Solution to Problem 4.2

We assume that the characteristics of the service duration is chosen (depending on the current value of *N*) at the instant service starts and is not modified even if there are more arrivals (i.e. *N* changes) during the time service is provided to a customer.

This problem is difficult to do for a general *N*. We consider the case of *N=2* next. However, the procedure outlined below for *N=2* may be extended to larger values of *N* as required.

In terms of notation, assume $b(t)$, $B(t)$, $L_B(s)$, $A(z)=L_B(l-lz)$, a_{i+1} for service started when the number of customers is less than *N*. When the number of customers is *N* or more when service starts, the corresponding notation used is $b^*(t)$, $B^*(t)$, $L_B^*(s)$, $A^*(z)=L_B^*(l-lz)$, a_{i+1}^*.

Case *N=2*

The Markov Chain at the departure instants may be written in the usual fashion as

$$
\begin{aligned}
n_{i+1} &= a_{i+1} && n_i = 0 \\
&= n_i + a_{i+1} - 1 && 0 < n_i < 2 \\
&= n_i + a_{i+1}^* - 1 && n_i \geq 2
\end{aligned}
$$

Taking the expectation of both sides of the above with $r = l\overline{X}$, $r^* = l\,\overline{X^*}$, we get that

$$N = p_0 r + p_1 r + (N - p_1) + (1 - p_0 - p_1)(r^* - 1)$$

or $$p_0(r - r^* + 1) + p_1(r - r^*) = 1 - r^* \qquad \text{(A)}$$

In addition, we can show that

$$P(z) = (p_0 + p_1)A(z) + \frac{A^*(z)}{z}[P(z) - p_0 - p_1 z]$$

$$P(z) = \frac{zA(z)(p_0 + p_1) - A^*(z)[p_0 + p_1 z]}{z - A^*(z)}$$

This may be differentiated to give

$$P'(z) = \frac{zA(p_0 + p_1) - A^*(p_0 + p_1 z)}{(z - A^*)^2}(A^{*/} - 1)$$
$$+ \frac{A(p_0 + p_1) + zA'(p_0 + p_1) - A^{*/}(p_0 + p_1 z) - A^* p_1}{(z - A^*)}$$

Evaluating the above at $z=0$, and using the results that $a_0=A(0)=L_B(l)$, $a_0^*=A^*(0)=L_B^*(l)$, $a_1=A^/(0)=L_B^/(l)$ and $a_1^*=A^{*/}(0)=L_B^{*/}(l)$ we get

$$P'(0) = p_1 = -\frac{p_0 a_0^*(a_1^* - 1)}{a_0^{*2}} - \frac{a_0(p_0 + p_1) - p_0 a_1^* - p_1 a_0^*}{a_0^*}$$
$$or \quad p_1 = p_0\left(\frac{1 - a_1^* - a_0 + a_1^*}{a_0^*}\right) - p_1\left(\frac{a_0 - a_0^*}{a_0^*}\right)$$

This gives $p_1 = p_0 \dfrac{1 - a_0}{a_0}$ (B)

Eqs. (A) and (B) can now be solved to get p_0 and p_1 as

$$p_0 = \frac{a_0(1 - r^*)}{a_0 + (r - r^*)} \qquad p_1 = \frac{(1 - a_0)(1 - r^*)}{a_0 + (r - r^*)}$$

These may now be substituted to get the actual expression for *P(z)*.