M/G/m/m Loss System

We can use our analysis of the finite capacity M/G/1/K system, to get results for the M/G/1/1 queue, considered as a special case of the M/G/m/m, m-server loss system discussed here.

Consider the M/G/1/K queue for the special case of $K=1$. For this, we get $p_{d,0}=1$, as the departing job will always leave the system empty. This leads to the equilibrium state probabilities at any arbitrary instant to be

$$p_d = \frac{\rho}{1+\rho} = P_{\text{b}} \quad p_0 = 1 - p_1 = \frac{1}{1+\rho} \quad \text{for } \rho = \lambda \bar{X} = \text{offered traffic}$$

The throughput (carried traffic) of the M/G/1/1 queue will be

$$\rho_c = \rho(1-P_{\text{b}}) = \frac{\rho}{1+\rho}$$

Note that this, $\rho(1+\rho)^{-1}$, will also be the mean number in the system. As expected, the mean delay $W$ through this queue will be merely its mean service time $\bar{X}$.

The equilibrium state distribution at an arbitrary time instant for the M/G/m/m queue is obtained subsequently. Rather surprisingly, it turns out that the state probabilities of this system are the same as that of the corresponding M/M/m/m system where the service times are exponentially distributed. This is the reason why it is sometimes stated that the state probabilities of a m-server loss system are independent of the actual state
distribution of its service times. It may be recalled that the M/M/m/m system was actually used to model a telephone exchange and its probability of blocking was given by the Erlang Blocking formula to be

\[
P_B = B(m, \rho) = \frac{\rho^m}{m!} \sum_{j=0}^{m} \frac{\rho^j}{j!} \quad \rho = \lambda \bar{X}
\]

where \( \rho \) is the load offered to the system. This would then also hold for a M/G/m/m system. Another implication of our earlier statement would then be that the Erlang Blocking formula may still be used to calculate the blocking (i.e. the grade of service) in a telephone system, even when the call duration is not an exponentially distributed random variable.

Consider a M/G/m/m queue, where the average arrival rate is \( \lambda \) and where the service time \( X \) has a pdf \( b(x) \) with cdf \( B(x) \). We can define the conditional distribution \( b_c(x) \) as the pdf of the service time \( X \), given that \( X > x \), such that

\[
b_c(x)dx = P[x < X < x + dx | X > x]
\]

Using the fact that the cdf \( B(x) = P[X \leq x] \) and Baye's rule, we get

\[
b_c(x) = \frac{b(x)}{1 - B(x)}
\]

This conditional distribution is required in the subsequent derivation.

Let \( N \) be the number in the system, with \( 0 \leq N \leq K \). We also define \( p_0 \) as the equilibrium probability that the system is empty, i.e. \( p_0 = P[N=0] \). Note that since this queue has no additional buffers other than the servers, the state \( N \) also represents the number of busy servers in the queue. An arrival is lost (i.e. blocked) if it finds the system in state \( N \). Therefore the probability of blocking \( P_B \) under equilibrium conditions will be the same as \( p_N \).

For a system in state \( k > 0 \) (i.e. \( N=k \), let the random variables \( X_i \) for \( i=1,...,k \) represent the elapsed service time for the job at the \( i \)th server. We define the joint probability density \( f_s(x_1,...,x_k) \) as
Using these, the following balance equations may be written:

\[ f_i(0)dx = \lambda p_0 dx \]
\[ f_i(0) = \lambda p_0 \] (4)

\[
\begin{align*}
    f_{k+1}(x_1, \ldots, x_k,0)(k+1)dx &= \lambda f_k(x_1, \ldots, x_k)dx \\
    f_{k+1}(x_1, \ldots, x_k,0) &= \frac{\lambda}{k+1} f_k(x_1, \ldots, x_k) \\
    k &= 1, \ldots, m-1
\end{align*}
\] (5)

\[
\begin{align*}
    f_k(x_1 + \Delta x, \ldots, x_k + \Delta x) &= f_k(x_1, \ldots, x_k)(1 - \lambda \Delta x) \prod_{n=1}^{k} [1 - b_c(x_n)] \Delta x \\
    f_m(x_1 + \Delta x, \ldots, x_m + \Delta x) &= f_m(x_1, \ldots, x_m) \prod_{n=1}^{m} [1 - b_c(x_n)] \Delta x \\
    k &= m
\end{align*}
\] (6)

It should be noted that for \( \Delta x \) small, i.e. when \( \Delta x \to 0 \)

\[
\begin{align*}
    f_k(x_1 + \Delta x, \ldots, x_k + \Delta x) &= f_k(x_1, \ldots, x_k) + \Delta x \sum_{n=1}^{k} \frac{\partial f_k(x_1, \ldots, x_k)}{\partial x} \\
    k &= m
\end{align*}
\] (7)

Using (6) and (7), we get that for \( \Delta x \to 0 \)
The conditional distributions \( f_k(x_1, \ldots, x_k) \) may be obtained by solving (8) subject to the boundary conditions given by (4) and (5). This solution will be

\[
f_k(x_1, \ldots, x_k) = f_k(0, \ldots, 0) \prod_{n=1}^{k} [1 - B(x_n)] \quad k = 1, \ldots, m \tag{9}
\]

subject to \( \lambda f_k(0, \ldots, 0) = (k + 1) f_{k+1}(0, \ldots, 0, 0) \quad k = 1, \ldots, m - 1 \tag{10} \)

Using (5) and (10) gives

\[
f_k(0, \ldots, 0) = p_0 \frac{\lambda^k}{k!} \quad k = 1, \ldots, m \tag{11}
\]

Note that

\[
\int_0^\infty [1 - B(x)] dx = [1 - B(x)] |_0^\infty - \int_0^\infty \frac{d[1 - B(x)]}{dx} dx = \int_0^\infty x b(x) dx = \bar{X}
\]

We use this and apply (3), (9) and (11) to the normalization condition, to get

\[
1 = \sum_{k=0}^{m} p_k = p_0 + \sum_{k=1}^{m} f_k(0, \ldots, 0)(\bar{X})^k = p_0 \sum_{k=0}^{m} \frac{\rho^k}{k!} \quad \rho = \lambda \bar{X} \tag{12}
\]

This gives the equilibrium state probabilities as

\[
p_k = \frac{\rho^k}{k!} \quad \text{for} \quad k = 0, 1, \ldots, m \tag{13}
\]
and the blocking probability $P_b = p_m$ leading to the same equation as (1), mentioned earlier.