes that the server goes for only one vacation when the queue becomes empty. Even if the queue is empty when it returns from the vacation, it stays at the queue waiting for a job to arrive. (In the multiple vacation model, it would have gone for another vacation in this case.) Other variations of this are also possible such as one where the server can go on multiple vacations and resumes service only when it finds L or more jobs waiting when it returns from a vacation. In the subsequent analysis, we first consider the multiple vacation model and then look at the case when we allow only a single vacation whenever the departure leaves behind an empty system.

Multiple Vacation Model

Following our usual notation, we assume that the arrivals come from a Poisson process with rate l and that the service times are generally distributed with pdf $b(t)$, cdf $B(t)$ and has $L_B(s)$ as the Laplace Transform of the pdf. Let the mean service time be $\overline{X} = m^{-1}$. As in Sections 4.1 and 4.2, we assume that a vacation interval has pdf $f_V(t)$, cdf $F_V(t)$ and has $L_V(s)$ as the L.T. of the pdf. Let the mean vacation interval be $\overline{V}$. As in Sections 4.1 and 4.2, we also assume that the service times and vacation times are i.i.d. random variables which are also independent of each other.

For this M/G/1/K queue with *exhaustive service* and *multiple vacations*, we consider the analysis using an imbedded Markov Chain approach. For this, the imbedded points are chosen to be at the time instants when either a job completes service or a vacation has ended. These have been illustrated in Fig. 1 using a typical plot of the residual service and vacation times with the imbedded time points marked with shaded circles.

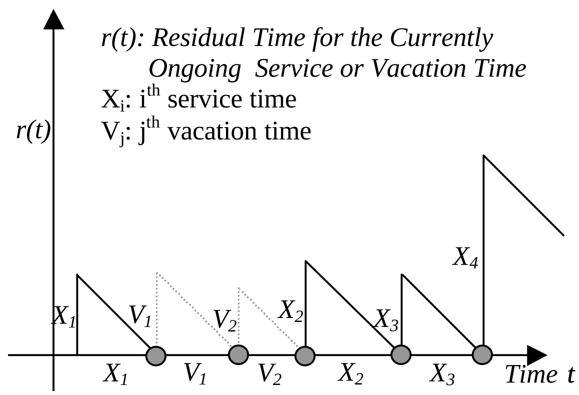

Figure 1. Imbedded Time Points for the M/G/1/K Queue with Multiple Vacations and Exhaustive Service

The system states at the imbedded points are represented by both the number in the system (waiting and in-service) immediately after the selected time instant and the nature of the imbedded point (i.e. whether it is a service completion or a vacation completion). The system state at the i^{th} imbedded point is represented by (n_i, f_i) where

n_i = number of jobs in the system just after the i^{th} imbedded point

and $f_i = 0$ if the i^{th} point was a vacation completion

$= 1$ if the i^{th} point was a service completion

Considering the system in equilibrium, let q_k, $k=0,1,.......,K$ be the probability of $(k,0)$ and r_k, $k=0,1,.....,(K-1)$ be the probability of $(k,1)$. (Note that just after a service completion, the system state cannot be K which is the reason why r_K is not defined.) Let f_j *j=0,1,......¥* be the probability of their being *j* jobs in the system just after a vacation interval. This will be given by

$$f_j = \int_0^{\infty} \frac{(lt)^j}{j!} e^{-lt} f_V(t)dt \qquad j=0,1,.......,¥ \qquad (1)$$

Let a_j *j=0,1,......¥* be similarly defined as the probability of *j* arrivals in a service time. This will be given by

$$a_j = \int_0^{\infty} \frac{(lt)^j}{j!!} e^{-lt} b(t)dt \qquad j=0,1,.......,¥ \qquad (2)$$

Considering the system state just after the imbedded points, the following transition equations may then be written.

$$q_k = (q_0 + r_0) f_k \qquad k=0,1,.......,(K-1) \qquad (3)$$

$$q_K = (q_0 + r_0) \sum_{k=K}^{\infty} f_k \qquad k=K \qquad (4)$$

$$r_k = \sum_{j=1}^{k+1} (q_j + r_j) a_{k-j+1} \qquad k=0,........,(K-2) \qquad (5)$$

$$r_{K-1} = q_K + \sum_{j=1}^{K-1} (q_j + r_j) \sum_{k=K-j}^{\infty} a_k \qquad k=K-1 \qquad (6)$$

Summing the probabilities of all possible states, we will also get

$$\sum_{k=0}^{K} q_k + \sum_{j=0}^{K-1} r_j = 1 \qquad (7)$$

We need to solve for q_k *k=0,1,.......,K* and r_k, *k=0,1,.....,(K-1)* using (3)-(7) along with the appropriately calculated values of f_j and a_j, *j=0,1,....,¥* . This is most conveniently done by defining an intermediate variable b_k, *k=0,1,.....,(K-1)* as

$$b_k = \frac{q_k + r_k}{q_0 + r_0} \qquad k=0,1,.......,(K\text{-}1) \qquad (8)$$

We can then get b_k, $k=0,1,.....,(K\text{-}1)$ using the following recursion

$$b_0 = 1$$

$$b_{k+1} = \frac{b_k - f_k - \sum_{j=1}^{k} b_j a_{k-j+1}}{a_0} \qquad k = 0,......,(K-2) \qquad (9)$$

Substituting (4) into (7) and simplifying gives

$$(q_0 + r_0)\left[\sum_{k=K}^{\infty} f_k + 1 + \sum_{k=1}^{K-1} b_k\right] = 1$$

and using $b_0=1$

$$q_0 + r_0 = \frac{1}{\left[\sum_{k=K}^{\infty} f_k + \sum_{k=0}^{K-1} b_k\right]} \qquad (10)$$

Using the value of (q_0+r_0) obtained from (10), we can find q_k $k=0,1,.......,K$ using (3) and (4). Using these values of q_k and the values of b_k obtained earlier, we can now find r_k, $k=0,1,.......,K$ using

$$r_k = (q_0 + r_0) b_k - q_k \qquad k=0,1,.......,(K\text{-}1) \qquad (11)$$

The probabilities q_k $k=0,1,.......,K$ and r_k, $k=0,1,.....,(K\text{-}1)$ may now be used to get some of the performance parameters of the system. We can see that if the state is either ($0,0$) or ($0,1$) at any imbedded point, then the time to the next imbedded point would correspond to a vacation. The probability of this would be (q_0+r_0) which would then also be the probability that a (single) vacation completion would follow an arbitrarily selected imbedded point (which could be either a vacation completion or a service completion). It would then also follow that ($1\text{-}q_0\text{-}r_0$) would be the probability that a job service completion imbedded point would follow an arbitrarily selected imbedded point. It may also be noted that by summing (3) and (4) for all $k=0,1,..,K$ we get

$$q_0 + r_0 = \sum_{k=0}^{K} q_k \qquad (12)$$

Let r_c be defined as the carried load, i.e. the probability that the server is busy at an arbitrary time. Note that if we look at all the intervals between successive imbedded points over a long time duration (say *T*), then we can easily conclude that

$$\begin{aligned} r_c &= \lim_{T \to \infty} \frac{\sum service\ \ times\ in\ T}{\sum vacation\ \ times\ in\ T \quad + \quad \sum service\ \ times\ in\ T} \\ &= \frac{(1 - q_0 - r_0)\overline{X}}{(q_0 + r_0)\overline{V} + (1 - q_0 - r_0)\overline{X}} \end{aligned} \qquad (13)$$

where the offered load r is defined as usual to be

$$r = l\overline{X} \qquad (14)$$

Using (13) and (14), the blocking probability P_B may be found using

$$r_c = r(1 - P_B)$$

to get

$$P_B = \frac{r - r_c}{r} \qquad (15)$$

Since a fraction P_B of the arrivals will be blocked and will not be allowed to enter the queue, the throughput (rate) g of the system will be given by

$$g = l(1 - P_B) \qquad (16)$$

Another useful quantity that may be obtained from the above analysis is the mean time *D* between successive imbedded points of the above analysis when the system is in equilibrium. It follows from the definitions that this will be

$$D = (q_0 + r_0)\overline{V} + (1 - q_0 - r_0)\overline{X} \qquad (17)$$

It follows from (13) that therefore

$$q_0 + r_0 = \frac{(1 - r_c)D}{\overline{V}} = 1 - \frac{r_c D}{\overline{X}} \tag{18}$$

It should be noted that the analysis given above leads only to the queue size distributions at the imbedded points corresponding to either a service completion or a vacation completion. (Actually, these are the distributions just after the imbedded point.) To analyse the system more completely, we need to actually find the corresponding distributions at an arbitrary instant of time as then we can use that to find the mean queue length and other related parameters at an arbitrary instant. Several methods for doing this are given in [Takagi2]. One of these methods is given next.

Consider the probabilities Q_k, *k=0,1,.....,K* and R_k, *k=0,.......,(K-1)* defined at arbitrary time instants as follows (These correspond to the probabilities q_k and r_k defined earlier at the imbedded time points.)

Q_k = P{*k* jobs in system, server currently in a vacation} *k=0,1,....,K*
R_k = P{*k* jobs in system, server currently serving a job} *k=0,....,(K-1)*

We define F_j as the probability that *j* or more jobs arrive to the system during a vacation time. This may be evaluated using (1) to be

$$\begin{aligned} F_j &= \sum_{i=j}^{\infty} f_i = \sum_{i=j}^{\infty} \int_{t=0}^{\infty} \frac{(lt)^i}{i!} e^{-lt} f_V(t)dt \\ &= \int_0^{\infty} \frac{(lt)^{j-1}}{(j-1)!} e^{-lt} \left[1 - F_V(t)\right] l\,dt \end{aligned} \tag{19}$$

where $F_V(t)$ is the cdf corresponding to the pdf $f_V(t)$ of a vacation interval. It may also be noted from the definitions that

$$\sum_{j=1}^{\infty} F_j = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} f_i = \sum_{i=1}^{\infty} i f_i = E\{\text{number of arrivals in a vacation interval}\}$$

Therefore

$$\sum_{j=1}^{\infty} F_j = l\overline{V} \tag{20}$$

We similarly define A_j as the probability that j or more jobs arrive during a service duration. Using a_j of (2) as the probability of j job arrivals in a service time, we get the following results which are similar to those given in (19) and (20).

$$\begin{aligned} A_j &= \sum_{i=j}^{\infty} a_i = \sum_{i=j}^{\infty} \int_{t=0}^{\infty} \frac{(lt)^i}{i!} e^{-lt} b(t)dt \\ &= \int_0^{\infty} \frac{(lt)^{j-1}}{(j-1)!} e^{-lt} \left[1 - B(t)\right] l\,dt \end{aligned} \tag{21}$$

$$\sum_{j=1}^{\infty} A_j = l\overline{X} = r \tag{22}$$

where r is the load offered to the system (which results in the carried load r_c, $r_c < r$).

In order to find Q_k, consider an arbitrary time instant that falls within a vacation such that there are k arrivals in the time interval (say x) between the start of that vacation interval and the time instant selected. Note that the probability of selecting a vacation interval would be $(1\text{-}r_c)$. The probability of k arrivals in the time interval x would be given by the Poisson distribution, i.e. $\dfrac{(lx)^k}{k!} e^{-lx}$. The pdf of the time interval x itself would be given by residual life type arguments to be $\dfrac{1 - F_V(x)}{\overline{V}}$. Using these, we can write

$$\begin{aligned} Q_k &= (1 - r_c) \int_0^{\infty} \frac{(lx)^k}{k!} e^{-lx} \frac{1 - F_V(x)}{\overline{V}} dx \qquad k = 0,.....,(K-1) \\ &= (1 - r_c) \sum_{k=K}^{\infty} \int_0^{\infty} \frac{(lx)^k}{k!} e^{-lx} \frac{1 - F_V(x)}{\overline{V}} dx \qquad k = K \end{aligned} \tag{23}$$

Using (19), we can simplify this to get

$$Q_k = \frac{(1-r_c)}{l\bar{V}} F_{k+1} \qquad k = 0,.....,(K-1)$$
$$= \frac{(1-r_c)}{l\bar{V}} \sum_{k=K+1}^{\infty} F_k \qquad k = K \tag{24}$$

Further simplification of (24) is possible by noting that by using (3) and (4), we can write

$$\sum_{j=k}^{K} q_j = (q_0 + r_0)F_k \qquad k = 1,......,K \tag{25}$$

Substituting this in (24) and using (18) and (20) leads to the final expression for Q_k as

$$Q_k = \frac{1}{lD} \sum_{j=k+1}^{K} q_j \qquad k = 0,.....,(K-1)$$
$$= 1 - r_c - \frac{1}{lD} \sum_{j=1}^{K} jq_j \qquad k = K \tag{26}$$

To find R_k, we similarly consider an arbitrary time instant that falls within a service time such that there are k arrivals in the time interval (say x) between the start of that service interval and the selected time instant. Note that the probability of selecting a service interval would be r_c. Given that the service interval will have to start with a non-empty queue, the probability of it starting with j jobs in the system will be $\frac{q_j + r_j}{1 - q_0 - r_0}$ for *j=1,....,(K-1)* and will be $\frac{q_K}{1 - q_0 - r_0}$ for $j=K$. The probability of m arrivals coming in the interval x will be $\frac{(lx)^m}{m!} e^{-lx}$. The pdf of the time interval x itself would be given by residual life type arguments to be $\frac{1 - B(x)}{\bar{X}}$. Using these, we get

$$
\begin{aligned}
R_k &= r_c \sum_{j=1}^{k} \left(\frac{q_j + r_j}{1 - q_0 - r_0} \right) \int_0^\infty \frac{(lx)^{k-j}}{(k-j)!} e^{-lx} \frac{1 - B(x)}{\overline{X}} dx \qquad k = 1, \ldots\ldots, (K-1) \\
&= r_c \left(\frac{q_K}{1 - q_0 - r_0} \right) \qquad\qquad (27) \\
&\quad + r_c \sum_{j=1}^{K-1} \left(\frac{q_j + r_j}{1 - q_0 - r_0} \right) \sum_{k=K-j}^{\infty} \int_0^\infty \frac{(lx)^k}{k!} e^{-lx} \frac{1 - B(x)}{\overline{X}} dx \qquad k = K
\end{aligned}
$$

Using (21) and $r = l\overline{X}$, we can simplify this to obtain

$$
\begin{aligned}
R_k &= \frac{r_c}{r} \sum_{j=1}^{k} \left(\frac{q_j + r_j}{1 - q_0 - r_0} \right) A_{k-j+1} \qquad k = 1, \ldots\ldots, (K-1) \\
&= r_c \left(\frac{q_K}{1 - q_0 - r_0} \right) + \frac{r_c}{r} \sum_{j=1}^{K-1} \left(\frac{q_j + r_j}{1 - q_0 - r_0} \right) \sum_{k=K-j+1}^{\infty} A_k \qquad k = K
\end{aligned} \qquad (28)
$$

We can further show using (2)-(6) that

$$
r_k = (q_0 + r_0) F_{k+1} + \sum_{j=1}^{k} (q_j + r_j) A_{k-j+1} \qquad k = 1, \ldots\ldots, (K-1) \qquad (29)
$$

This is most conveniently done by first showing that (29) holds for *k=1* and then using mathematical induction to prove it for a general value of *k*, assuming that it holds for *k-1*.

Now using (29) and (12) along with the normalisation condition of (7), we can prove that

$$
\sum_{j=1}^{K-1} (q_j + r_j) \sum_{k=K-j+1}^{\infty} A_k = \sum_{j=1}^{K} j q_j - (1 - r)(1 - q_0 - r_0) - r q_K \qquad (30)
$$

Simplifying (28) further using (29) and (30) and using (18) and (22), we get the final expression for R_k as

$$R_k = \frac{1}{lD}\left(r_k - \sum_{j=k+1}^{K} q_j \right) \qquad k = 1,......,(K-1)$$
$$= \frac{r_c(r-1)}{r} + \frac{1}{lD}\sum_{j=1}^{K} jq_j \qquad k = K \qquad (31)$$

Using the vlues of q_j $j=0,...,K$ and r_j $j=0,....,(K-1)$ (probabilities just after the imbedded Markov points) which were obtained earlier, we can now use (26) and (31) to calculate the probabilities Q_j $j=0,...,K$ and R_j $j=1,....,K$ at any arbitrarily chose time instant.

For a system in equilibrium, we define p_j $j=0,1,....,K$ as the probability of the system being in state j at an arbitrarily chosen time instant. Using the values of Q_j and R_j obtained above in (26) and (31), we can calculate the state probabilities p_j $j=0,1,....,K$ as

$$p_0 = Q_0 \qquad j = 0$$
$$p_j = Q_j + R_j = \frac{r_j}{lD} \qquad j = 1,.......,(K-1) \qquad (32)$$
$$p_K = Q_K + R_K = \frac{(r - r_c)}{r} \qquad j = K$$

These probabilities may now be used to find the usual queueing parameters as defined in Chapter 1. For example, the mean number N in the system and the mean time W spent in the system by a job which is not blocked will be given by

$$N = \frac{1}{lD}\sum_{j=1}^{K-1} jr_j + K\left(\frac{r - r_c}{r} \right) \qquad (33)$$

$$W = \frac{N}{l(1 - P_B)} \qquad (34)$$

It is also evident that the probability that the server is busy or idle (i.e. on a vacation) will be given by

$$\text{P\{server is busy\}} = \sum_{k=1}^{K} R_k = r_c = r(1 - p_K) \qquad (35)$$

$$\text{P}\{\text{server is on vacation}\} = \sum_{k=0}^{K} Q_k = 1 - r_c = 1 - r(1 - p_K) \qquad (36)$$

Single Vacation Model

In the *single vacation* model, the server still goes for vacation whenever the system becomes empty. However, unlike the multiple vacation case described earlier, once it comes back from this vacation, it does not go for another vacation even if it finds the queue empty on its return. It goes for its next vacation only after the system becomes empty once again following a "server busy" interval. If this model is being considered, the situation depicted in Fig. 1 will change and look as shown in Fig. 2.

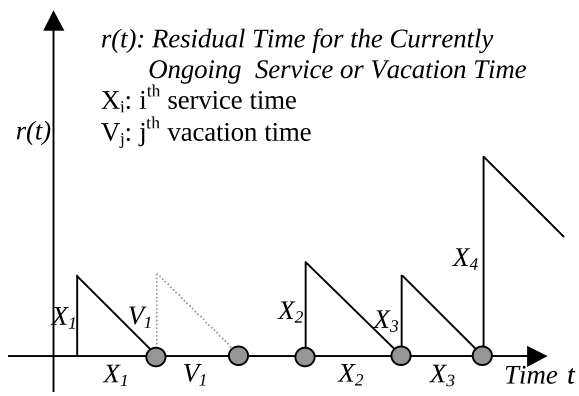

Figure 2. Imbedded Time Points for the M/G/1/K queue with Single Vacation and Exhaustive Service

The single vacation model with exhaustive service may be analysed in essentially the same fashion as the multiple vacation model considered earlier. We follow an imbedded Markov Chain approach with the imbedded points selected as the points corresponding to the departure instants of jobs that have finished service or the end of completed vacations. These imbedded points are the ones illustrated in Fig. 2 using shaded circles.

The system states at the imbedded points are represented by both the number in the system (waiting and in-service) immediately after the selected time instant and the nature of the imbedded point (i.e. whether it corresponds to a service completion or a vacation completion). We can write a Markov Chain for the system states, denoted in this fashion, between the imbedded points. The system state at the i^{th} imbedded point is represented by (n_i, f_i) where

n_i = number of jobs in the system just after the i^{th} imbedded point

and $f_i = 0$ if the i^{th} point was a vacation completion

$= 1$ if the i^{th} point was a service completion

Considering the system in equilibrium, let q_k, $k=0,1,.......,K$ be the probability of (k,0) and r_k, $k=0,1,.....,(K-1)$ be the probability of (k,1). These definitions are the same as the ones used for the multiple vacation model. The difference for the single vacation model is the situation when the vacation completes but the system is still empty. In that case, the next imbedded point is the one corresponding to the departure of the job that is the first to arrive after this vacation completion event.

We use here the same notation for characterising the Poisson arrival process (rate l), the service time distribution ($b(t)$, $B(t)$, $L_B(t)$ and $\overline{X}$) and the vacation time distribution ($f_V(t)$, $F_V(t)$, $L_V(t)$ and V) as used earlier for the multiple vacation model. Let f_j and a_j be as defined in (1) and (2), respectively.

Considering the system states just after the embedded points, the following equations may be written relating the transitions from one imbedded point to the next.

$$q_k = r_0 f_k \qquad k=0,1,.......,(K-1) \qquad (37)$$

$$q_K = r_0 \sum_{k=K}^{\infty} f_k \qquad k=K \qquad (38)$$

$$r_k = q_0 a_k + \sum_{j=1}^{k+1} (q_j + r_j) a_{k-j+1} \qquad k=0,........,(K-2) \qquad (39)$$

$$r_{K-1} = q_0 \sum_{k=K-1}^{\infty} a_k + \sum_{j=1}^{K-1} (q_j + r_j) \sum_{k=K-j}^{\infty} a_k + q_K \qquad k=K\text{-}1 \qquad (40)$$

Summing all the state probabilities at the imbedded points, we will also get

$$\sum_{k=0}^{K} q_k + \sum_{j=0}^{K-1} r_j = 1 \qquad (41)$$

Solving (37)-(41), we can obtain the equilibrium probabilities for q_k, $k=0,1,.......,K$ and r_k, $k=0,1,.....,(K-1)$ for this M/G/1/K queue with single vacations and exhaustive service. Summing (37) over $k=0,1,.....,(K-1)$ along with (38), we obtain that

$$r_0 = \sum_{k=0}^{K} q_k \tag{42}$$

This would correspond to the probability that a service completion instant leaves the queue empty, i.e. it is the start of a vacation. The time interval between successive imbedded points would be one of the following three possibilities.

(a) service time (with mean $\overline{X}$) with probability $1-r_0-q_0$

(b) vacation time (with mean $\overline{V}$) with probability r_0

(c) service time and inter-arrival time (with mean $\overline{X} + l^{-1}$) with probability q_0.

Defining D as before to be the mean time interval between successive imbedded points at equilibrium, we get

$$\begin{aligned} D &= (1 - r_0 - q_0)\overline{X} + r_0\overline{V} + q_0(\overline{X} + \frac{1}{l}) \\ &= (1 - r_0)\overline{X} + r_0\overline{V} + q_0 \frac{1}{l} \end{aligned} \tag{43}$$

Considering the mean time interval D between successive imbedded points, we can see that the interval actual corresponds to a job being served only with probability $(1-r_0)$. Therefore, the carried load r_c will be given by

$$r_c = \frac{(1 - r_0)\overline{X}}{D} \tag{44}$$

corresponding to an offered load of $r = l\overline{X}$. The blocking probability P_B can then be found as

$$P_B = 1 - \frac{r_c}{r} = 1 - \frac{1 - r_0}{lD} \tag{45}$$

It should be noted that the state probabilities q_k, $k=0,1,.......,K$ and r_k, $k=0,1,.....,(K-1)$ are only valid for the system at the imbedded points corresponding to either a service completion or a vacation completion. However, we can use them along with some additional analysis along the same lines as before to get the probabilities p_i $i=0,1,........,K$ for i jobs in the system (waiting and in service) at an arbitrary instant of time.

Considering the probability of the number in the system at an arbitrary instant of time, we can find the limiting values p_0 and p_K by a simple argument. Considering the interval between successive imbedded points, the probability p_0 will be given by the ratio of the mean time spent idle in the mean interval D. This gives

$$p_0 = \frac{r_0 \overline{V} + q_0 \frac{1}{l}}{D} \tag{46}$$

Using (43) and (44), this may be simplified to get

$$p_0 = 1 - r_c \tag{47}$$

which would be the expected result for the fraction of time the server will be idle in the system. Similarly, a job will be blocked and denied entry in the queue only when the system is full, i.e. in state K. Therefore,

$$p_K = P_B = 1 - \frac{r_c}{r} = 1 - \frac{1 - r_0}{lD} \tag{48}$$

Consider the probabilities Q_k, $k=0,1,.....,K$ and R_k, $k=0,.......,(K-1)$ defined at arbitrary time instants as follows (These correspond to the probabilities q_k and r_k defined earlier at the imbedded time points.)

Q_k = P{k jobs in system, server currently in a vacation} $k=0,1,....,K$
R_k = P{k jobs in system, server not on vacation} $k=0,....,(K-1)$

Note that when the server is not on vacation, it would either be busy serving a job or it would be waiting idle for the first job to arrive following a vacation which was completed without any job arrival. This is really the difference between the definition of R_k here and that given for the multiple vacation model earlier.

In order to find Q_k, consider an arbitrary time instant that falls within a vacation interval such that there are k arrivals in the time interval (say x) between the start of that vacation interval and the selected time instant. The probability of falling in a vacation interval would be $\frac{r_0 \bar{V}}{D}$. The probability of k arrivals in the time interval x would be given by the Poisson distribution, i.e. $\frac{(lx)^k}{k!} e^{-lx}$. The pdf of the time interval x itself would be given by residual life type arguments to be $\frac{1 - F_V(x)}{\bar{V}}$. Using these, we can write

$$
\begin{aligned}
Q_k &= \frac{r_0 \bar{V}}{D} \int_0^\infty \frac{(lx)^k}{k!} e^{-lx} \frac{1 - F_V(x)}{\bar{V}} dx & k = 0, \ldots\ldots, (K-1) \\
&= \frac{r_0 \bar{V}}{D} \int_0^\infty \sum_{k=K}^\infty \frac{(lx)^k}{k!} e^{-lx} \frac{1 - F_V(x)}{\bar{V}} dx & k = K
\end{aligned}
\tag{49}
$$

We define F_j as before to be the probability that j or more jobs arrive to the system in a vacation interval. Using (19) and (20), we can then simplify (49) to get

$$
\begin{aligned}
Q_k &= \frac{r_0 F_{k+1}}{lD} & k = 0, \ldots\ldots, (K-1) \\
&= \frac{r_0}{lD} \sum_{j=K+1}^\infty F_j = \frac{r_0}{lD} \left[l\bar{V} - \sum_{j=1}^K F_j \right] & k = K
\end{aligned}
\tag{50}
$$

To simplify this further, we note from (37) and (38), that

$$
F_{k+1} = \sum_{j=k+1}^\infty f_j = \frac{1}{r_0} \sum_{j=k+1}^K q_j \qquad k = 0, \ldots\ldots, (K-1) \tag{51}
$$

and that

$$\sum_{j=1}^{K} F_j = \sum_{j=1}^{K}\sum_{k=j}^{\infty} f_k = \frac{1}{r_0}\sum_{j=1}^{K}\sum_{k=j}^{K} q_k = \frac{1}{r_0}\sum_{j=1}^{K} jq_j \tag{52}$$

Substituting (51) and (52) in (50), we get our final expression for Q_k as

$$\begin{aligned} Q_k &= \frac{1}{lD}\sum_{j=k+1}^{K} q_j & k &= 0,......,\ (K-1) \\ &= \frac{r_0\overline{V}}{D} - \frac{1}{lD}\sum_{j=1}^{K} jq_j & k &= K \end{aligned} \tag{53}$$

To find R_k, we similarly consider an arbitrary time instant that falls within a time interval where the server is not on a vacation. Note that finding R_0 is particularly straightforward as this will correspond to the fraction of time, within the interval between successive imbedded points, when the server is idle although it is not on a vacation. This will therefore be

$$R_0 = \frac{q_0 \frac{1}{l}}{D} = \frac{q_0}{lD} \tag{54}$$

Note that we expect p_0 to be equal to Q_0+R_0. This may be verified using (54), (53) for $k=0$ and (42).

The procedure for finding R_k for the other values of k, i.e. $k=1,.....,K$ is similar to our earlier approach. Using this, we can write

$$\begin{aligned} R_k &= \frac{q_0\overline{X}}{D}\int_0^{\infty}\frac{(lx)^{k-1}}{(k-1)!}e^{-lx}\frac{1-B(x)}{\overline{X}}dx \\ &\quad + \sum_{j=1}^{k}\frac{(q_j+r_j)\overline{X}}{D}\int_0^{\infty}\frac{(lx)^{k-j}}{(k-j)!}e^{-lx}\frac{1-B(x)}{\overline{X}}dx \qquad k=1,....,K-1 \\ &= \frac{q_0\overline{X}}{D}\sum_{k=K-1}^{\infty}\int_0^{\infty}\frac{(lx)^{k}}{k!}e^{-lx}\frac{1-B(x)}{\overline{X}}dx \\ &\quad + \sum_{j=1}^{K-1}\frac{(q_j+r_j)\overline{X}}{D}\int_0^{\infty}\frac{(lx)^{k}}{k!}e^{-lx}\frac{1-B(x)}{\overline{X}}dx \\ &\quad + \frac{q_K\overline{X}}{D} \qquad\qquad k=K \end{aligned} \tag{55}$$

Using (21), this may be written as

$$
\begin{aligned}
R_k &= \frac{q_0}{lD} A_k + \sum_{j=1}^{k} \frac{(q_j + r_j)}{lD} A_{k-j+1} && k = 1,\ldots,K-1 \\
&= \frac{q_0}{lD} \sum_{k=K}^{\infty} A_k + \sum_{j=1}^{K-1} \frac{(q_j + r_j)}{lD} A_{k+1} + \frac{q_K \overline{X}}{D} && k = K
\end{aligned} \qquad (56)
$$

To simplify this further, we need the following result, which is similar to the one given earlier in (29) for the multiple vacation case. This states that

$$
r_k = q_0 A_k + \sum_{j=1}^{k} (q_j + r_j) A_{k-j+1} + \sum_{j=k+1}^{K} q_j \qquad k=1,\ldots,K-1 \qquad (57)
$$

where A_k is the probability that there are k or more arrivals in a service time as defined in (21). This result may be proved by first showing that it holds for $k=1$ and then showing that it would hold for $k+1$, given that it holds for k. Using (57), we can simplify (56) to get our final expressions for R_k.

$$
\begin{aligned}
R_k &= \frac{1}{lD}\left[r_k - \sum_{j=k+1}^{K} q_j \right] && k = 0,\ldots,(K-1) \\
&= r_c\left(1 - \frac{1}{r}\right) + \frac{1}{lD}\left[q_0 + \sum_{j=1}^{K} j q_j \right] && k = K
\end{aligned} \qquad (58)
$$

The probability p_k of finding k jobs in the system at an arbitrary instant of time for $k=1,\ldots,K$ may then be found as

$$
p_k = Q_k + R_k \qquad k = 1,\ldots,K \qquad (59)
$$

Using this and (54), we get our final result for the state probability at an arbitrary instant as

$$
\begin{aligned}
p_k &= \frac{r_0}{lD} && k = 0 \\
&= \frac{r_k}{lD} && k = 1,\ldots,K-1 \\
&= 1 - \frac{r_c}{r} && k = K
\end{aligned} \qquad (59)
$$

The mean number in the system N at an arbitrary instant of time may be found using (60) to be

$$N = \frac{1}{lD}\sum_{k=1}^{K-1} kr_k + K\left(1 - \frac{r_c}{r}\right) \tag{61}$$

The mean time W spent in system by a job which actually does enter the system (i.e. is not blocked and denied entry) may be obtained by applying Little's result to (61) to get

$$W = \frac{N}{l(1 - P_B)} \tag{62}$$

We also get that

$$\text{P\{server is on vacation\}} = \frac{r_0 \overline{V}}{D} \tag{63}$$

$$\text{P\{server is busy serving a job\}} = \frac{(1 - r_0)\overline{X}}{D} = r_c \tag{64}$$

$$\text{P\{server is not on vacation but is idle\}} = \frac{q_0}{lD} \tag{65}$$

$$\text{P\{server is idle\}} = p_0 = \frac{r_0 \overline{V}}{D} + \frac{q_0}{lD} \tag{66}$$