Solution to Problem 2.10

(a)

$$
\begin{aligned}
p'_0(t) &= -l_1 p_0 \\
p'_1(t) &= -l_1 p_1 + l_1 p_0 \\
p'_2(t) &= -l_1 p_2 + l_1 p_1 \\
&\ldots\ldots \\
&\ldots\ldots \\
p'_{K_1-1}(t) &= -l_1 p_{K_1-1} + l_1 p_{K_1-2} \\
p'_{K_1}(t) &= -l_2 p_{K_1} + l_1 p_{K_1-1} \\
p'_{K_1+1}(t) &= -l_2 p_{K_1+1} + l_2 p_{K_1} \\
&\ldots\ldots \\
&\ldots\ldots \\
p'_{K_1+K_2-1}(t) &= -l_2 p_{K_1+K_2-1_1} + l_2 p_{K_1+K_2-2} \\
p'_{K_1+K_2}(t) &= l_2 p_{K_1+K_2-1}
\end{aligned}
$$

Taking Laplace Transforms of both sides of the above equations and solving, we get the required time-dependent state probabilities.

$$
0 \le i \le K_1 - 1 \quad p_i(t) = \frac{(l_1 t)^i}{i!} e^{-l_1 t} \qquad p_{K_1}(t) = \left(l_1 \frac{(l_1 t)^{K_1-1}}{(K_1-1)!} e^{-l_1 t} \right) \otimes e^{-l_2 t}
$$

$$
K_1 + 1 \le i \le K_1 + K_2 - 1 \quad p_i(t) = \left(l_1 \frac{(l_1 t)^{K_1-1}}{(K_1-1)!} e^{-l_1 t} \right) \otimes \left(\frac{(l_2 t)^{i-K_1}}{(i-K_1)!} e^{-l_2 t} \right)
$$

$$
p_{K_1+K_2}(t) = 1 - \sum_{i=0}^{K_1+K_2-1} p_i(t)
$$

(b) Using the results of (a) with $K_2 = 0$, we get

$$
\begin{aligned}
p_0(t) &= e^{-l_1 t} \\
p_i(t) &= \frac{(l_1 t)^i}{i!} e^{-l_1 t} \qquad 1 \le i \le K_1 - 1 \\
p_{K_1}(t) &= 1 - \sum_{i=0}^{K_1-1} p_i(t)
\end{aligned}
$$

as expected.