M/G/m/m Loss System

We can use our analysis of the *finite capacity* M/G/1/K system, to get results for the M/G/1/1 queue, considered as a special case of the M/G/m/m, *m-server loss system* discussed here.

Consider the M/G/1/K queue for the special case of $K=1$. For this, we get $p_{d,0}=1$, as the departing job will always leave the system empty. This leads to the equilibrium state probabilities at any arbitrary instant to be

$$p_1 = \frac{r}{1+r} = P_B \quad p_0 = 1 - p_1 = \frac{1}{1+r} \quad \textit{for } r = l\overline{X} = \text{offered traffic}$$

The throughput (carried traffic) of the M/G/1/1 queue will be

$$r_c = r(1 - P_B) = \frac{r}{1+r}$$

Note that this, $r(1+r)^{-1}$, will also be the mean number in the system As expected, the mean delay W through this queue will be merely its mean service time $\overline{X}$.

The equilibrium state distribution at an arbitrary time instant for the M/G/m/m queue is obtained subsequently. Rather surprisingly, it turns out that the state probabilities of this system are the same as that of the corresponding M/M/m/m system where the service times are exponentially distributed. This is the reason why it is sometimes stated that the state probabilities of a *m-server loss system* are independent of the actual state distribution of its service times. It may be recalled that the M/M/m/m system was actually used to model a telephone exchange and its probability of blocking was given by the *Erlang Blocking* formula to be

$$P_B = B(m, r) = \frac{\dfrac{r^m}{m!}}{\sum_{j=0}^{m} \dfrac{r^j}{j!}} \qquad r = l\overline{X} \tag{1}$$

where r is the load offered to the system. This would then also hold for a M/G/m/m system. Another implication of our earlier statement would then be that the Erlang Blocking formula may still be used to calculate the blocking (i.e. the grade of service) in a telephone system, even when the call duration is not an exponentially distributed random variable.

Consider a M/G/m/m queue, where the average arrival rate is l and where the service time X has a pdf $b(x)$ with cdf $B(x)$. We can define the conditional distribution $b_c(x)$ as the pdf of the service time X, given that $X>x$, such that

$$b_c(x)dx = P\{x < X < x + dx \mid X > x\}$$

Using the fact that the cdf *B(x)=P{X£ x}* and Baye's rule, we get

$$b_c(x) = \frac{b(x)}{1 - B(x)} \tag{2}$$

This conditional distribution is required in the subsequent derivation.

Let N be the number in the system, with *0 £ N £ K*. We also define p_0 as the equilibrium probability that the system is empty, i.e. $p_0 = P\{N=0\}$. Note that since this queue has no additional buffers other than the servers, the state N also represents the number of busy servers in the queue. An arrival is lost (i.e. blocked) if it finds the system in state N. Therefore the probability of blocking P_B under equilibrium conditions will be the same as p_N.

For a system in state $k>0$ (i.e. $N=k$), let the random variables X_i $i=1,...,k$ represent the elapsed service time for the job at the i^{th} server. We define the joint probability density $f_k(x_1,.....,x_k)$ as

$$f_k(x_1,....,x_k)dx_1.....dx_k$$
$$\underline{\Delta} P\{N = k,\ x_1 < X_1 < x_1 + dx_1,.....,x_k < X_k < x_k + dx_k\} \quad k=1,...,m \quad (3)$$
$$p_k = \int_0^\infty .....\int_0^\infty f_k(x_1,....,x_k)dx_1.....dx_k$$

Using these, the following balance equations may be written

$$f_1(0)dx = l p_0 dx$$
$$f_1(0) = l p_0 \quad (4)$$

$$f_{k+1}(x_1,.....,x_k,0)(k+1)dx = l f_k(x_1,....x_k)dx$$
$$f_{k+1}(x_1,.....,x_k,0) = \frac{l}{k+1} f_k(x_1,....x_k) \quad k=1,.....,m-1 \quad (5)$$

$$f_k(x_1 + \Delta x,......., x_k + \Delta x) = f_k(x_1,......., x_k)(1 - l\Delta x)\prod_{n=1}^{k}[1 - b_c(x_n)]\Delta x$$
$$+ (k+1)\int_0^\infty f_{k+1}(x_1,......., x_k, x)b_c(x)dx\Delta x \qquad k = 1,......., m-1$$
$$f_m(x_1 + \Delta x,......., x_m + \Delta x) = f_m(x_1,......., x_m)\prod_{n=1}^{m}[1 - b_c(x_n)]\Delta x \quad k = m$$
$$(6)$$

It should be noted that for Δx small, i.e. when $\Delta x \to 0$

$$f_k(x_1 + \Delta x,......., x_k + \Delta x) = f_k(x_1,......., x_k) + \Delta x\sum_{n=1}^{k}\frac{\partial f_k(x_1,......., x_k)}{\partial x}$$
$$(7)$$

Using (6) and (7), we get that for $\Delta x \to 0$

$$\sum_{n=1}^{k} \frac{\partial f_k(x_1,.......,x_k)}{\partial x_n} + \left[l + \sum_{n=1}^{k} b_c(x_n) \right] f_k(x_1,.......,x_k)$$

$$= (k+1)\int_0^\infty f_{k+1}(x_1,.......,x_k, x) b_c(x)dx \qquad k = 1,.......,m-1 \qquad (8)$$

$$\sum_{n=1}^{m} \frac{\partial f_m(x_1,.......,x_m)}{\partial x_n} + \left[\sum_{n=1}^{k} b_c(x_n) \right] f_m(x_1,.......,x_m) = 0$$

The conditional distributions $f_k(x_1,....., x_k)$ may be obtained by solving (8) subject to the boundary conditions given by (4) and (5). This solution will be

$$f_k(x_1,.......,x_k) = f_k(0,.......,0) \prod_{n=1}^{k} [1 - B(x_n)] \qquad k = 1,.......,m \qquad (9)$$

subject to $l f_k(0,.......,0) = (k+1) f_{k+1}(0,.......,0,0) \qquad k = 1,.......,m-1 \qquad (10)$

Using (5) and (10) gives

$$f_k(0,.......,0) = p_0 \frac{l^k}{k!} \qquad k = 1,.......,m \qquad (11)$$

Note that $\int_0^\infty [1 - B(x)]dx = [1 - B(x)]x\Big|_0^\infty - \int_0^\infty \frac{d[1 - B(x)]}{dx} x dx = \int_0^\infty x b(x) dx = \overline{X}$

We use this and apply (3), (9) and (11) to the normalization condition, to get

$$1 = \sum_{k=0}^{m} p_k = p_0 + \sum_{k=1}^{m} f_k(0,.......,0)(\overline{X})^k = p_0 \sum_{k=0}^{m} \frac{r^k}{k!} \qquad r = l\overline{X} \qquad (12)$$

This gives the equilibrium state probabilities as

$$p_k = \frac{\frac{r^k}{k!}}{\sum_{n=0}^{m} \frac{r^n}{n!}} \qquad \text{for } k=0,1,.......,m \qquad (13)$$

and the blocking probability $P_B=p_m$ leading to the same equation as (1), mentioned earlier.