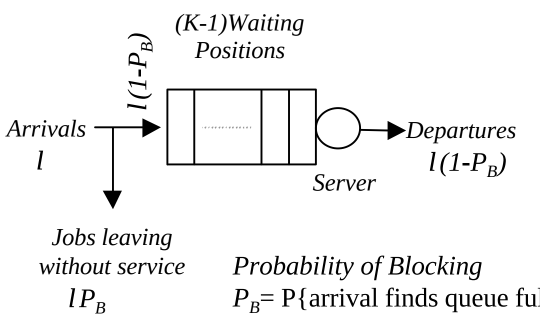
Analysis

of

A Finite Capacity, Single Server Queue

(M/G/1/K)



Probability of Blocking
P_B= P{arrival finds queue full}

A Finite Capacity, Single Server, M/G/1/K Queue

• We use the same notation as that used earlier for the M/G/1 queue in Section 3.2.

• The system state (i.e. the number in the system) at the imbedded points corresponding to the time instants just after a job completion will form a Markov Chain

$$\begin{aligned} n_{i+1} &= \min\{a_{i+1}, K-1\} && \textit{for} \quad n_i = 0 \\ &= \min\{n_i - 1 + a_{i+1}, K-1\} && \textit{for} \quad n_i = 1,......,(K-1) \end{aligned} \qquad (1)$$

n_i = Number left behind in the system by the i^{th} departure

Imbedded Points Û Departure Instants of Jobs after completing service

The "max" function in (1) will lead to loss of jobs which are denied entry into the queue when the system is full

Considering the Markov Chain of states at the imbedded points (corresponding to the departure instants), we will have at equilibrium -

State Probability at equilibrium:

$p_{d,k}$ = P{system in state k} *k=0,1,.....,(K-1)*

State Transition Probability at equilibrium

$$p_{d,jk} = P\{n_{i+1} = k \mid n_i = j\} \qquad 0 £ j, k£ (K\text{-}1)$$

Note that the system state at the departure instant can only be between *0* and *(K-1)*

a_k = P{k arrivals occurring in a service time}

$$a_k = \int_{t=0}^{\infty} \frac{(lt)^k}{k!} e^{-lt} b(t)dt \qquad (3)$$

Note that a_k may also be found as the coefficient of z^k in the expansion of $L_B(l - lz)$

The state transition probabilities $p_{d,jk}$ for this Markov Chain (at the departure instants) may then be found in terms of a_k as given in the next slide

State Transition Probabilities for the Departure Instants

$$p_{d,0k} = \begin{cases} a_k & 0 \le k \le K-2 \\ \sum_{m=K-1}^{\infty} a_m & k = K-1 \end{cases} \qquad j=0$$

$$p_{d,jk} = \begin{cases} a_{k-j+1} & j-1 \le k \le K-2 \\ \sum_{m=K-j}^{\infty} a_m & k = K-1 \end{cases} \qquad 1£ j£ (K\text{-}1)$$

(4)

K Balance Equations

$$p_{d,k} = \sum_{j=0}^{K-1} p_{d,j}\, p_{d,jk} \qquad k = 0,1,........K-1 \qquad (5)$$

Normalization Condition

$$\sum_{k=0}^{K-1} p_{d,k} = 1 \qquad (6)$$

As usual, we can solve for the equilibrium departure state probabilities $\{p_{d,j}\}$ $j=0,1,\ldots,(K-1)$ using any $(K-1)$ equations from (5) along with the normalization condition of (6).

Alternatively, we can solve for $\{p_{d,j}\}$ $j=0,1,\ldots,(K-1)$ using the following -

$$\left.\begin{aligned} & p_{d,k} = p_{d,0}\,a_k + \sum_{j=1}^{k+1} p_{d,j}\, a_{k-j+1} \qquad k = 0,1,........, K-2 \\ & \sum_{k=0}^{K-1} p_{d,k} = 1 \end{aligned}\right\} \quad (7)$$

See notes for another solution approach

We now need to use $\{p_{d,k}\}$ $k=0,1,\ldots,K-1$ to find the equilibrium state probabilities $\{p_k\}$ $k=0,1,\ldots,K$ at an arbitrary time instant. We would also like to find the probability P_B that an arrival finds the system full and is blocked, i.e. leaves without service.

We summarize our equilibrium state probability definitions as the following

$\{p_{d,k}\}$ $k=0,\ldots,K-1$	State probabilities at departure instants
$\{p_{a,k}\}$ $k=0,\ldots,K$	State probabilities at arrival instants regardless of whether the job joins the queue or is blocked
$\{p_{ac,k}\}$ $k=0,\ldots,K-1$	State probabilities at an arrival instant when the job actually does join the queue

Note that the “departure instant” implies the instant just after a departure and the “arrival instant” implies the instant just before the actual arrival.

PASTA ⇨ $p_k = p_{a,k}$ $\quad k = 0,1,\ldots\ldots,K$ (10)

Kleinrock’s Result ⇨ $p_{d,k} = p_{ac,k}$ $\quad k = 0,1,\ldots\ldots K-1$ (11)

Therefore

$$p_k = p_{a,k} = (1-P_B)\,p_{ac,k} = (1-P_B)\,p_{d,k} \qquad k = 0,1,\ldots\ldots,K-1 \qquad (12)$$

Arrival is not blocked and finds k in system

and

$$P_B = p_{a,K} = 1 - \sum_{k=0}^{K-1} p_{a,k}$$

Average arrival rate of jobs actually entering the system = l_c

$$l_c = l(1 - P_B)$$

Offered Traffic to the queue = $r = l\overline{X}$

Actual Throughput of the Queue = $r_c = r\ (1\text{-}P_B)$

$$\Downarrow$$

$$p_0 = 1 - r_c = 1 - r(1 - P_B)$$

probability of finding the system empty

But from (12), for $k=0$, we have $\qquad p_0 = (1 - P_B)\,p_{d,0}$

Therefore

$$1 - r(1 - P_B) = (1 - P_B)\,p_{d,0} \quad \Longrightarrow \quad P_B = 1 - \frac{1}{p_{d,0} + r} \qquad (14)$$

and

$$p_k = \frac{1}{p_{d,0} + r}\,p_{d,k} \qquad k = 0,1,\ldots\ldots K-1 \qquad (15)$$

Note that (15) implies that at equilibrium, for a given value of k in the range $k=0,\ldots,(K\text{-}1)$, the state probabilities at an arbitrary instant p_k and the state probabilities at the departure instant $p_{d,k}$ are strictly proportional.

Summarizing

• Find the state probabilities $\{p_{d,k}\}$ $k=0,\ldots,(K-1)$ at the departure instants using either (5) & (6) or (7)

• Find the blocking probability P_B using (14). This will also be the same as the probability p_K of observing the system to be in state K at an arbitrary time instant

• Find the state probabilities $\{p_k\}$ $k=0,\ldots,(K-1)$ at an arbitrary instant using (15). Note that $p_K=P_B$

Performance Results

$$N = \sum_{k=0}^{K} kp_k = \frac{1}{(p_{d,0} + r)} \sum_{k=0}^{K-1} kp_{d,k} + K\left(1 - \frac{1}{(p_{d,0} + r)}\right)$$

$$l_c = l(1 - P_B) = \frac{l}{(p_{d,0} + r)}$$

$$W_q = W - \overline{X} = \frac{1}{l} \sum_{k=0}^{K-1} kp_{d,k} + \frac{K}{l}(p_{d,0} + r - 1) - \overline{X}$$

An Alternate Analytical Approach for the M/G/1/K Queue

Consider the mean of the time interval between successive imbedded points (i.e. departure instants).

$\frac{1}{l} + \overline{X}$	queue empty at the previous departure instant	probability = $p_{d,0}$
$\overline{X}$	queue non-empty at the previous departure instant	probability =1- $p_{d,0}$

Therefore

$$p_0 = \frac{\left(\frac{1}{l}\right) p_{d,0}}{\left(\frac{1}{l} + \overline{X}\right) p_{d,0} + \overline{X}(1 - p_{d,0})} = \frac{p_{d,0}}{p_{d,0} + r}$$

Same as in (15)

To find p_k, for $k=1,\ldots,(K-1)$, consider when an arbitrarily chosen time instant falls within a service duration where *x is the amount of service already provided.*

when previous departure leaves system empty

$$p_k = r_c \left[p_{d,0} \int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx} \frac{1 - B(x)}{\overline{X}} dx \right] \qquad (28)$$

$$+ \, r_c \left[\sum_{j=1}^{k} p_{d,j} \int_0^\infty \frac{(lx)^{k-j}}{(k-j)!} e^{-lx} \frac{1 - B(x)}{\overline{X}} dx \right]$$

when previous departure leaves system non-empty

r_c — *P{chosen time instant will fall within a service time}*

$\frac{1 - B(x)}{\overline{X}}$ — *pdf of elapsed service time*

As before, let $A_k = \sum_{j=k} a_j = \sum_{j=k}^{\infty} \int_0^{\infty} \frac{(lx)^j}{j!} e^{-lx} b(x)dx$ $\quad$ *k=1,2,......,¥*

$$= \int_0^{\infty} \frac{(lx)^{k-1}}{(k-1)!} e^{-lx} [1 - B(x)]\, l\, dx \qquad (29)$$

where $\sum_{k=1}^{\infty} A_k = l\overline{X} = r$ $\qquad$ (30)

Using the expression for A_k, we can obtain

$$p_k = \frac{r_c}{r}\left[p_{d,0} A_k + \sum_{j=1}^{k} p_{d,j} A_{k-j+1} \right] \qquad k=1,2,......,¥ \qquad (31)$$

To simplify the expression for p_k further, we use the result

$$p_{d,k} = p_{d,0} A_k + \sum_{j=1}^{k} p_{d,j} A_{k-j+1}$$

Prove using recursion

Substituting this, we get the same result as obtained earlier in (12) and (15) -

$$p_k = \frac{r_c}{r} p_{d,k} = (1 - P_B) p_{d,k} \qquad k=0,1,\ldots,(K-1)$$

Note that we still need to find p_K, the probability of finding the system full at an arbitrary instant to complete the analysis.

This is done in the following slides.

For $k=K$, we need to take into account the fact that arrivals coming when the system is full are blocked and denied entry into the system.

$$\Downarrow$$

$$p_K = r_c \left[p_{d,0} \sum_{k=K-1}^{\infty} \int_0^{\infty} \frac{(lx)^k}{k!} e^{-lx} \frac{1-B(x)}{\overline{X}} dx \right] \qquad (33)$$
$$+ r_c \sum_{j=1}^{K-1} p_{d,j} \sum_{k=K-j}^{\infty} \int_0^{\infty} \frac{(lx)^k}{k!} e^{-lx} \frac{1-B(x)}{\overline{X}} dx$$

Using (29), this gives

$$p_K = \frac{r_c}{r} \left[p_{d,0} \sum_{k=K}^{\infty} A_k + \sum_{j=1}^{K-1} p_{d,j} \sum_{k=K-j+1}^{\infty} A_k \right] \qquad (34)$$

To simplify (34) further, we need the result

$$\left[p_{d,0} \sum_{k=K}^{\infty} A_k + \sum_{j=1}^{K-1} p_{d,j} \sum_{k=K-j+1}^{\infty} A_k \right] = r + p_{d,0} - 1 \qquad (35)$$

shown by summing $p_{d,k}$ over $k=1,\ldots,K-1$ and using (30)

Applying (35) to (34), and using $p_K = P_B$ and $r_c = r(1-P_B)$ we get our earlier result

$$p_K = P_B = 1 - \frac{1}{p_{d,0} + r}$$

