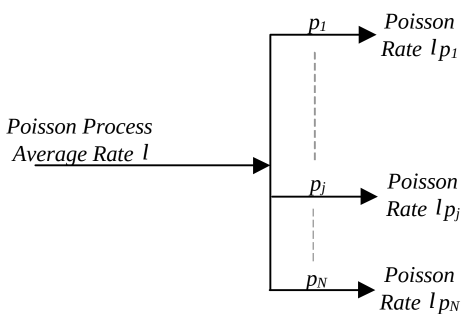
Open and Closed Networks

of

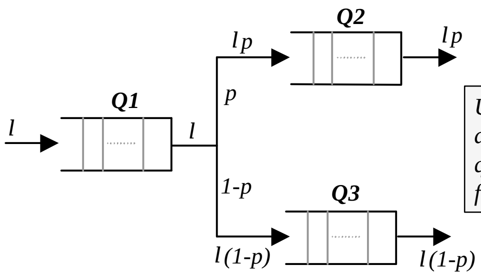
M/M/m Type Queues

(Jackson's Theorem for Open and Closed Networks)



$$\sum_{i=1}^{N} p_i = 1$$

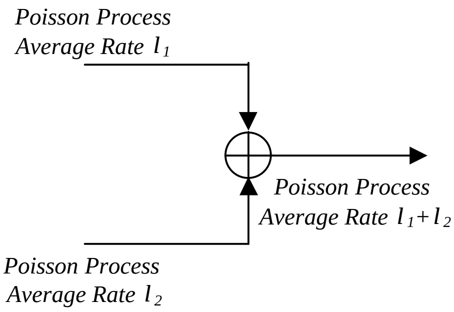
Splitting a Poisson process probabilistically (as in random, probabilistic routing) leads to processes which are also Poisson in nature.



Under equilibrium conditions, average flow leaving the queue will equal the average flow entering the queue.

Routing Probabilities are *p* and *(1-p)*

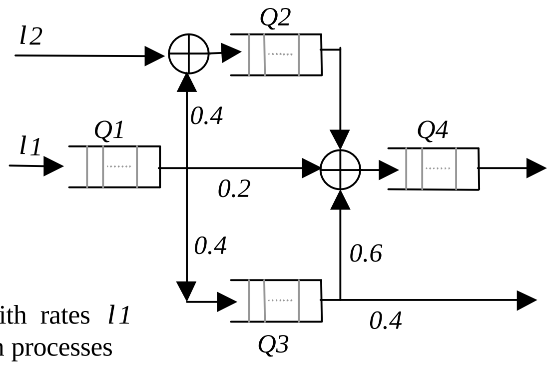
For M/M/m/∞ queues at equilibrium, Burke's Theorem (Section 2.7) assures us that the departure process of jobs from the network will also be Poisson. From *flow balance*, the average flow rate leaving the queue will also be the same as the average flow rate entering the queue.



Combining independent Poisson processes leads to a process which will also be Poisson in nature.

Example:

An Acyclic (Feedforward) Network of M/M/m Queues



External arrivals with rates *l1* and *l2* from Poisson processes

Probabilistic routing with the routing probabilities as shown

• Applying flow balance to each queue, we get

l_{Q1} = Average job arrival rate for Q1 = *l1*
l_{Q2} = Average job arrival rate for Q2 = 0.4*l1*+*l2*
l_{Q3} = Average job arrival rate for Q3 = 0.4*l1*
l_{Q4} = Average job arrival rate for Q4 = *0.84l1*+*l2*

• Burke's Theorem and the earlier quoted results on splitting and combining of Poisson processes imply that, under equilibrium conditions, the arrival process to each queue will be Poisson.

• Given the mean service times at each queue and using the standard results for M/M/m queues, we can then find the individual state probability distribution for each of the queues

• This process may be done for any system of M/M/m queues as long as there are no feedback connections between the queues

• It should be noted that this analysis can only give us the state distributions for each of the individual queues but cannot really say what will be the *joint state distribution* of the number of jobs in all the queues of the network.

• Jackson's Theorem, presented subsequently, is needed to get the *joint state distribution*. This gives the simple, and elegant result that -

$$P(n_1, n_2, n_3, n_4) = p_{Q1}(n_1)\, p_{Q2}(n_2)\, p_{Q3}(n_3)\, p_{Q4}(n_4)$$

Product Form Solution for
Joint State Distribution of the
Queueing Network

Jackson's Theorem for Open Networks

• Jackson's Theorem is applicable to a *Jackson Network*.

This is an arbitrary open network of M/M/m queues where jobs arrive from a Poisson process to one or more nodes and are probabilistically routed from one queue to another until they eventually depart from the system.

The departures may also happen from one or more queues

The M/M/m nodes are sometimes referred to as *Jackson Servers*

• Jackson's Theorem states that provided the arrival rate at each queue is such that equilibrium exists, the probability of the overall system state $(n_1 \ldots\ldots n_K)$ for K queues will be given by the product-form expression

$$P(n_1, \ldots\ldots, n_K) = \prod_{i=1}^{K} p_{Qi}(n_i)$$

Jackson Network: Network of K (M/M/m) queues, arbitrarily connected

External Arrival to Q_i: Poisson process with average rate L_i

At least one queue Q_i must be such that L_i*¹ 0*. Note that L_j*¹ 0* if there are no external arrivals to Q_j. This is because we are considering an *Open Network*. (Closed Networks are considered later).

Routing Probabilities: p_{ij} = P{a job served at Q_i is routed to Q_j}

$$\left[1 - \sum_{j=1}^{K} p_{ij}\right] = \text{P\{a job served at } Q_i \text{ exits from the network\}}$$

Arrival Process of Jobs to Q_i

= [External Arrivals, if any, to Q_i]

$+ \sum_{j=1}^{K}$ Jobs which finish service at Q_j and are then routed to Q_i for the next stage of service

Let l_i = Average Arrival Rate of Jobs to Q_i {external and rerouted}

Given the external arrival rates to each of the K queues in the system and the routing probabilities from each queue to another, the effective job arrival rate to each queue (at equilibrium) may be obtained by solving the *flow balance equations* for the network.

Flow Balance Conditions at Equilibrium imply that -

$$l_j = \Lambda_j + \sum_{i=1}^{K} l_i \, p_{ij} \qquad \text{for } j=1,\ldots, K \qquad (5.2)$$

• For an *Open Network*, at least one of the L_j*'s* will be non-zero (positive)

• The set of *K* equations in (5.2) can therefore be solved to find the effective job arrival rate to each of the *K* queues, under equilibrium conditions.

• The network will be at equilibrium if each of the *K* queues are at equilibrium. This can happen only if the effective traffic offered to each queue is less than the number of servers in the queue. i.e. $r_j = l_j / m_j < m_j \quad j=1,\ldots, K$ where m_j is the number of servers in Q_j.

For a network of this type with M/M/m/∞ queues (i.e. Jackson Servers) at each node, *Jackson's Theorem* states that provided the arrival rate at each queue is such that equilibrium exists, the probability of the overall system state ($n_1,\ldots\ldots, n_K$) will be -

$$P(\tilde{n}) = P(n_1,\ldots\ldots, n_K) = \prod_{j=1}^{K} p_j(n_j) \qquad (5.4)$$

with $p_j(n_j)$=P{n_j customers in Q_j}

This individual queue state probability may be found by considering the M/M/m/∞ queue at node *j* in isolation with its total average arrival rate l_j, its mean service time $1/m_j$ and the corresponding results for the steady state M/M/m/∞ queue

Stability requirement for the existence of the solution of (5.4) is that -

For each queue Q_j $j=1,\ldots, K$ in the network, the traffic offered should be such that

$$r_j = \left(l_j \big/ m_j \right) < m_j$$

where m_j is the number of servers in the M/M/m/∞ queue at Q_j

Implications of Jackson's Theorem - (extensions and generalizations considered subsequently)

• Once flow balance has been solved, the individual queues may be considered in isolation.

• The queues behave as if they are independent of each other *(even though they really are not independent of each other)* and the joint state distribution may be obtained as the continued product of the individual state distributions *(product-form solution)*

• The flows entering the individual queues behave as if they are Poisson, even though they may not really be Poisson in nature (i.e. if there is feedback in the network).

Note that Jackson's Theorem does require the external arrival processes to be Poisson processes and the service times at each queue to be exponentially distributed in nature with their respective, individual means.

Performance Measures

Total Throughput = $l = \sum_{j=1}^{K} \Lambda_j$ (5.5)

Average traffic load at node j (i.e. Q_j) = $r_j = \frac{l_j}{m_j}$ (5.6)

Visit Count to node j = $V_j = \frac{l_j}{l}$ (5.7)

These may also be obtained by directly solving the following *K* linear equations -

$$V_j = \frac{\Lambda_j}{l} + \sum_{i=1}^{K} V_i p_{ij} \qquad j = 1, \ldots\ldots, K \qquad (5.8)$$

Scaled Flow Balance Equations

Interpretation of the Visit Ratio V_j : Average number of times a job will visit Q_j every time it actually enters the (open) queueing network.

Average number of jobs at node j = $N_j = \sum_{k=0}^{\infty} k p_j(k)$ (5.9)

Average number of jobs in system = $N = \sum_{j=1}^{K} N_j$ (5.10)

Mean Sojourn Time (W): The mean total time spent in the system by a job before it leaves the network.

$$W = \frac{N}{l} = \sum_{j=1}^{K} \frac{N_j}{l} \qquad (5.11)$$

When does the *Product-Form Solution* hold?

The product-form expression for the joint state probabilities hold for any *open* or *closed* queueing network where local balance conditions are satisfied.

See Robertazzi's book for an interesting explanation for the conditions under which the product-form expression in valid.

 17

Specifically, *open* or *closed* networks with the following types of queues will have a product-form solution -

1. FCFS queue with exponential service times
2. LCFS queues with *Coxian* service times
3. Processor Sharing (PS) queues with *Coxian* service times
4. Infinite Server (IS) queues with *Coxian* service times

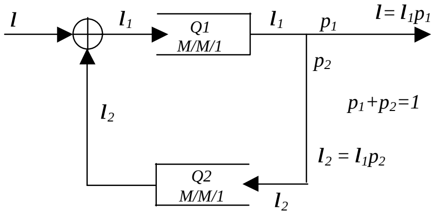
A *Coxian* service time has a distribution of the following type -

$$L_B(s) = g_1 + \sum_{i=1}^{L} b_1 b_2 \ldots\ldots b_i g_{i+1} \prod_{j=1}^{i} \frac{m_j}{s + m_j}$$

with $b_i = 1 - g_i$ for $1 \le i \le L$ and $g_{L+1} = 1$

 18

Example: Open Jackson Network



$l_1 = \frac{l}{p_1}$ $\quad l_2 = \frac{l(1-p_1)}{p_1}$

$r_1 = \frac{l}{m_1 p_1}$ $\quad r_2 = \frac{l(1-p_1)}{m_2 p_1}$

Service Rate of Q1=m_1
Service Rate of Q2=m_2

$$P(n_1, n_2) = r_1^{n_1}(1-r_1)\, r_2^{n_2}(1-r_2)$$

Mean Number in the Queues

$$N_1 = \frac{r_1}{1-r_1} \qquad N_2 = \frac{r_2}{1-r_2}$$

Mean Sojourn Time $\quad W = \frac{N}{l} = \frac{r_1}{l(1-r_1)} + \frac{r_2}{l(1-r_2)}$

See Section 5.3 for more examples

Extensions to Jackson's Theorem for Open Networks

[A] Jackson's Theorem with State dependent Service Rates at the Queuing Nodes

For this, assume that the service times at Q_j are exponentially distributed with mean $1/m_j(m)$ when there are m customers in Q_j just before the departure of a customer.

[B] Queuing Networks with Multiple Customer Classes

For this, we need to assume that the service time distribution at a node will be the same for all classes even though they may differ from one node to another. The service times may be state dependent.

The external arrival rates and routing probabilities will vary from one class of customers to another

See Section 5.4 for detailed formulation

Closed Queueing Networks

• K queues - Q_1, ..., Q_K in the queueing network

• M jobs of the same class circulating in the network

• p_{ij} be the routing probability from Q_i to Q_j (probabilistic routing)

Since network is a closed network $\sum_{j=1}^{K} p_{ij} = 1 \quad i = 1,......,K$

• No arrivals from outside and no departures from the network

• Flow balance conditions for this network may still be written as

$$l_j \quad \sum_i \quad _i \quad _{ij} \qquad ,......,$$

• The K equations of (5.12) are not independent. Hence, they cannot be solved to uniquely find the l_js for the K queues, $j=1,.....,K$

• Using any of the K-1 equations in (5.12), we can find the l_j*'s* up to a multiplicative constant

For this, assume that $a(M)$ is an (unknown) scalar quantity and let $\{l_j^*\}$ $j=1,...,K$ be a particular solution of (5.12) such that the *true average arrival rates* $\{l_j(M)\}$ $j=1,....,K$ are given by

$$l_j(M) = a(M)\, l_j^* \qquad j=1,.........,K \qquad (5.13)$$

•$a(M)$ and $\{l_j(M)\}$ $j=1,.........,K$ are both functions of the population size of M jobs circulating in the closed network

•However, $\{l_j^*\}$ $j=1,.......,K$ will be independent of M

An alternate, but equivalent approach would be to do the following -

• Choose any queue in the network (say Q_1) as the *reference queue* and assume that $l_1^* = a$

Any value of a *may* be chosen!

A convenient choice is $a = m_1$ *so that* $r_1 = l_1^*/m_1 = 1$

• Solve the flow balance equations of (5.13) to obtain the *relative throughputs* $(l_2^*, l_3^*, \ldots\ldots, l_K^*)$ in terms of a.

• Assuming the service times to be exponentially distributed (recall that we are assuming M/M/m type queues), we allow the actual service rates at each queue to be state dependent

$m_j(m)$ = service rate at Q_j when Q_j is in state m
(exponential service times assumed)

• Using the *relative throughputs* $\{l_j^*\}$ $j=1,\ldots\ldots,K$ found earlier, we define the *relative utilizations* $\{u_j\}$ $j=1,\ldots\ldots,K$ as -

$$u_j(m) = \frac{l_j^*}{m_j(m)} \qquad j=1,\ldots,K \quad m=1\ldots\ldots,M \tag{5.14}$$

Let n_i = Number in queue Q_i

State Probability Vector $\tilde{n} = (n_1, \ldots\ldots\ldots, n_K)$

such that $n_1 + \ldots\ldots\ldots + n_K = M$ Total number of jobs

Jackson's Theorem for Closed Networks of M/M/- Type Queues

$$P(\tilde{n}) = P(n_1, n_2, \ldots\ldots, n_K) = \frac{1}{G(M)} \prod_{i=1}^{K} \hat{P}_i(n_i) \qquad (5.17)$$

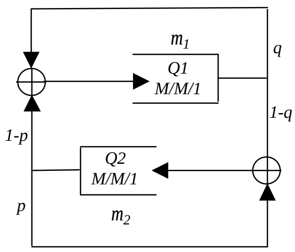
where
$$\hat{P}_j(n_j) = 1 \qquad n_j = 0$$
$$= u_j(1) u_j(2) \ldots\ldots u_j(n_j) \qquad n_j \geq 1$$

and
$$G(M) = \sum_{n_1 + \ldots + n_K = M} \hat{P}_1(n_1) \hat{P}_2(n_2) \ldots\ldots \hat{P}_K(n_K)$$

G(M) = Normalization Constant

 25

Example



Closed Network with M jobs

Choose $l_1^* = m_1$

Then $$l_2^* = l_1^* \frac{1-q}{1-p} = m_1 \frac{1-q}{1-p}$$

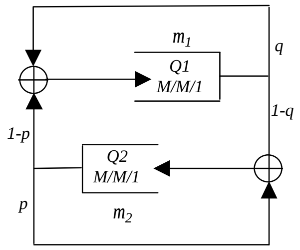
$$u_1 = 1 \qquad u_2 = \frac{1-q}{1-p} \frac{m_1}{m_2}$$

$$P(n_1, n_2) = P(M-n, n) = \frac{u_2^n}{G(M)}$$

Normalization Constant $$G(M) = \sum_{n=0}^{M} u_2^n = \frac{1 - u_2^{M+1}}{1 - u_2}$$

 26

Example



Closed Network with M jobs

P{Q_1 is busy} = 1-*P(0, M)* = $1-\frac{u_2^M}{G(M)}=\frac{G(M-1)}{G(M)}$

P{Q_2 is busy} = 1-*P(M, 0)* = $1-\frac{1}{G(M)}=u_2\frac{G(M-1)}{G(M)}$

Visit Ratios

The *visit ratio* V_i of the i^{th} queue Q_i in the queueing network is defined as the mean number of times Q_i is visited by a job for every visit it makes to a given reference queue, say Q_1.

Note that the definition is basically the same as for an open network.

With Q_1 as the reference queue, $V_i=\frac{l_i^*}{l_1^*}$ $i=1,\ldots,K$

The same result will be obtained by setting $V_1=1$ and solving the equations $\tilde{V}\cdot\tilde{P}=\tilde{V}$ with $\tilde{V}=(V_1,\ldots\ldots,V_K)$ and $\tilde{P}=[p_{ij}]$

Jackson's Theorem for Closed Networks of Multi-Server Queues

K exponential service queues in the closed network with probabilistic routing given by $\{p_{ij})\quad i, j = 1,.......,K$

Q_i has s_i servers $\quad \Rightarrow m_i(m) = \min\,(mm_i, s_i m_i) \qquad i=1,\ldots\ldots, K$

m_i = service rate of a single server at Q_i

$m_i\ (m)$ = overall (state dependent) service rate at Q_i when it has a total of m jobs (waiting and in-service)

Define $\quad u_i = \dfrac{l_i^*}{m_i}$ $\quad$ where l_i^* is the relative throughput for Q_i

 29

Jackson's Theorem for a Closed Network of Multi-Server Queues

Using these

$$P(\tilde{n}) = P(n_1,.......,n_K) = \frac{1}{G(M)}\left[\prod_{i=1}^{K}\frac{u_i^{n_i}}{b_i(n_i)}\right]$$

such that $\quad n_1+n_2+......+n_K = M$

$$b_i = n_i! \qquad n_i \le s_i$$

$$= s_i!\left(s_i\right)^{(n_i - s_i)} \qquad n_i > s_i$$

and $\quad G(M) = \sum_{n_1+....+n_K=M}\left(\prod_{i=1}^{K}\frac{u_i^{n_i}}{b_i(n_i)}\right)$

 30

These expressions may be written in a simpler form for a *Closed Network of Single Server Queues with Exponential Service*

Closed Network of Single Server Queues with Exponential Service Times

$$P(\tilde{n}) = P(n_1, \ldots\ldots, n_K) = \frac{1}{G(M)} \left[\prod_{i=1}^{K} u_i^{n_i} \right]$$

such that $n_1 + \ldots\ldots + n_K = M$

$$G(M) = \sum_{n_1 + \ldots + n_K = M} \left(\prod_{i=1}^{K} u_i^{n_i} \right)$$

• The major computational difficulty with finding the ***state probability distribution of a Closed Network*** is that of finding the value of the normalization constant *G(M)*

This complexity increases rapidly with larger networks (increasing values of K) and larger population of circulating jobs (increasing values of M)

• *G(M)* may be calculated directly only for very small networks with a very small number of circulating jobs. For larger networks, the ***Convolution Algorithm*** should be used to calculate *G(M)*.

• If mean performance parameters are desired (rather than the actual state probability), then the ***Mean Value Algorithm*** may be directly used to find these without finding *G(M)* at all.

For a closed network of K single server queues with M jobs circulating -

$$P(\tilde{n}) = P(n_1,.....,n_K) = \frac{1}{G(M)} \prod_{i=1}^{K} u_i^{n_i} \qquad n_1 + ... + n_K = M$$

$$u_i = \frac{l_i^*}{m_i} \quad i = 1,......., K$$

$$G(M) = \sum_{n_1+.....+n_K=M} \left(\prod_{i=1}^{K} u_i^{n_i} \right)$$

Then

$$P\{n_i \geq n\} = \sum_{\substack{n_1+...+n_K=M \\ n_i \geq n}} \left(\frac{1}{G(M)} \prod_{j=1}^{K} u_j^{\,n_j} \right)$$

$$= u_i^{\,n} \frac{1}{G(M)} \underbrace{\sum_{n_1+...+n_K=M-n} \left(\prod_{j=1}^{K} u_j^{\,n_j} \right)}$$

factoring out $(u_i)^n$

Normalization constant of the queueing network with n fewer customers, i.e. G(M-n)

Therefore $$P\{n_i \ge n\} = u_i^{\,n} \frac{G(M-n)}{G(M)}$$

and $$P\{n_i = n\} = P\{n_i \ge n\} - P\{n_i \ge (n+1)\}$$
$$= u_i^{\,n}\left[\frac{G(M-n)}{G(M)} - u_i \frac{G(M-n-1)}{G(M)}\right]$$

Marginal distribution of the i^{th} queue

Note that $$\sum_{n=1}^{M} nP\{n_i = n\} = \sum_{n=1}^{M} P\{n_i \ge n\}$$

and therefore $$E\{n_i\} = \sum_{n=1}^{M} u_i^{\,n} \frac{G(M-n)}{G(M)}$$

Mean number in Q_i

Note that the departure rate from Q_i will always be m_i whenever Q_i has one or more jobs.

Therefore, the actual throughput l_i of Q_i will be given by

$$l_i = m_i P\{n_i \ge 1\} = m_i u_i \frac{G(M-1)}{G(M)}$$

The actual utilization r_i of Q_i will then be

$$r_i = \frac{l_i}{m_i} \quad or \quad r_i = P\{n_i \ge 1\}$$
$$\Rightarrow \quad r_i = u_i \frac{G(M-1)}{G(M)}$$