EE 679, Queueing Systems (2000-01F)
Solutions to Test -4

Note: *The notation used below is the standard notation that has been used in the text*

1. (a) The imbedded Markov Chain at the departure instants may be written as -

$$
\begin{aligned}
n_{i+1} &= n_i + a_{i+1} - 1 && \text{for } n_i \geq 1 \\
&= a_{i+1} + j - U(j) && \text{for } n_i = 0
\end{aligned}
$$

where n_i is the number left behind in the system by a departing customer, a_{i+1} is the number of arrivals in the $(i+1)^{th}$ service time, and j is the number in the system at the end of the *only vacation* the server takes after the end of a *busy period* (i.e. for the case $n_i=0$) with pdf f_j and generating function $F(z)$

Note that-

$$F(z) = \tilde{F}_V(l - lz) \qquad f_0 = F(0) = \tilde{F}_V(l) \qquad F'(1) = l\overline{V} \qquad F''(1) = l^2\overline{V^2}$$

& $$A(z) = \tilde{B}(l - lz) \qquad A'(1) = l\overline{X} \qquad A''(1) = l^2\overline{X^2}$$

To find p_0 we can take expectation of both sides of the Imbedded Markov Chain equation to get -

$$\overline{n} = \overline{n} + l\overline{X} - (1 - p_0) + p_0 E\{j - U(j)\}$$

or $$1 - l\overline{X} = p_0[1 + l\overline{V} - (1 - f_0)] \qquad \Rightarrow \qquad p_0 = \frac{1 - l\overline{X}}{l\overline{V} + \tilde{F}_V(l)}$$

(b) $$P(z) = (1 - p_0)A(z)\left[\frac{1}{z(1 - p_0)}\{P(z) - p_0\}\right] + p_0 A(z) E\left\{f_0 + \sum_{j=1}^{\infty} f_j z^{j-1}\right\}$$

or $$P(z) = A(z)\left[\frac{P(z)}{z} - \frac{p_0}{z} + p_0 f_0 + p_0\frac{F(z)}{z} - p_0\frac{f_0}{z}\right]$$

$$P(z) = \left[\frac{1 - l\overline{X}}{l\overline{V} + \tilde{F}_V(l)}\right]\left[\frac{A(z)}{z - A(z)}\right]\left[\tilde{F}_V(l - lz) - (1 - z)\tilde{F}_V(l) - 1\right]$$

The expression for p_0 obtained in (a) may also be verified by using P(1)=1 in the above expression. Note that for this, L'Hospital's Rule will have to be used.

2. As in the usual M/G/1 queue, we can write $\overline{W_q} = R + l\,\overline{W_q}\,\overline{X}$

leading to $\overline{W_q} = \dfrac{R}{1 - l\overline{X}}$

Consider an interval *(0, t)* where *t* ® ¥ and let -

M(t) = Total Number of Arrivals in *(0, t)*
N(t) = Total Number of Busy Periods in *(0, t)*

If there was no *exceptional first service*, then the average busy period length will be $\dfrac{\overline{X}}{1 - l\overline{X}}$

Taking *exceptional first service* into account, the average busy period length will be $\left[\overline{\tilde{X}} + (l\overline{\tilde{X}})\left(\dfrac{\overline{X}}{1 - l\overline{X}}\right)\right] = \dfrac{\overline{\tilde{X}}}{1 - l\overline{X}}$

Therefore, the Mean Cycle Time = $\dfrac{1}{l} + \dfrac{\overline{\tilde{X}}}{1 - l\overline{X}} = \dfrac{1 + l\overline{\tilde{X}} - l\overline{X}}{l(1 - l\overline{X})} = T_C$ (say)

Then $N(t) = \dfrac{t}{T_C} = \dfrac{lt(1 - l\overline{X})}{1 + l\overline{\tilde{X}} - l\overline{X}}$

This yields -

$$R_t = \frac{1}{t}\int_0^t r(t)\,dt = \frac{1}{2}\left[\frac{1}{t}\sum_{i=1}^{M(t)-N(t)} X_i^2 + \frac{1}{t}\sum_{j=1}^{N(t)} \tilde{X}_j^2\right]$$

or
$$R_t = \frac{1}{2}\left[\frac{(M-N)}{t}\left(\frac{1}{M-N}\right)\sum_{i=1}^{M-N} X_i^2 + \frac{N}{t}\left(\frac{1}{N}\right)\sum_{j=1}^{N} \tilde{X}_j^2\right]$$

Taking limits as *t* ® ¥, we get -

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{l(1 - l\overline{X})}{1 + l\overline{\tilde{X}} - l\overline{X}}$$

& $$\lim_{t\to\infty} \frac{M(t) - N(t)}{t} = l - \frac{l(1 - l\overline{X})}{1 + l\overline{\tilde{X}} - l\overline{X}} = \frac{l^2\overline{\tilde{X}}}{(1 + l\overline{\tilde{X}} - l\overline{X})}$$

Therefore, $$R = \frac{1}{2}\frac{l^2\overline{\tilde{X}}}{(1 + l\overline{\tilde{X}} - l\overline{X})}(\overline{X^2}) + \frac{1}{2}\frac{l(1 - l\overline{X})}{1 + l\overline{\tilde{X}} - l\overline{X}}(\overline{\tilde{X}^2})$$

$$= \frac{1}{2}l\,\overline{\tilde{X}^2}\,\frac{(1 - l\overline{X})}{(1 + l\overline{\tilde{X}} - l\overline{X})} + \frac{1}{2}l\,\overline{X^2}\,\frac{[1 + l\overline{\tilde{X}} - l\overline{X} - (1 - l\overline{X})]}{(1 + l\overline{\tilde{X}} - l\overline{X})}$$

$$= \frac{1}{2}l\,\overline{X^2} + \frac{1}{2}l\,\frac{(1 - l\overline{X})}{(1 + l\overline{\tilde{X}} - l\overline{X})}(\overline{\tilde{X}^2} - \overline{X^2})$$

From this, we can get that $$\overline{W_q} = \frac{1}{2}\frac{l\,\overline{X^2}}{(1 - l\overline{X})} + \frac{1}{2}l\,\frac{(\overline{\tilde{X}^2} - \overline{X^2})}{(1 + l\overline{\tilde{X}} - l\overline{X})}$$

$$\text{Mean Service Time} = \overline{X}(1 - p_0) + p_0 \overline{\tilde{X}}$$

We can find p_0 as $$p_0 = \frac{1/l}{T_C} = \frac{1 - l\overline{X}}{1 + l\overline{X} - l\overline{\tilde{X}}}$$

Therefore, simplifying we get -

$$\text{Mean Service Time} = \frac{\overline{\tilde{X}}}{1 + l\overline{\tilde{X}} - l\overline{X}}$$

and $$\overline{W} = \overline{W_q} + \frac{\overline{\tilde{X}}}{1 + l\overline{\tilde{X}} - l\overline{X}}$$

Simplifying, we get -

$$\overline{W} = \frac{\overline{\tilde{X}}}{1 + l\overline{\tilde{X}} - l\overline{X}} + \frac{l\,\overline{X^2}}{2(1 - l\overline{X})} + \frac{l\,(\overline{\tilde{X}^2} - \overline{X^2})}{2(1 + l\overline{\tilde{X}} - l\overline{X})}$$