EE 679, Queueing Systems (2002-03F)
Solutions to Exam - II

1. (a) Considering a batch as one job, its service time is characterized by -

L.T.of batch service time pdf $L_B(s) = \frac{1}{2} L_a(s)[1 + L_b(s)]$

Mean of batch service time $\overline{X_B} = a^{(1)} + \frac{1}{2} b^{(1)}$

Second moment of batch service time $\overline{X_B^2} = \left(a^{(2)} + a^{(1)} b^{(1)} + \frac{1}{2} b^{(2)} \right)$

Offered Traffic $r = l\,\overline{X_B} = l\left[a^{(1)} + \frac{1}{2} b^{(1)} \right]$

For a batch considered as one job, we get

L.T. of pdf of batch queueing delay $L_{Wqb}(s) = \frac{s(1-r)}{s - l + l L_B(s)}$

Mean batch queueing delay $W_{qb} = \frac{l\,\overline{X_B^2}}{2(1-r)}$

Therefore

Mean queueing delay $W_q = W_{qb} + \frac{1}{3} a^{(1)}$

L.T. of pdf of queueing delay $L_{Wq}(s) = \frac{1}{3} L_{Wqb}(s)[2 + L_a(s)]$

(b) Mean Queueing Delay observed by the second job will be $W_{q2} = W_{qb} + a$

2. (a) Solving the flow balance equations, we get the following -

$$l_1 = 2.5l \quad l_2 = 1.5l \quad l_3 = 2.5l \quad l_4 = l \qquad \Rightarrow \quad P(n_1, n_2, n_3, n_4) = r^{n_1+n_2+n_3+n_4}(1-r)^4$$
$$r_1 = r_2 = r_3 = r_4 = r$$

(b) Mean number in the system $N = \frac{4r}{(1-r)}$

(c) Mean delay averaged over all arrivals $W_{overall} = \frac{4}{3m(1-r)}$

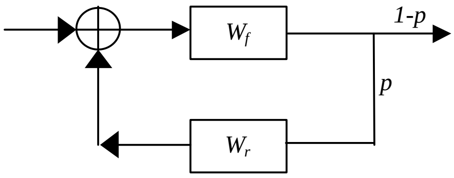
(d) The delay (queueing+service) at each of the queues will be as follows -

$$W_1 = \frac{2}{5m(1-r)} \quad W_2 = \frac{2}{3m(1-r)} \quad W_3 = \frac{2}{5m(1-r)} \quad W_4 = \frac{1}{m(1-r)}$$

The following result for a queueing system with a delay of W_f on the forward path and a delay of W_r on the reverse (feedback path) will be useful.

$$W_{total} = W_f + \sum_{j=1}^{\infty} [\,j(W_f + W_r)](1-p)p^j$$

$$= W_f + (W_f + W_r)\frac{p}{1-p}$$



Using this, we can easily get that

$$W_{(1\,entry)} = \frac{W_2 + W_3}{1-0.2} = \frac{4}{3m(1-r)}$$

The other delay term $W_{(2l\;entry)}$ required to be found may also be calculated explicitly in a similar manner. However, it is much simpler to note that $W_{(l\;entry)}$ is the same as the overall average $W_{overall}$. Hence, the delay term $W_{(2l\;entry)}$ must be the same as $W_{overall.}$

Therefore, we get that $W_{(2l\;entry)} = \dfrac{4}{3m(1-r)}$

(Alternate Solution Approach from *Nabhendra Bisnik*)

Let l_A be the input at **A**. Let l_{1A} be the flow in the i^{th} queue corresponding to this input flow. By solving the flow balance equations just for this flow alone, we can show that $l_{1A} = 1.25 l_A$, $l_{2A} = 0.125 l_A$, $l_{3A} = 0.625 l_A$, $l_{4A} = 0.5 l_A$ corresponding to visit ratios (for this flow alone) of $V_{1A} = 1.25, V_{2A} = 0.125, V_{3A} = 0.625, V_{4A} = 0.5$.
Therefore, we get that for jobs input from **A**, the mean time spent in the system before departure will be -

$$W_A = W_{(2l\;entry)} = \sum_{j=1}^{4} V_{jA}W_j = 1.25\left[\frac{2}{5m(1-r)}\right] + 0.125\left[\frac{2}{3m(1-r)}\right]$$

$$+ 0.625\left[\frac{2}{5m(1-r)}\right] + 0.5\left[\frac{1}{m(1-r)}\right] = \frac{4}{3m(1-r)}$$

Similarly, let l_B be the input at **B**. Let l_{1B} be the flow in the i^{th} queue corresponding to this input flow. By solving the flow balance equations just for this flow alone, we can show that $l_{1B} = 0, l_{2B} = l_{3B} = 1.25 l_B, l_{4B} = 0$ corresponding to visit ratios (for this flow alone) of $V_{1B} = V_{4B} = 0, V_{2B} = V_{3B} = 1.25$.
Therefore

$$W_B = W_{(l\;entry)} = \sum_{j=1}^{4} V_{jB}W_j = 1.25\left[\frac{2}{3m(1-r)}\right] + 1.25\left[\frac{2}{5m(1-r)}\right]$$

$$= \frac{4}{3m(1-r)}$$

3. In order to get the Norton's equivalent circuit, we should short *Q4* by setting its service time to zero (i.e. the service rate to infinity) in Fig. 3.1 and calculate the *actual throughput* l_4 under these conditions. This is done for *M=1,2,3,4* and the l_4 obtained would be the service rates of the FES in Figure 3.2. It would be convenient to do this using the MVA algorithm as given below

Visit Ratios $V_1 = 1 \quad V_2 = 1.25 \quad V_3 = 0.5 \quad V_4 = 0.5$

M=0 $N_1(0) = N_2(0) = N_3(0) = N_4(0) = 0$

M=1
$$\begin{cases} W_1(1) = 1 \quad W_2(1) = 0.5 \quad W_3(1) = 1 \\ l(1) = \dfrac{1}{1 + 0.625 + 0.5} = 0.4706 \quad \Rightarrow \quad l_4(1) = 0.2353 \\ N_1(1) = 0.4706 \quad N_2(1) = 0.2941 \quad N_3(1) = 0.2353 \end{cases}$$

M=2
$$\begin{cases} W_1(2) = 1.4706 \quad W_2(2) = 0.64705 \quad W_3(2) = 1.2353 \\ l(2) = 0.6903 \quad \Rightarrow \quad l_4(2) = 0.3452 \\ N_1(2) = 1.0151 \quad N_2(2) = 0.5583 \quad N_3(2) = 0.4624 \end{cases}$$

M=3
$$\begin{cases} W_1(3) = 2.0151 \quad W_2(3) = 0.7792 \quad W_3(3) = 1.4264 \\ l(3) = 0.8103 \quad \Rightarrow \quad l_4(3) = 0.4051 \\ N_1(3) = 1.6328 \quad N_2(3) = 0.7892 \quad N_3(3) = 0.5779 \end{cases}$$

M=4
$$\begin{cases} W_1(4) = 2.6328 \quad W_2(4) = 0.8946 \quad W_3(4) = 1.5779 \\ l(4) = 0.8810 \quad \Rightarrow \quad l_4(4) = 0.4405 \\ N_1(4) = 2.3195 \quad N_2(4) = 0.9852 \quad N_3(4) = 0.6951 \end{cases}$$

From the above, we get that the FES of Fig. 3.2 must have the following state dependent service rates

$$m_{FES}(1) = l_4(1) = 0.2353$$
$$m_{FES}(2) = l_4(2) = 0.3452$$
$$m_{FES}(3) = l_4(3) = 0.4051$$
$$m_{FES}(4) = l_4(4) = 0.4405$$

4. (a) *Late Arrival Model*
$$n_{i+1} = a_{i+1} \qquad n_i = 0$$
$$= n_i + a_{i+1} - 1 \qquad n_i \geq 1$$

Early Arrival Model
$$n_{i+1} = \tilde{a}_{i+1} \qquad n_i = 0$$
$$= n_i + a_{i+1} - 1 \qquad n_i \geq 1$$

Give a graphical argument to justify these equations where a_{i+1} is the number of arrivals in the $(i+1)^{th}$ service time and $\tilde{a}_{i+1}$ is the number of arrivals in the $(i+1)^{th}$ service time minus one slot.

(b) To find p_o directly for the two cases, consider the Markov Chain at equilibrium and take the expectation of both left-hand and right-hand sides. This gives the following.

Late Arrival Model

$$N = p_0[lb] + N + (1 - p_0)[lb - 1]$$
$$\Rightarrow \quad p_0 = 1 - lb$$

Early Arrival Model

$$N = p_0[l(b-1)] + N + (1 - p_0)[lb - 1]$$
$$\Rightarrow \quad p_0 = \frac{1 - lb}{1 - l}$$

Note that p_0 is higher in the case of the *Early Arrival Model*. This is because in this model, at the job departure instants, the arrivals that may come at that slot boundary are not taken into account. Hence, the system would have a higher probability of being observed to be empty.

(c) Note that the queue has been assumed FCFS in nature. Therefore, the number seen in the system at a job's departure instant would be the number arriving while the job was in the system. It may also be noted that for the time spent in system in the *Early Arrival Model* the slot in which the job arrives and the slot in which it leaves will both be counted. The model also implies that for the arrival and subsequent departure of a job, other jobs may only arrive in one less slot than the total number of slots that the job spends in the system.

Let $g_w(j)$ be the probability that a job spends j slots in the system with $G_W(z) = \sum_{j=1}^{\infty} g_w(j) z^j$. Therefore -

$$P_E(z) = \sum_{j=1}^{\infty} g_w(j) \sum_{k=0}^{j-1} \binom{j-1}{k} l^k (1-l)^{j-1-k} z^k = \sum_{j=1}^{\infty} g_w(j)(1 - l + lz)^{j-1} = \frac{G_W(1 - l + lz)}{(1 - l + lz)}$$

5. We consider each class separately, starting from the highest priority class

Class 3: Mean Residual Lifetime $R_3 = \frac{1}{2}\left(l_2 \overline{X_2^2} + l_3 \overline{X_3^2}\right)$

$$W_{q3} = R_3 + \overline{X}_3 N_{q3} \quad \Rightarrow \quad W_{q3} = \frac{R_3}{(1 - r_3)} \quad \text{with } r_3 = l_3 \overline{X}_3$$

$$W_3 = \overline{X}_3 + \frac{1}{2(1 - r_3)}\left(l_2 \overline{X_2^2} + l_3 \overline{X_3^2}\right)$$

Class 2: Mean Residual Lifetime $R_2 = R_3 = \frac{1}{2}\left(l_2 \overline{X_2^2} + l_3 \overline{X_3^2}\right)$

$$W_{q2} = R_2 + N_{q3}\overline{X_3} + \overline{X_3}\, l_3 W_{q2} + \overline{X_2} N_{q2}$$

$$\Rightarrow \quad W_{q2} = \frac{R_2 + r_3 W_{q3}}{(1 - r_2 - r_3)} \qquad \text{with } r_2 = l_2 \overline{X_2}$$

$$W_2 = \overline{X_2} + \frac{1}{(1 - r_2 - r_3)}\left[\frac{\left(l_2 \overline{X_2^2} + l_3 \overline{X_3^2}\right)}{2} + r_3 W_{q3}\right]$$

Class 1: Mean Residual Lifetime $R_1 = \frac{1}{2}\left(l_1 \overline{X_1^2} + l_2 \overline{X_2^2} + l_3 \overline{X_3^2}\right)$

$$W_1 = \overline{X_1} + \frac{R_1}{(1 - r_1 - r_2 - r_3)} + \overline{X_3}\, l_3 W_1 + \overline{X_2}\, l_2 W_1 \qquad \text{with } r_1 = l_1 \overline{X_1}$$

$$\Rightarrow \quad W_1 = \frac{\overline{X_1}(1 - r_1 - r_2 - r_3) + R_1}{(1 - r_2 - r_3)(1 - r_1 - r_2 - r_3)}$$