Solution to Problem 4.1

Following the residual life approach of Chapter 4, we get

\[ W_q = R + \lambda W_q \bar{X} \quad \text{or} \quad W_q = \frac{R}{1 - \lambda \bar{X}} \]

where \( \bar{X} \) is the mean service time and \( R \) is the mean residual service time. The exceptional first service time is the random variable \( X^* \). This may be alternatively expressed as \( X + \Delta \) where \( \Delta \) is a random variable indicating the additional service required by the first customer starting a busy period.

To find \( R \), we consider a time interval of length \( t \) where we will subsequently let \( t \to \infty \). Let \( M(t) \) be the number of arrivals in this interval and \( N(t) \) the number of busy periods. We note that -

Mean Busy Period Length (without exceptional first service) is \( \frac{\bar{X}}{1 - \lambda \bar{X}} \) and the actual mean busy period length \( BP \) will then be given as

\[ BP = X^* + \lambda X^* \frac{\bar{X}}{1 - \lambda \bar{X}} = \frac{X^*}{1 - \lambda \bar{X}} = \frac{X + \Delta}{1 - \lambda \bar{X}} \]

Using this, the mean cycle time \( T_C \) will be given by

\[ T_C = \frac{1}{\lambda} + \frac{X^*}{1 - \lambda \bar{X}} = \frac{(1 + \lambda \Delta)}{\lambda(1 - \lambda \bar{X})} \]

\[ N(t) = \frac{t}{T_C} = \frac{\lambda t (1 - \lambda \bar{X})}{(1 + \lambda \Delta)} \]

We can define the mean residual service time \( R_t \) measured over the time duration \((0, t)\) as the following as a good approximation (which gets better as \( t \to \infty \)).

\[ R_t = \frac{1}{t} \int_0^t r(\tau) d\tau = \frac{1}{t} \sum_{i=1}^{M(t)-N(t)} \frac{X_i^2}{2} + \frac{1}{t} \sum_{j=1}^{N(t)} X_j^2 \]

\[ = \frac{1}{2} \left[ \left( \frac{M - N}{t} \right) \left( \frac{1}{(M - N)} \sum_{i=1}^{M-N} X_i^2 \right) + \left( \frac{N}{t} \right) \left( \frac{1}{N} \sum_{j=1}^{N} X_j^2 \right) \right] \]
For $t \to \infty$, we observe the following

\[
\begin{align*}
\lim_{t \to \infty} R_t &= R \\
\lim_{t \to \infty} \frac{N(t)}{t} &= \frac{\lambda(1 - \lambda \bar{X})}{1 + \lambda \Delta} \\
\lim_{t \to \infty} \frac{M(t)}{t} &= \lambda \\
\lim_{t \to \infty} \frac{M(t) - N(t)}{t} &= \lambda - \frac{\lambda(1 - \lambda \bar{X})}{1 + \lambda \Delta} = \frac{\lambda^2 (\bar{X} + \Delta)}{(1 + \lambda \Delta)} = \frac{\lambda^2 \bar{X}}{(1 + \lambda \Delta)}
\end{align*}
\]

Substituting, we get

\[
R = \frac{\lambda^2 \bar{X}}{2} + \frac{\lambda^2 \bar{X}^2}{1 + \lambda \Delta}
\]

\[
= \frac{\lambda \bar{X}^2}{2} \left[ (1 - \lambda \bar{X}) \right] + \frac{\lambda \bar{X}^2}{2} \left[ (1 + \lambda \Delta) - (1 - \lambda \bar{X}) \right]
\]

\[
= \frac{\lambda \bar{X}^2}{2} + \frac{\lambda(1 - \lambda \bar{X})(\bar{X}^2 - \bar{X}^2)}{2(1 + \lambda \Delta)}
\]

with \( \bar{X}^2 = \bar{X} + \Delta \left[ \bar{X}^2 = \bar{X} + 2\Delta \bar{X} + \Delta^2 \right] \)

and therefore \( W_q = \frac{\lambda \bar{X}^2}{2(1 - \lambda \bar{X})} + \frac{\lambda(\bar{X}^2 - \bar{X}^2)}{2(1 + \lambda \Delta)} \)