The $M^{[X]}/G/1$ Queue

Single Server Queue

with

Batch Arrivals

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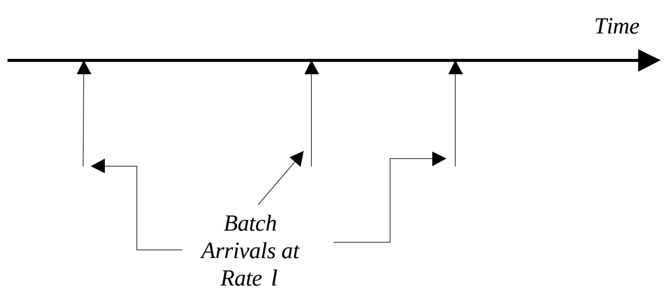

Number of jobs in a batch $= r$ (random variable) $1 \le r \le \infty$

b_r = P{r jobs in a batch}

$$b(z) = \sum_{r=1}^{\infty} b_r z^r$$

$$E\{r\} = \bar{r} = b'(1) = \sum_{r=1}^{\infty} r b_r$$

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Solving the $M^{[X]}/G/1$ Queue (Number in the System, Delay)

Method I

Markov Chain imbedded at the individual job departure instants

Method II

Imbedded Markov Chain approach considering each batch as a job

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Method I: Considering the imbedded Markov Chain at the departure instant of each job

In this case, we can show that -

Generating Function of the number of arrivals in an interval of length *t*

$$E\{z^N\} = \sum_{g=0}^{\infty} b^g(z) \frac{(lt)^g}{g!} e^{-lt} = e^{-lt[1-b(z)]} \qquad (4.24)$$

Generating Function of the number of arrivals in a service time

$$A(z) = \int_{t=0}^{\infty} e^{-lt[1-b(z)]} b(t)dt = L_B(l - lb(z)) \qquad (4.25)$$

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Q(z) = Generating Function of number left behind in the system as seen by a departing job

Q(z) may be obtained in the same way as *P(z)* for the simple M/G/1 queue

Applying the *P-K Transform Equation* of Eq. (3.14) with *A(z)* as given by Eq. (4.25) gives

$$Q(z) = \frac{(1-\rho)(1-z)L_B(\lambda - \lambda\beta(z))}{L_B(\lambda - \lambda\beta(z)) - z} \qquad (4.26)$$

with $\rho = A'(1) = \lambda\bar{\beta}\,\overline{X} = \lambda\beta'(1)\overline{X}$

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Unfortunately, this derivation cannot be taken much further.

Since the arrival process is a batch Poisson process, the upward transitions of the system state may be +1, +2, +3……

This implies that Kleinrock's result will not be applicable to this queue and hence *Q(z)* will not be the generating function as seen by an arrival to the queue.

Moreover, PASTA will also not be applicable since the arrival process of jobs to the queue is also not a Poisson process.

The overall implication of this is that *Q(z)* will not be the generating function of the number in the system at equilibrium and cannot be used as we had used *P(z)* for the M/G/1 queue

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Method II: Considering each batch as a single job, deriving the batch queueing delay and then adding to that the delay within a batch.

Batch Service Time: Random variable X^* with distribution given by $b^*(t)$, $L_{B*}(s)$

$$L_{B*}(s) = \sum_{r=1}^{\infty} b_r \left(L_B(s)\right)^r = b(L_B(s)) \qquad (4.27)$$

$$\overline{X*} = \bar{r}\bar{X} = \bar{X}b'(1) \qquad (4.28)$$

$$\overline{X*^2} = \overline{X^2}\bar{r} + (\bar{X})^2[\overline{r^2} - \bar{r}] \qquad (4.29)$$

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In this case, the generating function *A(z)* of the *number of batches* arriving within a *batch service time* will be given by

$$A(z) = L_{B*}(l - lz) = b(L_B(l - lz))$$

Using this *A(z)* in Eq. (3.14), we get

Q(z) = Generating Function of number in the system at the batch departure instants

$$= \frac{(1-r)(1-z)[b(L_B(l - lz))]}{[b(L_B(l - lz))] - z} \qquad r = l\bar{r}\bar{X} \qquad (4.32)$$

Q(z) will also be the generating function of the number of batches in the system both at the batch arrival instants and at an arbitrary time instant at equilibrium. (*See Section 4.4.1*)

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W_{qb} = Mean waiting time in queue for service to start to a batch (*Batch Queueing Delay*)

We can obtain this either from direct residual life-time arguments or from *Q(z)*

$$W_{qb} = \frac{l}{2(1-r)}\overline{X^{*2}} \qquad (4.33)$$

with

$$r = l\overline{r}\overline{X} \qquad (4.31)$$

$$\overline{X^{*2}} = (\overline{X^2} - \overline{X}^2)\overline{r} + \overline{r^2}\,\overline{X}^2 = s_X^2\,\overline{r} + \left(\overline{r}^2 + s_r^2\right)\overline{X}^2 \qquad (4.34)$$

See (4.35) in Section 4.4.1 for another form of expression for W_q

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We now need the *queueing delay inside a batch* to complete the derivation for the *queueing delay for a job*

g_k = P{a job is served in the k^{th} order in a batch}

$$g_k = \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^{n} y_i(k)}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \frac{P\{X_i \ge k\}}{E\{X_i\}} = \frac{1}{\overline{r}}\sum_{i=k}^{\infty} b_i \qquad (4.36)$$

For the job served in the k^{th} order, the mean queueing delay will be $W_{qb}+kE\{X\}$

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w_2 = Queueing delay (random) for a job given that service to its batch has started

Probability density function $f_{W2}(t)$ with L.T. $L_{W2}(s)$

$$L_{W2}(s) = \sum_{k=1}^{\infty}[L_B(s)]^{k-1} g_k$$
$$= \frac{1}{\bar{r}}\sum_{k=1}^{\infty}[L_B(s)]^{k-1}(b_k + b_{k+1} + b_{k+2} + \ldots\ldots\infty)$$
$$= \frac{1}{\bar{r}}\sum_{i=1}^{\infty} b_i \frac{1-[L_B(s)]^i}{1-L_B(s)}$$
$$= \frac{[1-b(L_B(s))]}{\bar{r}(1-L_B(s))} \qquad (4.37)$$

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Using the *Moment Generating Propertyi*, we can use $L_{W2}(s)$ to find the mean queueing delay W_2 within the batch

$$W_2 = L_{W2}(s)\Big|_{s=1} = \frac{\bar{X}[\overline{r^2} - \bar{r}]}{2\bar{r}} \qquad (4.38)$$

Using the batch queueing delay obtained earlier, we get the overall queueing delay W_q for an arriving job

$$W_q = W_{qb} + W_2$$
$$= \frac{l[s_X^2 \bar{r} + (\bar{r}^2 + s_r^2)\bar{X}^2}{2(1-r)} + \frac{\bar{X}[\overline{r^2} - \bar{r}]}{2\bar{r}} \qquad (4.39)$$

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