Solution to Problem 2.4

(a) The expected length of time customer A spends waiting for service $= \dfrac{(n+1)}{mm}$

(b) The expected length of time from A's arrival to the time when the system becomes empty $= \dfrac{(n+1)}{mm} + \dfrac{1}{m}\sum_{i=1}^{m}\dfrac{1}{i}$

(c) For $k=1,......,(n+1)$ $P\{X=k\} = 0$

For $k=n+2$,

$P\{X=k\}$ = P{X_A < residual service time of each X_i, $i=1,..,(m-1)$ in queue}

$$= \int_0^\infty me^{-mt}\,[P\{X > t\}]^{m-1}dt = \int_0^\infty me^{-mt}\,e^{-(m-1)mt}\,dt = \frac{1}{m}$$

For $k=n+3$,

$P\{X=k\}$ = P{one residual service time is less than X_A while the other $(m-2)$ are greater than X_A} $= \int_0^\infty me^{-mt}\,(m-1)(1-e^{-mt})e^{-(m-2)mt}\,dt = \dfrac{1}{m}$

In general, for $k=n+2+i$, $i=0,......(m-1)$, using $x=e^{-mt}$

$$P\{X=k\} \begin{aligned} &= \binom{m-1}{i}\int_0^1 (1-x)^i\,x^{m-1-i}dx \\ &= \binom{m-1}{i}\int_0^1 \sum_{j=0}^{i}(-1)^j\binom{i}{j}x^{m-1-i+j}dx = \binom{m-1}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^j}{m-i+j} = \frac{1}{m}\end{aligned}$$

For proving the above, use the result that for $i=0,...., (m-1)$

$I(m-1,i) = \int_0^1 \binom{m-1}{i}(1-x)^i\,x^{m-1-i}dx = I(m-1,i-1)$ and that $I(m-1,0\} = I\{m-1,1\} = 1$

(d) Service to the customer served before customer A and the service to customer A will be as shown in Fig. 1.1 when A finishes service before the former. Here t is the time interval between the start of service for these two customers in the queue. Let X_A be the duration of service for customer A and let X_1 be the service duration of the other customer.

The probability P that we need to find is then $P=P\{t+X_A<X_1\}$ as obtained next where $f_t(t)=(m-1)me^{-(m-1)mt}$ and $f_{XA}(t)=me^{-mt}$ for $t ³ 0$. Let $Y=t+X_A$ and since $t \wedge X_A$, we can write that

$$L_Y(s)=L_t(s)L_{X_A}(s)=\frac{m^2(m-1)}{(s+m)[s+(m-1)m]}$$
$$=\frac{m(m-1)}{(m-2)}\left[\frac{1}{s+m}+\frac{1}{s+(m-1)m}\right]$$

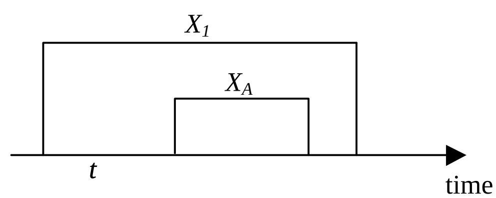

Figure 1.1. Service to A finishes before the service ends for the earlier customer

Therefore $f_Y(y)=\frac{m(m-1)}{(m-2)}\left[e^{-my}-e^{-(m-1)my}\right]$ for $y ³ 0$

Using this, we can find

$$P\{Y<t\}=\int_0^t f_Y(y)dy=\left(\frac{m-1}{m-2}\right)\left[(1-e^{-mt})-\frac{1}{(m-1)}(1-e^{-(m-1)mt}\right]$$

and therefore

$$P=\int_{t=0}^{\infty}P\{Y<t\}me^{-mt}dt=\left(\frac{m-1}{m-2}\right)\left[1-\frac{1}{2}-\frac{1}{(m-1)}\left(1-\frac{1}{m}\right)\right]=\frac{1}{2}\left(1-\frac{1}{m}\right)$$

(e) From the definition of the Erlang-n distributions as the sum of *(n+1)* i.i.d exponentially distributed random variables, we get

$$P\{w\leq x\}=\int_0^x\frac{m(ma)^n}{n!}e^{-ma}da$$