Solution to Problem 2.3

(a) Infinite System Capacity (N=¥)

We have $l_j = a^{-j}, m_j = jm$ for *j=0,1,2,.......¥*

The differential equations for each state may be written as

$$\frac{dp_0(t)}{dt} = -a^{-0} p_0(t) + m\, p_1(t)$$
$$\frac{dp_j(t)}{dt} = -(a^{-j} + jm) p_j(t) + a^{-(j-1)} p_{j-1}(t) + (j+1)m\, p_{j+1}(t) \qquad j \geq 1$$

We define *P(z,t)* as the generating function of the state of the system at time t as given next.

$$P(z,t) = \sum_{j=0}^{\infty} p_j(t) z^j$$

Multiplying the j^{th} equation by z^j and summing the L.H.S. and RHS for all values of *j*, we will get

$$\frac{\partial P(z,t)}{\partial t} = -\sum_{j=0}^{\infty} a^{-j} z^j p_j(t) + \sum_{j=1}^{\infty} a^{-(j-1)} z^j p_{j-1}(t) - \sum_{j=1}^{\infty} jm z^j p_j(t)$$
$$+ \sum_{j=0}^{\infty} (j+1) m z^j p_{j+1}(t)$$

This may be simplified to get the final result

$$\frac{\partial P(z,t)}{\partial t} = (z-1)\left[P\left(\frac{z}{a}, t\right) - m \frac{\partial P(z,t)}{\partial z} \right]$$

This may be solved with the desired initial conditions to get the corresponding transient solution.

(b) Finite System capacity N

In this case, the differential equations for the system's state probabilities will become

$$\frac{dp_0(t)}{dt} = -a^{-0} p_0(t) + m\, p_1(t)$$
$$\frac{dp_j(t)}{dt} = -(a^{-j} + jm) p_j(t) + a^{-(j-1)} p_{j-1}(t) + (j+1)m\, p_{j+1}(t) \quad 1 \le j < N$$
$$\frac{dp_N(t)}{dt} = -Nm\, p_N(t) + a^{-(N-1)} p_{N-1}(t)$$

From the above, multiplying the j^{th} equation by z^j and summing the L.H.S. and RHS for all values of $j=0, 1....., N$, we will get

$$\frac{\partial P(z,t)}{\partial t} = -\sum_{j=0}^{N-1} a^{-j} z^j p_j(t) + \sum_{j=1}^{N} a^{-(j-1)} z^j p_{j-1}(t) - \sum_{j=1}^{N} jm z^j p_j(t)$$
$$+ \sum_{j=0}^{N-1} (j+1) m z^j p_{j+1}(t)$$

This may be simplified to get the final result for the finite capacity (of N) case as

$$\frac{\partial P(z,t)}{\partial t} = (z-1)\left[P\left(\frac{z}{a}, t\right) - m \frac{\partial P(z,t)}{\partial z} - \left(\frac{z}{a}\right)^N p_N(t) \right]$$