Stochastic Processes

Markov Processes and Markov Chains

Birth Death Processes

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Stochastic Process $X(t)$

Process takes on random values, $X(t_1)=x_1, \ldots\ldots\ldots, X(t_n)=x_n$ *at times* $t_1, \ldots\ldots, t_n$

The random variables $x_1, \ldots\ldots, x_n \ldots$ are specified by specifying their joint distribution.

One can also choose the time points $t_1, \ldots\ldots, t_n \ldots$ (where the process $X(t)$ is examined) in different ways.

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Markov Processes

X(t) satisfies the Markov Property (memoryless) which states that -

$$P\{X(t_{n+1})=x_{n+1} \mid X(t_n)=x_n \ldots\ldots X(t_1)=x_1\}=P\{X(t_{n+1})=x_{n+1} \mid X(t_n)=x_n\}$$

for any choice of time instants t_i, $i=1,\ldots\ldots, n$ *where* $t_j>t_k$ *for* $j>k$

Memoryless property as the state of the system at future time t_{n+1} is decided by the system state at the current time t_n and does not depend on the state at earlier time instants $t_1,\ldots\ldots, t_{n-1}$

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Restricted versions of the Markov Property leads to -

(a) Markov Chains over a Discrete State Space

(b) Discrete Time and Continuous Time Markov Processes and Markov Chains

Markov Chain	State Space is discrete (e.g. set of non-negative integers)
Discrete Time	State changes are pre-ordained to occur only at the integer points $0, 1, 2, \ldots\ldots, n$ (that is at the time points $t_0, t_1, t_2, \ldots\ldots, t_n$)
Continuous Time	State changes may occur anywhere in time

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In the analysis of simple queues, the state of the queue may be represented by a single random variable $X(t)$ which takes on integer values $\{i, i=0, 1\ldots..,\}$ at any instant of time.

The corresponding process may be treated as a

Continuous Time Markov Chain

since time is continuous but the state space is discrete

Homogenous Markov Chain

A Markov Chain where the transition probability

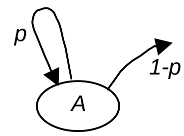
$$P\{X_{n+1}=j \mid X_n=i\}$$

is the same regardless of n.

Therefore, the transition probability p_{ij} of going from state i to state j may be written as

$$p_{ij} = P\{X_{n+1}=j \mid X_n=i\} \text{ "}n.$$

It should be noted that for a Homogenous Markov chain, the transition probabilities depend only on the terminal states (i.e. the initial state i and the final state j) but do not depend on when the transition (i® j) actually occurs.



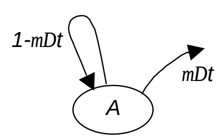
Discrete Time Markov Chain

Transition from State A

P{system stays in state A for N time units | given that the system is currently in state A} = p^N

P{system stays in state A for N time units before exiting from state A} $= p^N(1-p)$

Distribution is Geometric, which is memoryless in nature



Continuous Time Markov Chain

Transition from State A

P{system in state A for time T | system currently in state A}

$$= (1 - m\Delta t)^{T/\Delta t} \to e^{-mT} \qquad Dt®0$$

This is the (1- cdf) of an exponential distribution

Distribution is Exponential, which is memoryless in nature

Important Observation

"*In a homogenous Markov Chain, the distribution of time spent in a state is (a) Geometric for discrete time or (b) Exponential for continuous time*"

Semi-Markov Processes

In these processes, the distribution of time spent in a state can have an arbitrary distribution but the one-step memory feature of the Markovian property is retained.

Discrete-Time Markov Chains

The sequence of random variables X_1, X_2, ... forms a Markov Chain if for all n $(n=1, 2, ...)$ and all possible values of the random variables, we have that -

$$P\{X_n=j \mid X_1=i_1..........X_{n-1}=i_{n-1}\}=P\{X_n=j \mid X_{n-1}=i_{n-1}\}$$

Note that this once again, illustrates the *one step memory* of the process

Homogenous Discrete-Time Markov Chain

The homogenity property additionally implies that the state transition probability

$$p_{ij} = P\{X_n=j \mid X_{n-1}=i\}$$

will also be independent of *n*, i.e. the instant when the transition actually occurs.

In this case, the state transition probability will only depend on the value of the initial state and the value of the next state, regardless of when the transition occurs.

Homogenous Discrete-Time Markov Chain

The homogenity property also implies that we can define a *m-step* state transition probability as follows

$$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\} = \sum_{\forall k} p_{ik}^{(m-1)} p_{kj} \qquad m=2, 3, \ldots\ldots$$

where $p_{ij}^{(m)}$ is the probability that the state changes from state *i* to state *j* in *m* steps.

This may also be written in other ways, though the value obtained in each case will be the same. For example, we can also write

$$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\} = \sum_{\forall k} p_{ik}\, p_{kj}^{(m-1)}$$

Irreducible Markov Chain

This is a Markov Chain where every state can be reached from every other state in a *finite* number of steps.

This implies that k exists such that $p_{ij}^{(k)}$ for $\forall i, j$.

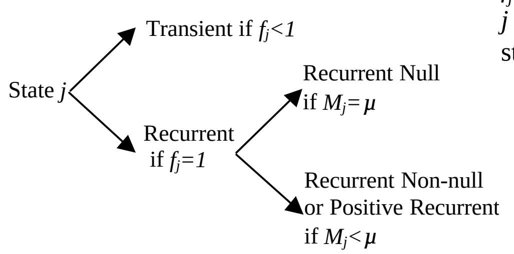
If a Markov Chain is *not irreducible*, then -

(a) it may have one or more *absorbing states* which will be states from which the process cannot move to any of the other states,

or

(b) it may have a subset of states A from where one cannot move to states outside A, i.e. in A^C

Classification of States for a Markov Chain



$f_j^{(n)}$ = P{system returns to state j exactly n steps after leaving state j}

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)}$$

$$M_j = \sum_{n=1}^{\infty} n f_j^{(n)}$$

f_j = P{system returns to state j some time after leaving state j}

M_j = Mean recurrence time for state j

= Mean number of steps to return to state j after leaving state j

State *j* is *periodic* with respect to *a (a >1)*, if the only possible steps in which state *j* may occur are *a, 2a, 3a*...............

In that case, the *recurrence time* for state j has *period a*.

State j is said to be *aperiodic* if *a=1*

A recurrent state is said to be *ergodic* if it is both positive recurrent and aperiodic.

An ergodic Markov chain will have all its states as ergodic.

An *Aperiodic, Irreducible, Markov Chain with a finite number of states* will always be ergodic.

The states of an Irreducible Markov Chain are either all transient, or all recurrent null or all recurrent positive. If the chain is periodic, then all states have the same period *a*.

In an irreducible, aperiodic, homogenous Markov Chain, the limiting state probabilities p_j=P{state j} always exist and these are independent of the initial state probability distribution

and

either

All states are transient, or all states are recurrent null - in this case, the state probabilities p_j's are zero for all states and no stationary state distribution will exist.

or

All states are recurrent positive - in this case a *stationary distribution* giving the equilibrium state probabilities exists and is given by $p_j = 1/M_j$ *"j*.

The *stationary distribution* of the states (i.e. the *equilibrium state probabilities*) of an irreducible, aperiodic, homogenous Markov Chain (which will also be ergodic), may be found by solving a set of simultaneous linear equations and a normalization condition.

$$p_j = \sum_{\forall i} p_i p_{ij} \qquad "j \qquad \text{Balance Equations}$$

$$\sum_{\forall i} p_i = 1 \qquad \text{Normalization Condition}$$

If system has N states, $j=0, 1, \ldots, (N\text{-}1)$, then we need to solve for $p_0, p_1, \ldots, p_{N-1}$ using the Normalization Condition and any $(N\text{-}1)$ equations from the N Balance Equations

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Birth-Death Processes

Homogenous, aperiodic, irreducible (discrete-time or continuous-time) Markov Chain where state changes can only happen between neighbouring states.

If the current state (at time instant n) is $X_n=i$,then the state at the next instant can only be $X_{n+1}= (i+1), i$ or $(i\text{-}1)$.

Note that states are represented as integer-valued without loss of generality.

Pure Birth

No decrements, only increments

Pure Death

No increments, only decrements

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Continuous-Time Birth-Death Markov Chain

Let l_k be the birth rate in state k

m_k be the death rate in state k

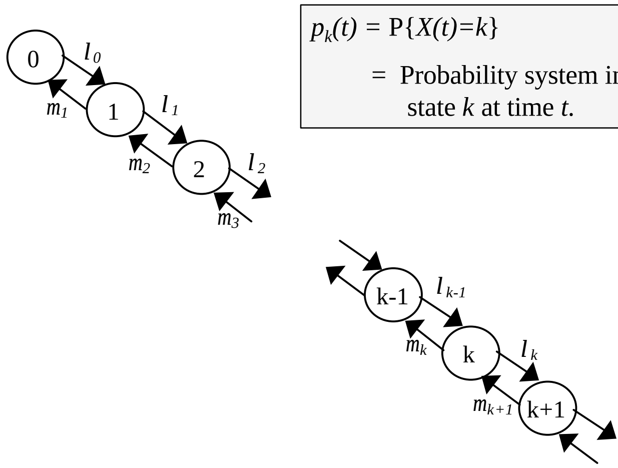
Then
P{state k to state $k+1$ in time Dt} = $l_k(Dt)$
P{state k to state k-1 in time Dt} = $m_k(Dt)$
P{state k to state k in time Dt} = 1 - $(l_k + m_k)(Dt)$
P{other transitions in Dt} = 0

(all four as Dt®0)

System State $X(t)$ = Number in the system at time t

= Total Births - Total Deaths in $(0, t)$
assuming system starts from state 0 at $t=0$

> The initial condition will not matter when we are only interested in the equilibrium state distribution.



> $p_k(t)$ = P{$X(t)=k$}
> = Probability system in state k at time t.

State Transition Diagram for a Birth-Death Process

The state transitions from time *t* to *t+Dt* will then be governed by the following equations -

$$p_0(t+\Delta t) = p_0(t)[1 - l_0\Delta t] + p_1(t)m_1\Delta t$$

$$p_k(t+\Delta t) = p_k(t)[1-(l_k + m_k)\Delta t] + p_{k-1}(t)\,l_{k-1}\Delta t + p_{k+1}(t)m_{k+1}\Delta t$$

$$\sum_{k=0}^{\infty} p_k(t) = 1$$

Dt®0

$$\frac{dp_0(t)}{dt} = -l_0 p_0(t) + m_1 p_1(t)$$

$$\frac{dp_k(t)}{dt} = -(l_k + m_k)p_k(t) + l_{k-1}p_{k-1}(t) + m_{k+1}p_{k+1}(t) \qquad (2.6)$$

$$\sum_{k=0}^{\infty} p_k(t) = 1$$

Obtain the equilibrium solutions by setting

$$\frac{dp_i(t)}{dt} = 0 \quad \forall i$$

and obtaining the state distribution p_i $"i$ such that the normalization condition

$$\sum_{i=0}^{\infty} p_i = 1$$

is satisfied.

This yields the following equations to be solved for the state probabilities under equilibrium conditions-

$l_0 p_0 = m_1 p_1 \qquad k=0$

$l_{k-1} p_{k-1} + m_{k+1} p_{k+1} = (l_k + m_k) p_k \qquad k=1,2,3,\ldots\ldots,$

$$\sum_{i=0}^{\infty} p_i = 1$$

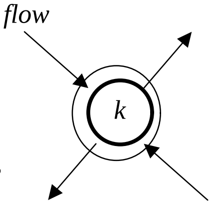
Product Form Solution

The solution is -

$$p_k = p_0 \left[\prod_{i=0}^{k-1} \frac{l_i}{m_{i+1}} \right] \qquad (2.7)$$

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{l_i}{m_{i+1}}} \qquad (2.8)$$

Instead of writing differential equations, one can obtain the solution in a simpler fashion by directly considering *flow balance* for each state.



(a) Draw the state transition diagram

(b) Draw closed boundaries and equate flows across this boundary. Any closed boundary may be chosen for this.

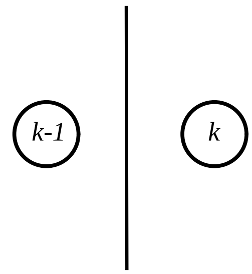
If the closed boundary encloses state k, then we get

Flow entering state $k = l_{k-1} p_{k-1} + m_{k+1} p_{k+1} = (l_k + m_k) p_k$
=Flow leaving state k

Global Balance Equation for state k

as the desired equation for state k.

(c) Solve the equations in (b) along with the normalization condition to get the equilibrium state distribution.



It would be even simpler in this case to consider a closed boundary which is actually closed at infinity

This would lead to the following equation

Flow from state k-1 to k = Flow from state k to k-1

$$l_{k-1}p_{k--1}=m_k p_k$$

Detailed Balance Equation

The solution for this will be the same as that obtained earlier

In general, the equations expressing flow balance in a Birth-Death Chain of this type will be -

$$\sum_{i \neq j} p_i p_{ij} = p_j \sum_{i \neq j} p_{ji}$$

Global Balance Equations

Closed boundary encircling each state *j*

$$p_i p_{ij} = p_j p_{ji}$$

Detailed Balance Equations

Equates flows between states *i* and *j*, in a pair-wise fashion.

Boundary between states *i* and *j*, closed at +∞ and -∞

Conditions for Existence of Solution for Birth-Death Chain

$$a = \sum_{k=0}^{\infty}\left[\prod_{i=0}^{k-1}\frac{l_i}{m_{i+1}}\right]$$

$$b = \sum_{k=0}^{\infty}\frac{1}{l_k\prod_{i=0}^{k-1}\frac{l_i}{m_{i+1}}}$$

(a) All states are *transient*, if and only if $a = ¥, b < ¥$

(b) All states *recurrent null*, if and only if $a = ¥, b = ¥$

(c) All states *ergodic*, if and only if $a < ¥, b = ¥$

Equilibrium State Distribution will exist only for the case where all the states are ergodic and hence, the chain itself is also ergodic