Solution to Problem 4.1

Following the residual life approach of Chapter 4, we get

$$W_q = R + l W_q \overline{X} \quad \text{or} \quad W_q = \frac{R}{1 - l\overline{X}}$$

where $\overline{X}$ is the mean service time and R is the mean residual service time. The exceptional first service time is the random variable X^*. This may be alternatively expressed as $X+D$ where D is a random variable indicating the additional service required by the first customer starting a busy period.

To find R, we consider a time interval of length t where we will subsequently let t ® ¥. Let *M(t)* be the number of arrivals in this interval and *N(t)* the number of busy periods. We note that -

Mean Busy Period Length (without exceptional first service) is $\frac{\overline{X}}{1 - l\overline{X}}$ and the actual mean busy period length $\overline{BP}$ will then be given as

$$\overline{BP} = \overline{X^*} + l\,\overline{X^*}\frac{\overline{X}}{1 - l\overline{X}} = \frac{\overline{X^*}}{1 - l\overline{X}} = \frac{\overline{X} + \overline{\Delta}}{1 - l\overline{X}}$$

Using this, the mean cycle time T_C will be given by

$$T_C = \frac{1}{l} + \frac{\overline{X^*}}{1 - l\overline{X}} = \frac{(1 + l\overline{\Delta})}{l(1 - l\overline{X})}$$

$$N(t) = \frac{t}{T_C} = \frac{l t(1 - l\overline{X})}{(1 + l\overline{\Delta})}$$

We can define the mean residual service time R_t measured over the time duration (*0,t*) as the following as a good approximation (which gets better as t ® ¥).

$$R_t = \frac{1}{t}\int_0^t r(t)\,dt = \frac{1}{t}\sum_{i=1}^{M(t)-N(t)}\frac{X_i^2}{2} + \frac{1}{t}\sum_{j=1}^{N(t)}\frac{X_j^{*2}}{2}$$
$$= \frac{1}{2}\left[\left(\frac{M-N}{t}\right)\left(\frac{1}{(M-N)}\sum_{i=1}^{M-N}X_i^2\right) + \left(\frac{N}{t}\right)\left(\frac{1}{(N)}\sum_{j=1}^{N}X_j^{*2}\right)\right]$$

For t ® ¥, we observe the following

$$\lim_{t\to\infty} R_t = R$$

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{l(1-l\overline{X})}{(1+l\overline{\Delta})}$$

$$\lim_{t\to\infty} \frac{M(t)}{t} = l$$

$$\lim_{t\to\infty} \frac{M(t)-N(t)}{t} = l - \frac{l(1-l\overline{X})}{(1+l\overline{\Delta})} = \frac{l^2(\overline{X}+\overline{\Delta})}{(1+l\overline{\Delta})} = \frac{l^2\overline{X^*}}{(1+l\overline{\Delta})}$$

Substituting, we get

$$R = \frac{\overline{X^{*2}}}{2}\left[\frac{l(1-l\overline{X})}{1+l\overline{\Delta}}\right] + \frac{\overline{X^2}}{2}\left[\frac{l^2\overline{X^*}}{1+l\overline{\Delta}}\right]$$

$$= \frac{l\overline{X^{*2}}}{2}\left[\frac{(1-l\overline{X})}{1+l\overline{\Delta}}\right] + \frac{l\overline{X^2}}{2}\left[\frac{(1+l\overline{\Delta})-(1-l\overline{X})}{1+l\overline{\Delta}}\right]$$

$$= \frac{l\overline{X^2}}{2} + \frac{l(1-l\overline{X})(\overline{X^{*2}}-\overline{X^2})}{2(1+l\overline{\Delta})}$$

$$\textit{with} \qquad \overline{X^*} = \overline{X} + \overline{\Delta} \qquad \overline{X^{*2}} = \overline{X^2} + 2\overline{\Delta}\overline{X} + \overline{\Delta^2}$$

and therefore $W_q = \dfrac{l\overline{X^2}}{2(1-l\overline{X})} + \dfrac{l(\overline{X^{*2}}-\overline{X^2})}{2(1+l\overline{\Delta})}$