EE 679, Queueing Systems (2002-03F)
Solutions to Exam - I

1. Let the random variable *x*, $x^{3}0$, be the length of the inter-arrival time, i.e. the time between successive arrivals, where the earlier arrival occurs at time $t=0$. We examine the system at time t measured from this earlier arrival, when no arrivals have occurred in the time interval $(0, t)$. Let $Y=\{(X-t) \mid X>t\}$ be the remaining time to the next arrival (measured from t), given that there have been no arrivals between 0 and t.

Then $P\{(X-t)>y , X>t\} = P\{(X-t)>y \mid X>t\}P\{X>t\}$

Since y and t are both positive, we have $P\{(X-t)>y , X>t\}=P\{(X-t)>y\}$

Therefore $P\{(X-t)>y \mid X>t\} = \dfrac{P\{(X-t)>y\}}{P\{X>t\}} = \dfrac{1-F_X(t+y)}{1-F_X(t)}$

Since the inter-arrival times are exponentially distributed, we have $f_X(x) = l e^{-lx} \quad x \geq 0$ and therefore $F_X(x) = 1-e^{-lx} \quad x \geq 0$.

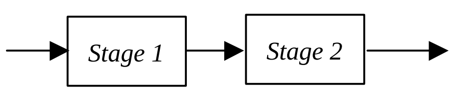
Substituting, we get that $\quad P\{(X-t)>y \mid X>t\} = e^{-ly} \quad t, y \geq 0$

Therefore, the cdf of Y is $\quad F_Y(y) = P\{(X-t) \leq y \mid X>t\} = 1-e^{-ly} \quad t, y \geq 0$

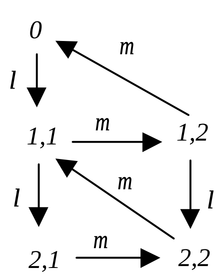
and the pdf of Y is $\quad f_Y(y) = l e^{-ly} \quad y \geq 0$

Note that the remaining time Y to the next arrival has the same distribution (pdf, cdf) as the inter-arrival time measured between successive arrivals. This demonstrates the memoryless property for this case.

2. One can easily argue that as far as computing the probabilities of the number in the system are concerned, the server may be modeled as follows where each stage provides exponential service at rate *m*.



(a)



The state of the system (with this server model) is represented as (m, j) where m is the number in the system and j is the stage in which the currently serverd customer is being served, $m=0,1,2$ and $j=1,2$

The following balance equations may be written

$$\begin{aligned} mp_{12} &= lp_0 \\ (l+m)p_{12} &= mp_{11} \\ lp_{11} &= mp_{21} \\ p_{11}+p_{21} &= p_{12}+p_{22} \end{aligned} \quad \Rightarrow \quad \begin{aligned} p_{12} &= rp_0 \\ p_{11} &= r(1+r)p_0 \\ p_{21} &= r^2(1+r)p_0 \\ p_{22} &= r^2(2+r)p_0 \end{aligned} \quad \text{with} \quad p_0 = \frac{1}{1+2r(1+r)^2}$$

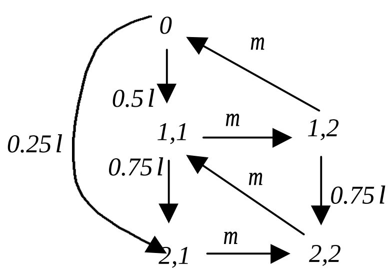
The other values of p_i where i is the number in the system will be

$$p_1 = p_{11}+p_{12} = \frac{r(2+r)}{1+2r(1+r)^2} \quad \text{and} \quad p_2 = p_{21}+p_{22} = \frac{r^2(3+2r)}{1+2r(1+r)^2}$$

At equilibrium, the average departure rate l_D of jobs from the system will be the same as the average arrival rate of jobs actually entering the system. Therefore

$$l_D = l[1-p_2] = l\,\frac{1+2r+r^2+4r^3}{1+2r(1+r)^2}$$

(b)



The state transition diagram is as shown. The balance equations for this are given below.

$$\begin{aligned} mp_{12} &= 0.75lp_0 \\ (0.75l+m)p_{12} &= mp_{11} \\ 0.25lp_0+0.75lp_{11} &= mp_{21} \\ p_{11}+p_{21} &= p_{12}+p_{22} \end{aligned}$$

Solving these, we get

$$\begin{aligned} p_{12} &= 0.75rp_0 \\ p_{11} &= (0.75r)(1+0.75r)p_0 \\ p_{21} &= [0.25r+(0.75r)^2(1+0.75r)]p_0 \\ p_{22} &= [(0.75r)(1+0.75r)^2-0.5r]p_0 \end{aligned} \quad \Rightarrow \quad p_0 = \frac{1}{\left(1+0.5r+(1.5r)(1+0.75r)^2\right)}$$

Therefore

$$p_1 = \frac{(0.75r)(2+0.75r)}{\left(1+0.5r+(1.5r)(1+0.75r)^2\right)}$$

$$p_2 = \frac{(0.75r)(1+0.75r)(1+0.75r)-0.25r}{\left(1+0.5r+(1.5r)(1+0.75r)^2\right)}$$

Once again, the departure rate of jobs will be the same as the arival rate of jobs which actually enter the system. Using this, we get

$$l_D = p_0[0.5l+0.5l]+p_1[0.5l+0.25l] = l[p_0+0.75p_1]$$

3. For notational simplicity, use $q=1-p$

(a) Effective Service Time Distribution (L.T.) = $L_{B*}(s)$

$$L_{B*}(s)=\sum_{k-1}^{\infty} pL_B(s)\left[qL_B(s)\right]^{k-1}=\frac{pL_B(s)}{1-qL_B(s)}$$

with Mean Effective Service Time $\overline{X^*}=\sum_{k=1}^{\infty} pq^{k-1})(k\overline{X})=\frac{p\overline{X}}{(1-q)^2}=\frac{\overline{X}}{p}$

and Effective Traffic = $r^*=l\,\overline{X^*}=\frac{l\overline{X}}{p}=\frac{r}{p}$

Drawing an analogy with the basic M/G/1 queue, we can then write

$$P(z)=p_0\frac{(1-z)L_{B*}(l-lz)}{L_{B*}(l-lz)-z} \quad \text{with} \quad p_0=1-r^*$$

Simplifying, we get

$$P(z)=\left[1-\frac{l\overline{X}}{p}\right]\frac{p(1-z)L_B(l-lz)}{pL_B(l-lz)-z[1-qL_B(l-lz)]}=\left[1-\frac{l\overline{X}}{p}\right]\frac{p(1-z)L_B(l-lz)}{(p+qz)L_B(l-lz)-z}$$

with $p_0=1-\frac{l\overline{X}}{p}=\frac{p-r}{p}$

(b) For job completions at the server, the following Markov Chain may be written for the corresponding imbedded points

For $n_i=0$	$n_{i+1}=a_{i+1}$	probability p
	$=a_{i+1}+1$	probability q
For $n_i>0$	$n_{i+1}=n_i+a_{i+1}-1$	probability p
	$=n_i+a_{i+1}$	probability q

Therefore

$$P(z)=A(z)p_0[p+qz]+A(z)pz^{-1}[P(z)-p_0]+A(z)q[P(z)-p_0]$$
$$zP(z)=zp_0(p+qz)A(z)+(p+qz)A(z)[P(z)-p_0]$$
$$P(z)[(p+qz)A(z)-z]=p_0(1-z)(p+qz)A(z)$$
$$P(z)=p_0\frac{(1-z)(p+qz)A(z)}{[(p+qz)A(z)-z]}$$

Directly taking means of the LHS and the RHS of the Markov Chain expressions at equilibrium and using $E(n_i\}=E\{n_{i+1}\}=N$ and $E\{a_{i+1}\}=l\overline{X}=r$, we get

$$N=N+l\overline{X}+p_0q-(1-p_0)p$$
$$0=r+p_0-p$$

therefore $p_0=p-r$

Note that this is different from the p_0 obtained in part (a)

4. To prove $A_k = \int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx}[1-B(x)]\, l\, dx$ for $k=1,2,,..........$¥ **(A)**

Note that since A_k=P{k or more arrivals in the time interval}
and A_{k+1}=P{$k+1$ or more arrivals in the time interval}

Therefore $A_k = A_{k+1}$ + P{k arrivals in the time interval}

$$= A_{k+1} + \int_{x=0}^{\infty} \frac{(lx)^k}{k!} e^{-lx} b(x)dx$$

or
$$A_{k+1} = A_k - \int_0^\infty \frac{(lx)^k}{k!} e^{-lx} b(x)dx \qquad \textbf{(B)}$$

We prove **(A)** by mathematical induction by first showing that it holds for $k=1$ and then using the recursion of **(B)** to show that if it holds for k then it will also hold for $k+1$.

For $k=1$
$$A_1 = \sum_{j=1}^{\infty} \int_0^\infty \frac{(lx)^j}{j!} e^{-lx} b(x)dx = \int_0^\infty (1-e^{-lx}) b(x)dx = 1 - \int_0^\infty e^{-lx} b(x)dx$$

Integrating by parts, we can show that
$$\int e^{-lx} b(x)dx = e^{-lx}B(x) + l\int e^{-lx}B(x)dx \qquad \textbf{(C)}$$

and hence
$$\int_0^\infty e^{-lx} b(x)dx = e^{-lx}B(x)\Big|_0^\infty + l\int_0^\infty e^{-lx}B(x)dx = l\int_0^\infty e^{-lx}B(x)dx$$

Moreover
$$1 = l\int_0^\infty e^{-lx}dx$$

Therefore $A_1 = \int_0^\infty e^{-lx}[1-B(x)]\, l\, dx$ as given by **(A)** for the case $k=1$

Using the recursion of **(B)** and assuming **(A)** holds for k, we get the following for $k+1$

$$A_{k+1} = l\int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx}[1-B(x)]dx - \int_0^\infty \frac{(lx)^k}{k!} e^{-lx} b(x)dx \qquad \textbf{(D)}$$

Integrating by parts, we can show that

$$\int_0^\infty \frac{(lx)^{k-1}}{(k-1)!} e^{-lx}\, l[1-B(x)]dx$$

$$= e^{-lx}\, l[1-B(x)]\frac{1}{l}\frac{(lx)^k}{k!}\Bigg|_0^\infty + \int_0^\infty l\left[l e^{-lx}\left(1-B(x)\right) + e^{-lx}b(x)\right]\frac{1}{l}\frac{(lx)^k}{k!}dx \qquad \textbf{(E)}$$

$$= \int_0^\infty \frac{(lx)^k}{k!} e^{-lx}\left[l\left(1-B(x)\right) + b(x)\right]dx$$

Substituting **(E)** in **(D)**, we get the desired result

$$A_{k+1} = \int_0^\infty \frac{(lx)^k}{k!} e^{-lx} \left(\left[l\left(1 - B(x)\right) + b(x)\right] - b(x)\right) dx$$
$$= \int_0^\infty \frac{(lx)^k}{k!} e^{-lx}\, l\,[1 - B(x)]dx$$

Q.E.D.