Another Look at the M/G/1 Queue

Analyzing the M/G/1 Queue Using the Method of Supplementary Variables

The M/G/1 queue may also be analysed using the *Method of Supplementary Variables* as described in [Kle75], [Takagi2]. This method may also be used to study several variations of the basic M/G/1 queue [Takagi2]. The results obtained using this approach is the same as that obtained through the *Imbedded Markov Chain* based analysis given earlier. However, this method would provide a new insight in to the system and may therefore lead to a better understanding of the overall operation of the M/G/1 queue.

Consider a M/G/1 queue which has $N(t)$ users at time t. Note that $N(t)$ will not be a Markov Process by itself. This was the reason why the *Imbedded Markov Chain* approach had to imbed the chain at the special time points, i.e. the departure instants of jobs, where the Markovian property would hold. However, if we assume that $X_0(t)$ is the service time already received by the user currently in service, then the joint process $[N(t), X_0(t)]$ would be a *Continuous Time Markov Process*. This process, including the *elapsed service time* $X_0(t)$ as the supplementary variable, helps in the analysis of the queue using the method of supplementary variables. Note that, by definition, we have $X_0(t)=0$ when $N(t)=0$, i.e. when the system is empty. We assume that jobs arrive to the queue from a Poisson process with average arrival rate λ.

As usual, we define $P_k(t)=P\{N(t)=k\}$ to be the probability of finding the system in state k at time t. Let $p_k=P_k(t)$ as $t \to \infty$ be the equilibrium state probability distribution for $k=0,1,......,\ \infty$. We also define the joint probability density $f_k(t, x)$ as

$$f_k(t,x)dx = P\{N(t)=k, x < X_0(t) \leq x+dx\} \tag{1}$$

Considering this at an arbitrary time instant (i.e. under equilibrium) as $t \rightarrow \infty$, we define

$$\begin{aligned} &f_k(x)dx = P\{N=k, x < X_0 \leq x+dx\} = \lim_{t\to\infty} f_k(t,x\}dx \\ &f_0(x) = 0 \end{aligned} \tag{2}$$

Consider a job which requires a service of duration X with pdf $b(x)$ and cdf $B(x)$. Let $b_c(x)$ be the pdf of the service time X given that $X>x$, such that

$$b_c(x)dx = P\{x < X < x+dx \mid X > x\} \tag{3}$$

Using the fact that the cdf $B(x)=P\{X \leq x\}$ and Baye's rule, we get

$$b_c(x) = \frac{b(x)}{1-B(x)} \tag{4}$$

Under equilibrium conditions, we can equate the flow from state 0 to state 1 and vice versa. This gives

$$\lambda p_0 = \int_0^\infty f_1(x) b_c(x) dx \tag{5}$$

For the higher states, $k=1,......,\infty$, we can similarly show that

$$\begin{aligned} f_k(x+\Delta x)dx = {} & \lambda \Delta x[1-b_c(x)\Delta x] f_{k-1}(x)dx \\ & + (1-\lambda \Delta x)[1-b_c(x)\Delta x] f_k(x)dx \end{aligned} \qquad k=1,......,\infty \tag{6}$$

To see this, consider the definition of $f_k(x)$ as given in (2) along with the definition of the *elapsed service time* (with pdf $b_c(x)$) and the arrival rate λ of new jobs to the system. Consider the event $[N(t+\Delta x)=k,\ X_0(t+\Delta x)=x+\Delta x\}$ which examines the system at time $t+\Delta x$ to find k jobs in the system (including the one being served) and that the job in service has already obtained a service of duration $x+\Delta x$. This event can occur in two ways -

(a) at time t, the state was $[N(t)=k, X_0=x]$ and that there were no arrivals (probability = $1-\lambda\Delta x$) and no service completion (probability = $1-b_c(x)\Delta x$) during the interval Δx.

or

(b) at time t, the state was $[N(t)=k-1, X_0=x]$ and that there was an arrival in the interval Δx but there was no service completion.

We also assume that $t\rightarrow\infty$. This is so that we may assume that the system has reached equilibrium conditions provided the traffic $\rho=\lambda E\{X\}$ is such that $\rho<1$. Retaining only the Δx terms and dropping those with higher powers of Δx (in anticipation of the fact that we would eventually let $\Delta x\rightarrow 0$), we get

$$f_k(x+\Delta x)=\lambda\Delta x f_{k-1}(x)+\left[1-\Delta x\left(\lambda+b_c(x)\right)\right]f_k(x) \qquad k=1,......,\infty \qquad (7)$$

Taking the limits as $\Delta x\rightarrow 0$ in (7) gives

$$\frac{df_k(x)}{dx}+\left[\lambda+b_c(x)\right]f_k(x)=\lambda f_{k-1}(x) \qquad k=1,......,\infty \qquad (8)$$

In order to solve for the probability densities $f_k(x)$ using (8), we would need appropriate boundary conditions. Using the earlier arguments for $x=0$, we can write these as

$$\begin{aligned} f_1(0)&=\lambda p_0+\int_0^\infty f_2(x)b_c(x)dx && k=1 \\ f_k(0)&=\int_0^\infty f_{k+1}(x)b_c(x)dx && k=2,.....,\infty \end{aligned} \qquad (9)$$

The corresponding *normalisation condition* may also be written as

$$\sum_{k=0}^{\infty}p_k=p_0+\sum_{k=1}^{\infty}\int_0^\infty f_k(x)dx=1 \qquad (10)$$

The actual equilibrium solution p_k $k=0,1,.....,\infty$ may be obtained by solving the equations (5), (8) and (9) along with the normalisation condition of (10). A way to solve this is given next.

We define $F(z,x)$ as

$$F(z,x)=\sum_{k=1}^{\infty}f_k(x)z^k$$

Multiplying (8) by z^k for each *k=1,.....,¥* and summing over *k*, we get

$$
\begin{aligned}
&\frac{\partial F(z,x)}{\partial x} + \left[l + b_c(x) \right] F(z,x) = lzF(z,x) \\
&or \quad \frac{\partial F(z,x)}{\partial x} = \left[lz - l - b_c(x) \right] F(z,x)
\end{aligned}
\tag{11}
$$

Note that for obtaining (11), we have used the fact that $f_0(x)=0$ from (2). Using (9) in a similar way, i.e. by multiplying the k^{th} equation by z^k and summing over *k=1,.....,¥*, we get

$$
\begin{aligned}
&F(z,0) = lzp_0 + \sum_{k=1}^{\infty} z^k \int_0^{\infty} f_{k+1}(x) b_c(x) dx \\
&or \quad zF(z,0) = lz(z-1)p_0 + \int_0^{\infty} b_c(x) F(z,x) dx
\end{aligned}
\tag{12}
$$

It is easier to solve this by doing a change of variables defining $g_k(x)$ and its generating function $G(z,x)$ as follows

$$
\begin{aligned}
&g_k(x) = \frac{f_k(x)}{1-B(x)} \qquad k = 1,......,\infty \\
&g_0(x) = 0
\end{aligned}
\tag{13}
$$

$$
G(z,x) = \sum_{k=1}^{\infty} g_k(x) z^k = \frac{F(z,x)}{1-B(x)}
\tag{14}
$$

where $B(x)$ is the cdf of the service time distribution corresponding to the pdf $b(x)$. Using (11), we then get

$$
\begin{aligned}
&[1-B(x)] \frac{\partial G(z,x)}{\partial x} - b(x) G(z,x) + [1-B(x)][l + b_c(x)] G(z,x) \\
&\qquad = lz[1-B(x)] G(z,x) \\
&\frac{\partial G(z,x)}{\partial x} + l(1-z) G(z,x) = 0
\end{aligned}
\tag{15}
$$

The solution to (15) may be written as

$$G(z,x) = G(z,0)e^{-l(1-z)x} \tag{16}$$

where the initial condition *G(z,0)* may be found using (12). For this, note that *F(z,0)=G(z,0)* and $f_k(0)=g_k(0)$ for *k=0,1,.....,¥*. Using this in (12) gives

$$\begin{aligned} zG(z,0) &= lz(z-1)p_0 + \int_0^{\infty} b(x)G(z,0)e^{-l(1-z)x}dx \\ &= lz(z-1)p_0 + G(z,0)L_B(l - lz) \end{aligned}$$

Note that $L_B(s)$ is the Laplace Transform of the pdf *b(x)* of the service time. Simplifying yields

$$G(z,0) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} \tag{17}$$

$$G(z,x) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} e^{-l(1-z)x} \tag{18}$$

Therefore,

$$F(z,0) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} \tag{19}$$

$$F(z,x) = \frac{lz(1-z)p_0}{L_B(l - lz) - z}[1-B(x)]e^{-l(1-z)x} \tag{20}$$

We define $F(z) = \int_{x=0}^{\infty} F(z,x)dx$ which can then be obtained using (20) to be

$$\begin{aligned} F(z) &= \left(\frac{lz(1-z)p_0}{L_B(l - lz) - z}\right)\left(\frac{1 - L_B(l - lz)}{l(1-z)}\right) \\ or \quad F(z) &= \frac{zp_0[1 - L_B(l - lz)]}{L_B(l - lz) - z} \end{aligned} \tag{21}$$

Note that, in order to obtain (21), we have used the result

$$\int_{x=0}^{\infty} B(x)e^{-l(1-z)x}dx = \frac{L_B(l - lz)}{l(1-z)}$$

which may be shown by direct integration.

Note that the state probabilities $P_k(t)$ of the system at time t may be obtained from the definition of $f_k(t,x)$ in (1), as

$$P_k(t) = \int_{x=0}^{\infty} f_k(t,x)dx \qquad k=1,.....,¥ \qquad (22)$$

The corresponding equilibrium state probabilities, p_k may be obtained from (22) as

$$p_k = \lim_{t\to\infty} P_k(t) = \int_{0}^{\infty} f_k(x)dx \qquad k=1,.....,¥ \qquad (23)$$

where the equilibrium probability p_0 of the system being empty will have to be found by applying the normalisation condition of (10). It may also be noted that (23) may be used to observe that

$$\sum_{k=1}^{\infty} p_k z^k = \sum_{k=1}^{\infty} z^k \left(\int_{x=0}^{\infty} f_k(x)dx \right) = \int_{x=0}^{\infty} F(z,x)dx = F(z)$$

Therefore, evaluating $F(z)$ at $z=1$,we will get $F(1)=1-p_0$ corresponding to the required normalisation condition. Using this in (21) gives

$$\begin{aligned} 1 - p_0 &= F(z)\big|_{z=1} = p_0 \frac{(-l\overline{X})}{l\overline{X} - 1} \\ or \quad p_0 &= (1 - l\overline{X}) = (1 - r) \qquad with \quad r = l\overline{X} \end{aligned} \qquad (24)$$

The generating function $P(z)$ of the system state at equilibrium will then be given by

$$\begin{aligned} P(z) = p_0 + F(z) = p_0\left[1 + \frac{z[1 - L_B(l - lz)]}{L_B(l - lz) - z}\right] \\ = \frac{(1-z)(1-r)L_B(l - lz)}{L_B(l - lz) - z} \end{aligned} \tag{25}$$

Note that this is the same as the *P-K Transform Equation* result for the M/G/1 queue obtained in (3.14) using the *Imbedded Markov Chain Approach.*

The Elapsed Service Time Approach for the M/G/1 Queue

Consider once again the M/G/1 queue with infinite buffers. Following the same approach as in Section 3.2, we consider once again the imbedded time points at the departure instants of the jobs after service completion. These correspond to the time instants marked with the shaded circles in Fig. 1. As in our earlier analysis of Section 3.2, we consider the Markov Chain of system states at these imbedded points where the state of the system is represented by the number left behind in the queue by a departing job.

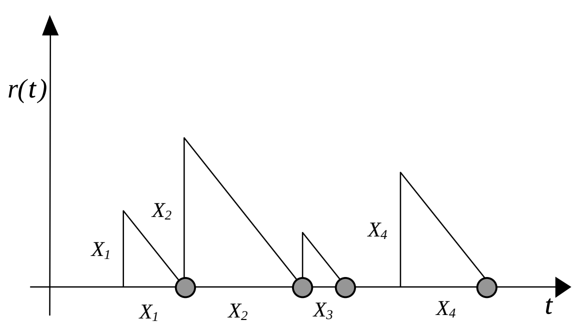

Figure 1. Imbedded Points at the Job Departure Instants of the M/G/1 Queue

In the equilibrium analysis of Section 3.2, we directly obtained the generating function of the system states at these imbedded points (actually just after the imbedded points). Since the number of jobs in this system can change by at most ±1, we then used Kleinrock's principle to claim that this will also be the generating function of the system states at the arrival instants of jobs. Finally, the PASTA property was used to claim that this generating function will also be what will be observed at an arbitrary instant in the queue.

We consider a slightly different analytical approach in this section. We still obtain the state distribution at the imbedded points as before. Instead of obtaining the generating function directly, we actually obtain a system of equations that may be solved to obtain the state probabilities at the job departure instants. (These may also be used to find the same generating function as before.) Let q_i be the probability of there being i jobs in the system as observed by a departing job. We use this and result from residual life arguments to obtain the state probability p_i of there being i jobs in the system at an arbitrary time instant between successive imbedded points. This would give us the expected result that $p_i = q_i$, $i=0,1,.......¥$. It may however be noted that in this case, we get the desired results without invoking Kleinrock's principle or PASTA.

We define a_k as the probability of k arrivals in a service time. Note that the service times are considered to be random variables with pdf $b(t)$, cdf $B(t)$ and with the Laplace Transform of the pdf given by $L_B(s)$. Let $\overline{X}$ be the mean service time. Since the arrivals come from a Poisson process at rate l, we get that

$$a_k = \int_{x=0}^{\infty} \frac{(lx)^k}{k!} e^{-lx} b(x) dx \qquad k = 0,1,.......\infty \tag{26}$$

It may be noted that the z-transform of a_k, defined as $A(z)$, may be obtained as in (3.1).

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = L_B(l - lz) \tag{27}$$

This will be the generating function of the number of job arrivals in a service time. We also define A_k, $k=0,1,.......,¥$ as the probability of there being k or more arrivals in a service time defined as

$$\begin{aligned} A_0 &= 1 \\ A_k &= \sum_{k}^{\infty} a_k \qquad k = 0,1,.......\infty \end{aligned} \tag{28}$$

We also have that

$$a_k = A_k - A_{k+1} \qquad k = 0,1,.......\infty \tag{29}$$

Focussing on the imbedded points under equilibrium conditions, let q_{ij} $i,j=0,1,.........., ¥$ be the transition probability of the system going from state i to state j from one imbedded point to the next. These transition probabilities will be given by

$$\begin{aligned} q_{jk} &= a_k && j = 0 \\ &= a_{k-j+1} && j = 1,2,.......,\infty \end{aligned} \qquad (30)$$

Using these, we can obtain the equilibrium state probabilities at the departure instants by solving

$$q_k = \sum_{j=0}^{\infty} q_j q_{jk} \qquad k = 0,1,........,\infty \qquad (31)$$

along with the normalisation condition

$$\sum_{k=0}^{\infty} q_k = 1 \qquad (32)$$

Note that, using (30), we can also write (31) as

$$\begin{aligned} q_o &= q_0 a_0 + q_1 a_0 && k = 0 \\ q_1 &= q_0 a_1 + q_2 a_0 + q_1 a_1 && k = 1 \\ &............ \\ &............ \\ q_k &= q_0 a_k + q_{k+1} a_0 + \sum_{j=1}^{k} q_j a_{k-j+1} && k = 1,......,\infty \end{aligned} \qquad (33)$$

Multiplying the k^{th} equation in (33) by z^k and summing all the left-hand sides and the right-hand sides from $k=0$ to $k=¥$, we get

$$\begin{aligned} Q(z) &= q_0 A(z) + q_1 A(z) + q_2 z A(z) + q_3 z^2 A(z) + ... \\ &= q_0 A(z) + \frac{A(z)}{z}\left[Q(z) - q_0\right] \end{aligned}$$

where $Q(z)$ is the generating function of the number left behind in the system by a departing job under equilibrium conditions. Rearranging terms gives us

$$Q(z) = \frac{q_0(1-z)A(z)}{A(z)-z} \tag{34}$$

Using the property that *Q(1)=1* (evaluated in the same way as done for *P(1)* in Section 3.2) will give us

$$q_0 = 1 - r \qquad \text{for } \; r = l\overline{X} \tag{35}$$

Using q_0 as obtained from (35), we can evaluate q_k from (33) as

$$\begin{aligned} q_1 &= \frac{1}{a_0}\left[q_0(1-a_0)\right] && k=1 \\ q_k &= \frac{1}{a_0}\left[q_{k-1} - \sum_{j=1}^{k-1} q_j a_{k-j} - q_0 a_{k-1}\right] && k = 2,......,\infty \end{aligned} \tag{36}$$

Having obtained the state probabilities at the imbedded points, we can now use these results to get the state probabilities at an arbitrary instant of time. For this, we first note that the mean time interval *D* between successive imbedded points will be given by

$$\begin{aligned} D &= q_0\left(\frac{1}{l} + \overline{X}\right) + (1-q_0)\overline{X} \\ &= \overline{X} + q_0\frac{1}{l} \end{aligned} \tag{37}$$

The probability p_0 of examining the system at an arbitrary time instant and finding it empty will be the fraction of time the system stays idle in the time interval between successive imbedded points. This implies that

$$p_0 = \frac{q_0\frac{1}{l}}{\overline{X} + q_0\frac{1}{l}} = \frac{q_0}{q_0 + r} \tag{38}$$

Substituting for q_0 using (35), gives us the expected result that

$$p_0 = q_0 = 1 - r \tag{39}$$

To prove the similar results, i.e. $p_k = q_k$ for $k=1,2,.......,¥$, we consider the event of examining the system at an arbitrary time instant and finding k jobs in the system. Note that since we are not considering $k=0$, this arbitrary instant of time will not be one when the system is empty, i.e. in the time duration from the last departure which left the system empty to an instant after the next arrival, which starts the server once again.) Two cases may arise where the arbitrary time instant chosen falls in a service time, i.e. when the server is busy. These are

(a) The time instant chosen falls in a service time following an imbedded point where the queue was empty. The probability of this will be $\dfrac{q_0 \overline{X}}{\overline{X} + q_0 \dfrac{1}{l}}$.

Let x be the time interval between the arrival of the first customer following the last imbedded point (where the system became empty) and the time instant chosen. The system will have k jobs at the chosen time instant if there are k job arrivals in this time interval where the arrivals come from a Poisson process. The pdf of the interval x itself may be obtained from residual life arguments to be $\dfrac{1 - B(x)}{\overline{X}}$

(b) The time instant chosen falls in a service time following an imbedded point where the queue was not empty. This implies that the system state at that earlier imbedded point may be j where j may range from 1 to k. The probability of choosing an arbitrary time instant within a service time and with a particular $j ³ 1$, will be $\dfrac{q_j \overline{X}}{\overline{X} + q_0 \dfrac{1}{l}}$. As in (a), let x be the time interval between the last imbedded point and the time instant chosen. If there were j jobs in the system at the last imbedded point, we need $(k\text{-}j)$ job arrivals in this time interval where the arrivals come from a Poisson process; this is so that the system is in state k at the chosen instant. The pdf of the interval x itself may be obtained from residual life arguments to be $\dfrac{1 - B(x)}{\overline{X}}$ as in (a).

Using (a) and (b), we get

$$p_k = \left(\frac{q_0 \overline{X}}{\overline{X} + q_0 \frac{1}{l}} \right) \int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx} \frac{1-B(x)}{\overline{X}} dx$$

$$\sum_{j=1}^{k} \left(\frac{q_j \overline{X}}{\overline{X} + q_0 \frac{1}{l}} \right) \int_0^\infty \frac{(lx)^{k-j}}{(k-j)!} e^{-lx} \frac{1-B(x)}{\overline{X}} dx \qquad k = 1, \ldots\ldots, \infty \qquad (40)$$

To simplify (40) further, we need the result

$$A_k = \sum_{j=k} a_j = \sum_{j=k}^{\infty} \int_0^\infty \frac{(lx)^j}{j!} e^{-lx} b(x) dx$$
$$= \int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx} [1 - B(x)] l dx \qquad k=1,2,\ldots\ldots,¥ \qquad (41)$$

and $\sum_{k=1}^{\infty} A_k = l\overline{X} = r$ (42)

Applying these to (40) gives

$$p_k = \left(\frac{q_0}{q_0 + r} \right) \left[A_k + \sum_{j=1}^{k} \left(\frac{q_j}{q_0} \right) A_{k-j+1} \right]$$
$$= p_0 \left[A_k + \sum_{j=1}^{k} \left(\frac{q_j}{q_0} \right) A_{k-j+1} \right] \qquad k=1,2,\ldots\ldots,¥ \qquad (43)$$
$$= q_0 \left[A_k + \sum_{j=1}^{k} \left(\frac{q_j}{q_0} \right) A_{k-j+1} \right]$$

We can show that

$$\left[A_k + \sum_{j=1}^{k} \left(\frac{q_j}{q_0} \right) A_{k-j+1} \right] = \frac{q_k}{q_0} \qquad k=1,2,\ldots\ldots,¥ \qquad (44)$$

For showing (44), subtract q_{k+1} from q_k using the relevant expression for both from (33). Manipulation of the resultant expression would lead to (44).

Substituting (44) in (43) gives us the desired result that $p_k=q_k$ also for $k=1,2,......,$¥. Note that this result has been proved without using PASTA and Kleinrock's principle.