Solution to Problem 2.20

The state transition diagram for this system will be as shown in Fig. 1.15. The state is represented as \((m, j)\) where \(m\) is the number of jobs in the system and \(j\) is the stage that the job is currently in.

![State Transition Diagram](image)

Choosing convenient closed boundaries, the balance equations for this system may be written as

\[\begin{align*}
\lambda p_0 &= \mu p_{11} = (\lambda + \mu) p_{12} \\
\lambda (p_{11} + p_{12}) &= \mu p_{21} \\
p_{22} + p_{12} &= p_{11} + p_{21}
\end{align*}\]

which gives

\[\begin{align*}
p_{11} &= \rho p_0 \\
p_{12} &= \frac{\rho}{1+\rho} p_0 \\
p_{21} &= \frac{\rho^2 (2+\rho)}{1+\rho} p_0 \\
p_{22} &= \frac{\rho^2 (3+\rho)}{1+\rho} p_0
\end{align*}\]

Applying the normalisation condition to this, we get

\[p_0 = \frac{1+\rho}{1+3\rho + 6\rho^2 + 2\rho^3}\]

The probability \(P_{B}\) that an arrival leaves without service is \((p_{21}+p_{22})\). This leads to

\[P_{B} = \frac{\rho^2 (5+2\rho)}{(1+\rho)} p_0 \quad \text{and} \quad \lambda_{\text{eff}} = \lambda (1-P_{B}) = \lambda - \frac{1+3\rho + \rho^2}{1+3\rho + 6\rho^2 + 2\rho^3}\]
Since \( N_q = p_{21} + p_{22} = \frac{\rho^2 (5 + 2\rho)}{1 + 3\rho + 6\rho^2 + 2\rho^3} \), we get \( W_q = \frac{\rho^2 (5 + 2\rho)}{\lambda (1 + 3\rho + \rho^2)} \).

The Laplace Transform of the effective overall service distribution will be given by

\[
L_B(s) = 0.5 \left( \frac{2\mu}{s + 2\mu} \right) \sum_{i=0}^{\infty} 0.5 \left( \frac{2\mu}{s + 2\mu} \left( \frac{\mu}{s + \mu} \right) \right)^i
= \frac{\mu (s + \mu)}{(s^2 + 3\mu s + \mu^2)}
\]