EE 679, Queueing Systems (2001-02F)
Solutions to Test -5

1. $b(z) = \sum_{r=1}^{\infty} (1-q)q^{r-1}z^r = \dfrac{(1-q)z}{1-qz}$

$$b'(z) = \frac{(1-q)}{1-qz} + \frac{q(1-q)z}{(1-qz)^2} = \frac{(1-q)}{(1-qz)^2} \qquad b''(z) = \frac{2q(1-q)}{(1-qz)^3}$$

Therefore $\quad \bar{r} = b'(1) = \dfrac{1}{1-q} \qquad \overline{r^2} - \bar{r} = b''(1) = \dfrac{2q}{(1-q)^2}$

Since $L_{B*}(s) = b(L_B(s))$, we have

$$\overline{X^*} = -L_{B*}'(s)\Big|_{s=0} = \bar{X}\,\bar{r} \qquad \overline{X^{*2}} = L_{B*}''(s)\Big|_{s=0} = \overline{X^2}\,\bar{r} + (\bar{X})^2(\overline{r^2} - \bar{r})$$

Therefore

$$W_{qb} = \frac{l}{2(1-r)}\overline{X^{*2}} \qquad \text{with } \; r = l\bar{r}\,\bar{x}$$

and

$$W_2 = \frac{\bar{X}(\overline{r^2} - \bar{r})}{2\bar{r}}$$

The total queueing delay is

$$W_q = W_{qb} + W_2$$

2. From Problem 1, we can get that

$$b(z) = \frac{(1-q)z}{1-qz} \qquad \bar{r} = \frac{1}{1-q} \qquad \overline{r^2} - \bar{r} = \frac{2q}{(1-q)^2}$$

In this case, we have

$$L_{B*}(s) = e^{-s\Delta}\, b(L_B(s))$$

Differentiating this and using the moment generating property, we get the moments of the batch service time as

$$\overline{X^*} = \overline{X}\,\overline{r} + \Delta$$
$$\overline{X^{*2}} = \Delta^2 + \overline{r}\left[\overline{X^2} - (\overline{X})^2\right] + 2\Delta\overline{r}\,\overline{X} + \overline{r^2}(\overline{X})^2$$

$$W_{qb} = \frac{l\,\overline{X^{*2}}}{2(1-r)} \qquad r = l\,\overline{X^*}$$

and $$W_2 = \frac{(\overline{r^2} - \overline{r})}{2\overline{r}}\overline{X} + q\Delta$$
$$W_q = W_{qb} + W_2$$

3. Note that, as derived in Section 4.5.1, the mean queueing delay $W_{q(k)}$ of customers of priority class k will be given by

$$W_{q(k)} = \frac{R}{(1-r_n-....-r_{k+1})(1-r_n-....-r_k)}$$

This may be rewritten as

$$W_{q(k)} = \frac{R}{r_k}\left[\frac{1}{(1-r_n-....-r_k)} - \frac{1}{(1-r_n-....-r_{k+1})}\right]$$

Therefore $$r_k W_{q(k)} = R\left[\frac{1}{(1-r_n-....-r_k)} - \frac{1}{(1-r_n-....-r_{k+1})}\right]$$

We can now write the sum $\sum_{k=1}^{n} r_k W_{q(k)}$ as

$$\sum_{k=1}^{n} r_k W_{q(k)} = \frac{Rr_n}{1-r_n} + \sum_{k=1}^{n-1} r_k W_{q(k)}$$
$$= \frac{Rr_n}{1-r_n} + R\left[\frac{1}{(1-r_n-....-r_1)} - \frac{1}{(1-r_n-....-r_2)}\right]$$
$$+ R\left[\frac{1}{(1-r_n-....-r_2)} - \frac{1}{(1-r_n-....-r_3)}\right]$$
$$+ ....$$
$$+ R\left[\frac{1}{(1-r_n-r_{n-1})} - \frac{1}{(1-r_n)}\right]$$
$$= \frac{Rr_n}{1-r_n} + R\left[\frac{1}{(1-r)} - \frac{1}{(1-r_n)}\right] \qquad r = r_1 + ...+ r_n$$

Therefore

$$\sum_{k=1}^{n} r_k W_{q(k)} = \frac{R}{(1-r)} - R = \frac{Rr}{(1-r)} \qquad \text{Q.E.D.}$$

An interesting alternate approach may also be considered. Assume that the queue is examined at an arbitrary time instant and let R be the mean residual service time observed at that time with U as the mean unfinished work in the queue. We can then write that

$$U = R + \sum_{k=1}^{n} N_{q(k)} \overline{X_k}$$

Applying Little's result individually for each priority class gives

$$U = R + \sum_{k=1}^{n} l_k W_{q(k)} \overline{X_k}$$

$$= R + \sum_{k=1}^{n} r_k W_{q(k)}$$

Since both U and R are not dependent on the priority order of the classes, $\sum_{k=1}^{n} r_k W_{q(k)}$ will also be independent of this. Moreover, for a single class queue, we can write that

$$U = R + r\,\frac{R}{1-r} = \frac{R}{1-r}$$

Therefore, we get that for n priority classes as well $U = \frac{R}{1-r}$. Using this, we get

$$\sum_{k=1}^{n} r_k W_{q(k)} = U - R = \frac{Rr}{1-r} \qquad \text{Q.E.D.}$$