3.5 Delay Analysis for a LCFS M/G/1 Queue

In Section 3.3, we had done the delay analysis for the FCFS M/G/1 queue to obtain the distributions of the total time spent in the system by an arriving customer and the queueing delay encountered by a job (before its service is started). In this section, we consider the delay analysis for a LCFS M/G/1 queue to obtain the same distributions. Note that the distributions of the delay will depend on the service discipline even though the means will remain the same.

For this derivation, we consider a customer A arriving to the system. Two cases are possible here. The first case is when customer A arrives to an empty queue, with probability $(1-r)$; in this case, the queueing delay is zero and service to customer A starts immediately. The second case is more complicated and arises when customer A arrives to a non-empty queue (with probability r).

This second case has been shown in Figure 3.7. We consider this in more detail. Note that in this case, the total time T spent in the system by Customer A will be the sum of its queueing time Q and its service time X, i.e. $T = Q + X$. (In the first case, the queueing time would be zero as service would start immediately for customer A, immediately upon arrival.)

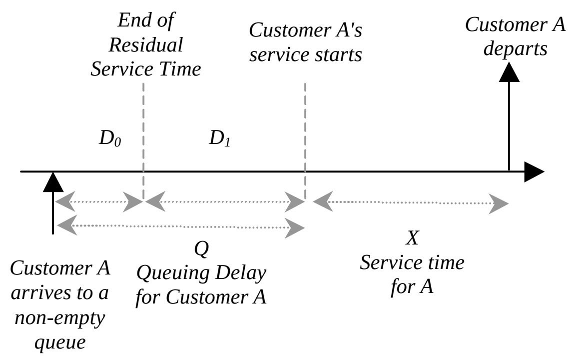

Figure 3.7. Customer Arrival and Departure from a LCFS M/G/1 Queue

Note that when it arrives to a non-empty queue, customer A will actually come in the middle of the ongoing service time of whichever customer is currently being served. Let D_0 be the residual service time for this service to complete. The Queueing Delay Q is shown to consist of D_0 and another component D_1. Since the queue service is LCFS, this second component D_1 will consist of *sub busy periods*, one associated with each of the customer arrivals in D_0. (Since the queue is LCFS, these will have to be served first before Customer A can be served.) Note that *D_0 and D_1 are not independent of each other*. This is evident as the length of D_0 affects the number arriving in it, which in turn affects the length D_1.

When customer A arrives to a non-empty queue, it will essentially sample an ongoing service where D_0 is the corresponding residual service time. From our earlier results on residual life, as given in Eq. (3.7), we can then state that

$$f_{D_0}(t) = \frac{1 - B(t)}{\bar{X}} \tag{3.20}$$

where $B(t)$ is the cumulative distribution function of the service time X whose mean is $\bar{X}$. Taking transforms, we get

$$L_{D_0}(s) = \frac{1 - L_B(s)}{s\bar{X}} \tag{3.21}$$

Let N_0 be the number of arrivals in D_0. Then, for customer A coming to a non-empty queue, we get

$$E\left\{e^{-sD_1} \mid D_0 = y, N_0 = n\right\} = [L_{BP}(s)]^n$$

& $E\left\{e^{-sQ} \mid D_0 = y, N_0 = n\right\} = e^{-sy}[L_{BP}(s)]^n$

where $L_{BP}(s)$ is the L.T. of the probability density function of the busy period which can be found using Eq. (3.19). Therefore for the case where customer A arrives to a non-empty queue, we will have

$$E\left\{e^{-sD_1} \mid D_0 = y\right\} = \sum_{n=0}^{\infty} \frac{(ly)^n}{n!} e^{-ly}[L_{BP}(s)]^n = \exp[-y\{l - lL_{BP}(s)\}]$$

and

$$E\left\{e^{-sQ} \mid D_0 = y\right\} = \sum_{n=0}^{\infty} \frac{(ly)^n}{n!} e^{-ly} e^{-sy}[L_{BP}(s)]^n$$
$$= \exp[-y\{s + l - lL_{BP}(s)\}]$$

Using the probability density function (and its L.T.) of D_0, from Eqs. (3.20) and (3.21), we get that for the case of customer A coming to a non-empty queue, we will have

$$E\left\{e^{-sD_1}\right\} = \frac{1 - L_B(l - lL_{BP}(s))}{\overline{X}(l - lL_{BP}(s))} \tag{3.22}$$

and

$$E\left\{e^{-sQ}\right\} = \frac{1 - L_B(s + l - lL_{BP}(s))}{\overline{X}(s + l - lL_{BP}(s))} = \frac{1 - L_{BP}(s)}{\overline{X}(s + l - lL_{BP}(s))} \tag{3.23}$$

Therefore, considering both the cases where Customer A finds the queue empty and non-empty, we get

$$L_Q(s) = (1 - r) + r\,\frac{1 - L_{BP}(s)}{(s + l - lL_{BP}(s))\overline{X}} \tag{3.24}$$

as the L.T. of the probability density function of the queueing delay Q for the LCFS M/G/1 queue.

The L.T. of the probability density function of the overall delay T for the LCFS M/G/1 queue may also be found. This follows from the fact that $T=Q+X$ where X is the service time and that Q and X will be independent of each there. Therefore, we have

$$L_T(s) = L_Q(s)L_B(s) \tag{3.25}$$

Note that $L_{BP}(s)$ may be obtained by first solving Eq. (3.19). This may be used in Eq. (3.24) to get $L_Q(s)$. Since $L_B(s)$ is given (as it is the L.T. of the probability density function of the service time X), we can then use Eqs. (3.24) and (3.25) to obtain $L_T(s)$.