M/G/1 Queue

with

Vacations

Vacation: After a *busy period*, the server goes on *vacation* of random length. It examines the queue once again when it returns from the vacation

Multiple Vacations (possibly)

If system still empty when the server returns from a vacation, it goes for another vacation. This continues until it finds system non-empty on return from vacation; it then resumes service normally

Single Vacation (per idle)

After a busy period ends, server goes on only one vacation. If system is still empty when in returns, it stays and waits for a job to arrive.

Other models are also possible, i.e. server goes on (possibly multiple) vacations following the busy period until there are *K* waiting jobs

Analysis of M/G/1 Queue with (Multiple) Vacations

Vacation Interval random (i.i.d) and independent of service times with

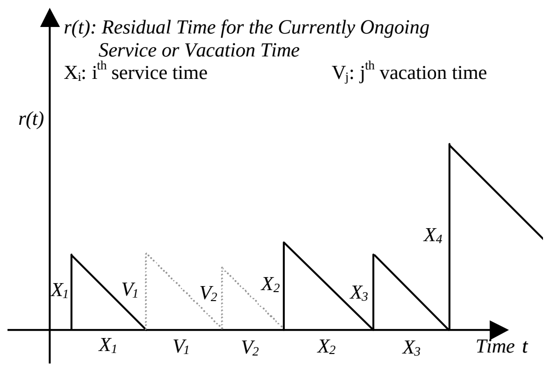
- *moments* $\overline{V}$, $\overline{V^2}$
- *pdf* $f_V(t)$
- *cdf* $F_V(t)$
- *L.T.* $L_V(s)$

→ Residual Life Approach

→ Imbedded Markov Chain Approach

 3

Analysis using the Residual Life based Approach



Residual Time $r(t)$ - service time or vacation - for a M/G/1 Queue with (multiple) Vacations

 4

Time Average of *r(t)* over *(0, t)* $= \frac{1}{t}\int_0^t r(x)dx = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 + \frac{1}{t}\sum_{j=1}^{L(t)} \frac{1}{2} V_j^2$

where

- *M(t)* = Number of arrivals in the interval *(0,t)*
- *L(t)* = Number of vacation intervals in the interval *(0,t)*

For *t* ® ¥

$$R = \lim_{t\to\infty} \frac{1}{t}\int_0^t r(x)dx \qquad \lim_{t\to\infty} \frac{1}{M(t)}\sum_{i=1}^{M(t)} X_i^2 = \overline{X^2}$$

$$\lim_{t\to\infty} \frac{M(t)}{t} = l \qquad \lim_{t\to\infty} \frac{1}{L(t)}\sum_{j=1}^{L(t)} V_j^2 = \overline{V^2}$$

$$\lim_{t\to\infty} \frac{t(1-r)}{L(t)} = \overline{V}$$

As for the basic M/G/1 queue considered earlier, this leads to

Mean Residual Time (service or vacation)

$$R = \frac{1}{2} l \overline{X^2} + \frac{1}{2}(1-r)\frac{\overline{V^2}}{\overline{V}} \qquad (4.2)$$

Writing $W_q = N_q \overline{X} + R = l W_q \overline{X} + R$

gives

$$W_q = \frac{l \overline{X^2}}{2(1-r)} + \frac{\overline{V^2}}{2\overline{V}} \qquad r = l\overline{X} \qquad (4.3)$$

as the mean waiting time in queue seen by an arriving customer

Knowing W_q, the other parameters N_q, N and W may be found

Analysis using the Imbedded Markov Chain Approach

As for the basic M/G/1 queue, imbed Markov Chain of system states (denoting the number in the system) at the time instants t_i $i=1, 2, 3,$ .... when the i^{th} customer departs from the system

n_i = Number of jobs left behind in the system by the i^{th} departure

a_i = Number of job arrivals during the i^{th} service time

$$A(z) = L_B(l - lz)$$

with $A(1) = 1 \quad A'(1) = r = l\overline{X} \quad A''(1) = l^2\,\overline{X^2}$

j = Number of jobs waiting for service when a busy period begins, j ³ 1

f_j = P{j customers starting the busy period} $j=1,2,.....,$¥

Generating Function for j
(see Sec. 4.1.1 for details)

$$F(z) = \sum_{j=1}^{\infty} f_j z^j = E\{z^j\}$$

$$= \frac{L_V(l - lz) - L_V(l)}{1 - L_V(l)}$$

with $F(1) = 1 \quad F'(1) = \dfrac{l\overline{V}}{1 - L_V(l)} \quad F''(1) = \dfrac{l^2\,\overline{V^2}}{1 - L_V(l)}$

Relating the state at the i^{th} and $(i+1)^{th}$ instants, we get

$$\begin{aligned} n_{i+1} &= a_{i+1} + j - 1 \qquad for \;\; n_i = 0 \\ &= n_i + a_{i+1} - 1 \qquad for \;\; n_i \geq 1 \end{aligned} \tag{4.5}$$

or

$$n_{i+1} = n_i + a_{i+1} - 1 + j[1 - U(n_i)] \tag{4.4}$$

$$P(z) = E\{z^{n+a-1+j[1-U(n)]}\} = E\{z^a\}E\{z^{n-1+j[1-U(n)]}\}$$

$$P(z) = A(z)E\{p_0 z^{j-1} + \sum_{n=1}^{\infty} z^{n-1} p_n\}$$

$$\Rightarrow \quad P(z) = p_0 A(z)\frac{1-F(z)}{A(z)-z}$$

Evaluating $P(z) = p_0 A(z)\frac{1-F(z)}{A(z)-z}$ at $z=1$, i.e. using $P(1)=1$, *gives*

$$p_0 = \frac{1-r}{F'(1)} \tag{4.9}$$

and therefore

$$P(z) = (1-r)(1 - L_V(l - lz))\left(\frac{L_B(l - lz)}{z - L_B(l - lz)}\right) \tag{4.10}$$

Note that though *P(z)* was derived for the customer departure instants, it will also hold for the arrival instants and at an arbitrary time instant under equilibrium conditions.

• From *P(z)*, we can find the system state distribution either by inverting the generating function *P(z)* or by expanding it in powers of *z*

• The moments of the number in the system may be found directly using the moment generating properties of the generating function *P(z)*.

• Specifically, we get $N = P'(1) = l\overline{X} + \frac{l^2 \overline{X^2}}{2(1 - l\overline{X})} + \frac{l\overline{V^2}}{2\overline{V}}$

• Knowing *N*, we can obtain *W*, W_q and N_q following our usual approach

For example $$W_q = \frac{l\overline{X^2}}{2(1-r)} + \frac{\overline{V^2}}{2\overline{V}} \qquad (4.11)$$

M/G/1 Queue with only one Vacation after Idle (Section 4.2)

$$P(z) = p_0\left(\frac{L_B(l - lz)}{z - L_B(l - lz)}\right)\left(L_V(l - lz) - (1-z)L_V(l) - 1\right)$$

with $$p_0 = \frac{1 - l\overline{X}}{l\overline{V} + L_V(l)}$$

Using Imbedded Markov Chain

Directly using Residual Life Approach

$$W_q = \frac{l\overline{X^2}}{2(1 - l\overline{X})} + \frac{\overline{V^2}}{2\left(\overline{V} + \frac{1}{l}L_V(l)\right)}$$

M/G/1 Queue with Exceptional First Service (Section 4.3)

In this queue, the first customer starting a busy period requires a service time with a different distribution, i.e. $b^*(t)$ and $L_{B^*}(s)$ with moments $\overline{X^*}$ and $\overline{X^{*2}}$

An imbedded Markov Chain analysis will give

$$P(z) = \frac{p_0[L_B(l - lz) - zL_{B^*}(l - lz)]}{L_B(l - lz) - z} \qquad (4.19)$$

with

$$p_0 = \frac{1 - l\overline{X}}{1 - l\overline{X} + l\overline{X^*}} \qquad (4.18)$$

The delay distribution for the FCFS case, may be found using

$$P(z)=L_T(l - lz).$$

This may then be used to find W and W_q

Alternatively, these may be found using a *Residual Life Approach*

$$W = \frac{\overline{X^*}}{1 - l\overline{X} + l\overline{X^*}} + \frac{l\overline{X^2}}{2(1 - l\overline{X})} + \frac{l(\overline{X^{*2}} - \overline{X^2})}{2(1 - l\overline{X} + l\overline{X^*})} \qquad (4.21)$$

$$W_q = \frac{l\overline{X^2}}{2(1 - l\overline{X})} + \frac{l(\overline{X^{*2}} - \overline{X^2})}{2(1 - l\overline{X} + l\overline{X^*})} \qquad (4.22)$$

