Analyzing the M/G/1 Queue

using

The Method of Supplementary Variables

$N(t)$ Number of jobs in system at time t

$X_0(t)$ Elapsed service time for job currently in service at time t

($X_0(t)=0$ if $N(t)=0$ at the time instant t)

As noted earlier, $N(t)$ would not form a Markov Chain for the M/G/1 queue.

However, the joint process $[N(t), X_0(t)]$ would be a Continuous Time Markov Process.

The *Method of Supplementary Variables* focusses on solving the joint process $[N(t), X_0(t)]$ under equilibrium conditions as $t \to \infty$. Eliminating the variable $X_0(t)$ by averaging over its distribution gives the required state probabilities

Road Map of Approach to be Followed

For k = 1,………………, ¥

- $f_k(t,x)dx = P\{N(t) = k, x < X_0(t) \le x + dx\}$
- $\begin{cases} f_k(x)dx = P\{N = k, x < X_0 \le x + dx\} = \lim_{t\to\infty} f_k(t,x\}dx \\ f_0(x) = 0 \end{cases}$
- $P_k(t) = \int_{x=0}^{\infty} f_k(t,x)dx$
- $p_k = \lim_{t\to\infty} P_k(t) = \int_0^{\infty} f_k(x)dx$

 3

Consider a job which requires a service of duration *X* with pdf *b(x)* and cdf *B(x)*.

Let $b_c(x)$ be the pdf of the service time *X* given that *X>x*, such that

$$b_c(x)dx = P\{x < X < x + dx \mid X > x\}$$
$$= \frac{b(x)}{(1 - B(x))}$$

 4

Equating flows between state *k* and state (*k-1*) at equilibrium, *t*® ¥

$$l p_0 = \int_0^\infty f_1(x) b_c(x) dx \qquad k=0 \qquad (5)$$

$$f_k(x+\Delta x)dx = l\Delta x[1-b_c(x)\Delta x] f_{k-1}(x)dx$$
$$+ (1-l\Delta x)[1-b_c(x)\Delta x] f_k(x)dx \qquad k=1,\ldots,¥ \qquad (6)$$

Note that for *k*³*1*, there cannot be a transition in *Dx* from the state {*N(t)=k+1*, $X_0(t)=x$} to the state {*N(t+ Dx)=k*, $X_0(t)=x+ Dx$}. This is because a departure here would make it impossible for the new job starting service to have an elapsed service time of *x+ Dx*

For *k=1,......,¥*, Eq. (6) leads to

$$f_k(x+\Delta x) = l\Delta x f_{k-1}(x) + \left[1-\Delta x\left(l + b_c(x)\right)\right] f_k(x) \qquad (7)$$

and with *Dx*®*0*

$$\frac{df_k(x)}{dx} + \left[l + b_c(x)\right] f_k(x) = l f_{k-1}(x) \qquad (8)$$

Boundary Conditions

$$\begin{cases} f_1(0) = l p_0 + \int_0^\infty f_2(x) b_c(x) dx & k=1 \\ f_k(0) = \int_0^\infty f_{k+1}(x) b_c(x) dx & k = 2,.....,\infty \end{cases} \qquad (9)$$

The boundary conditions are obtained by noting that $f_k(0)$ is the flow rate at which the system enters state k when sevice to a job has just started, i.e. when the elapsed service time is zero, $x=0$

Normalization Condition

$$\sum_{k=0}^{\infty} p_k = p_0 + \sum_{k=1}^{\infty} \int_0^\infty f_k(x) dx = 1 \qquad (10)$$

using the definition of p_k for $k=1,\ldots,¥$

We can obtain $f_k(x)$ $k=1,\ldots,¥$ using (8) along with the initial conditions of (9).

A convenient strategy for this is to solve for the generating function $F(z,x)$

$$F(z,x) = \sum_{k=1}^{\infty} f_k(x) z^k$$

Multiplying the k^{th} equation of (8) by z^k and summing gives

$$\frac{\partial F(z,x)}{\partial x} = [lz - l - b_c(x)] F(z,x) \qquad (11)$$

Initial condition using (9)

$$F(z,0) = lzp_0 + \sum_{k=1}^{\infty} z^k \int_0^\infty f_{k+1}(x) b_c(x) dx$$

or

$$zF(z,0) = lz(z-1) p_0 + \int_0^\infty b_c(x) F(z,x) dx \qquad (12)$$

F(z,x) may be solved using (11) and (12)

A convenient approach for this *(refer to notes)* is to define a new variable *G(z,x)* as

$$\left.\begin{aligned} g_k(x) &= \frac{f_k(x)}{1-B(x)} \quad k=1,\ldots\ldots,\infty \\ g_0(x) &= 0 \end{aligned}\right\} \quad G(z,x) = \sum_{k=1}^{\infty} g_k(x) z^k = \frac{F(z,x)}{1-B(x)}$$

We can then write (11) as

$$\frac{\partial G(z,x)}{\partial x} + l(1-z)G(z,x) = 0 \qquad (15)$$

with solution

$$G(z,x) = G(z,0)e^{-l(1-z)x} \qquad (16)$$

Using (16) and *F(z,0)=G(z,0)* and $f_k(0)=g_k(0)$ for *k=0,1,.....,¥* in (12)

$$zG(z,0) = lz(z-1)p_0 + \int_0^{\infty} b(x)G(z,0)e^{-l(1-z)x}dx$$

$$= lz(z-1)p_0 + G(z,0)L_B(l - lz)$$

Solving, we get

$$G(z,0) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} \qquad (17)$$

$$G(z,x) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} e^{-l(1-z)x} \qquad (18)$$

From (18) and the definition of *G(z,x)* given earlier, we get -

$$F(z,0) = \frac{lz(1-z)p_0}{L_B(l - lz) - z} \qquad (19)$$

$$F(z,x) = \frac{lz(1-z)p_0}{L_B(l - lz) - z}[1 - B(x)]e^{-l(1-z)x} \qquad (20)$$

as the solution for *F(z,x)*.

This may be inverted or expanded in terms of z^k to get $f_k(x)$

The state probabilities $P_k(t)$ *k=1,…...,¥* of the system at time *t* may be obtained from $f_k(t,x)$ as

$$P_k(t) = \int_{x=0}^{\infty} f_k(t,x)dx$$

and the corresponding equilibrium state probabilities p_k as

$$p_k = \lim_{t\to\infty} P_k(t) = \int_{0}^{\infty} f_k(x)dx$$

Defining *F(z)* as $F(z) = \int_{x=0}^{\infty} F(z,x)dx$ we can then observe that

$$\sum_{k=1}^{\infty} p_k z^k = \sum_{k=1}^{\infty} z^k \left(\int_{x=0}^{\infty} f_k(x)dx \right) = \int_{x=0}^{\infty} F(z,x)dx = F(z)$$

and $P(z) = \sum_{k=0}^{\infty} p_k z^k = p_0 + F(z)$

Since $F(z) = \int_{x=0}^{\infty} F(z,x)dx$ we can use (20) to get

$$F(z) = \left(\frac{lz(1-z)p_0}{L_B(l - lz) - z} \right) \left(\frac{1 - L_B(l - lz)}{l(1-z)} \right)$$

or

$$F(z) = \frac{zp_0[1 - L_B(l - lz)]}{L_B(l - lz) - z} \qquad (21)$$

Evaluating *F(z)* at *z=1*, we will get *F(1)=(1-p_0)*

$$\begin{cases} 1 - p_0 = F(z)\big|_{z=1} = p_0 \dfrac{(-l\overline{X})}{l\overline{X} - 1} \\ \qquad or \\ p_0 = (1 - l\overline{X}) = (1 - r) \quad with \quad r = l\overline{X} \end{cases} \qquad (24)$$

 13

This leads to our final result, the generating function *P(z)* of the system state probabilities at equilibrium, as

$$P(z) = p_0 + F(z) = p_0 \left[1 + \frac{z[1 - L_B(l - lz)]}{L_B(l - lz) - z} \right]$$
$$= \frac{(1-z)(1-r)L_B(l - lz)}{L_B(l - lz) - z} \qquad (25)$$

Note that, as expected, this is the same as the *P-K Transform Equation* obtained for the M/G/1 queue using the imbedded Markov Chain approach.

 14