Stabilized Formulations and Smagorinsky Turbulence Model for Incompressible Flows

J. Ed Akin*, Tayfun Tezduyar, and Mehmet Ungor
Mechanical Engineering, Rice University – MS 321
6100 Main Street, Houston, Texas 77005, USA
e-mail: akin@rice.edu, tezduyar@rice.edu, ungormeh@rice.edu

Sanjay Mittal
Aerospace Engineering, IIT Kanpur
Kanpur 208016, INDIA
e-mail: smittal@iitk.ac.in

Key words: stabilized formulations, stabilization parameters, element length, Smagorinsky turbulence model

Abstract
We investigate the stabilization parameters used in the streamline-upwind/Petrov-Galerkin and pressure-stabilizing/Petrov-Galerkin formulations for flow problems. We present a comparative study of the stabilization parameters defined in different ways. The stabilization parameters are closely related to the local length scales ("element length"). Our comparisons include parameters defined based on the element-level matrices and vectors, some earlier definitions of element lengths, extensions of these to higher-order elements, and calculations for quadrilateral and triangular elements with different shapes. We also compare the numerical viscosities generated by these stabilized formulations with the eddy viscosity associated with a Smagorinsky turbulence model that is based on element length scales.
1 Introduction

In recent decades, there has been a substantial emphasis on using stabilized formulations in flow computations with the finite element method. Streamline-upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows [1], SUPG formulation for compressible flows [2], Galerkin/least-squares (GLS) formulation [3], and pressure-stabilizing/Petrov-Galerkin (PSPG) formulation for incompressible flows [4] are some of the most significant stabilized formulations that found usage in a wide range of applications. These stabilized formulations became attractive primarily because they stabilize the method without introducing excessive numerical dissipation.

The SUPG, GLS and PSPG formulations all include a stabilization parameter that is mostly referred to in the literature as \( \tau \). In general, this parameter might involve a measure of the local length scale (i.e. the "element length") and other factors such as the local Reynolds and Courant numbers. Various element lengths and \( \tau \)s were proposed for the SUPG formulation, starting with those proposed in [1] and [2], and followed by the one introduced in [5]. More element lengths and \( \tau \)s were prescribed for the SUPG, GLS and PSPG methods reported later. Some other \( \tau \)s, dependent upon spatial and temporal discretizations, were introduced and tested in [6]. Later, \( \tau \)s which are applicable to higher-order elements were proposed in [7].

Recently, new ways of computing the \( \tau \)s based on the element-level matrices and vectors were introduced in [8]. These new definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They automatically take into account the local length scales, advection field and the element-level Reynolds number. Based on these definitions, a \( \tau \) can be calculated for each element, or even for each element node or degree of freedom or element equation. It was also shown in [8] that these \( \tau \)s, when calculated for each element, yield values quite comparable to those calculated based on the definition introduced in [5]. In conjunction with these stabilization parameters, in [9], a Discontinuity-Capturing Directional Dissipation stabilization was introduced as a potential alternative or complement to the LSIC (least-squares on incompressibility constraint) stabilization. A second element length scale based on the solution gradient was also introduced in [9]. This new element length scale would be used together with the element length scales already defined (directly or indirectly) in [8]. New stabilization parameters for the diffusive limit were introduced in [10]. These new parameters are closely related to the second element length scale that was introduced in [9]. That second element length scale can be recognized in [10] as a diffusion length scale.

In this paper we carry out a comparative investigation of the stabilization parameters and element length scales defined in [5, 8], as well as the element length scales defined in [1, 11]. These comparisons include extensions of all these stabilization parameters and element length scales to higher-order elements, and calculations for quadrilateral and triangular elements with different shapes. In addition, we compare the numerical viscosities generated by the SUPG stabilization with the eddy viscosity introduced by a Smagorinsky turbulence model [12], specifically one that is based on element length scales [13].

In Section 2, we describe the stabilized formulations we use for an advection-diffusion equation and the Navier-Stokes equations of incompressible flows, and provide the definitions for the \( \tau \)s based on the element-level matrices and vectors. In Section 3, we report the results from our comparative investigation of the element length definitions. We also report results from the comparison of the numerical viscosities generated by the SUPG stabilization with the eddy viscosity introduced by the Smagorinsky turbulence model. We present our concluding remarks in Section 4.
2 Formulations and Stabilization Parameters

2.1 Advection-Diffusion Equation

Consider over a domain \( \Omega \) with boundary \( \Gamma \) the following time-dependent advection-diffusion equation, written on \( \Omega \) and \( \forall t \in (0, T) \) as

\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \cdot (\nabla \phi) = 0,
\]

where \( \phi \) represents the transported quantity, \( u \) is a divergence-free advection field, and \( \nu \) is the diffusivity. The essential and natural boundary conditions associated with Eq. (1) are

\[
\phi = g \quad \text{on } \Gamma_g, \quad n \cdot \nabla \phi = h \quad \text{on } \Gamma_h,
\]

where \( \Gamma_g \) and \( \Gamma_h \) are complementary subsets of the boundary \( \Gamma \), \( n \) is the unit normal vector, and \( g \) and \( h \) are given functions. A function \( \phi_0(x) \) is specified as the initial condition.

Given suitably-defined finite-dimensional trial solution and test function spaces \( S_h^\phi \) and \( V_h^\phi \), the stabilized finite element formulation of Eq. (1) can be written as follows: find \( \phi^h \in S_h^\phi \) such that \( \forall w^h \in V_h^\phi \):

\[
\int_\Omega w^h \left( \frac{\partial \phi^h}{\partial t} + u^h \cdot \nabla \phi^h \right) d\Omega + \int_\Omega \nabla w^h \cdot \nu \nabla \phi^h d\Omega - \int_{\Gamma_h} w^h h d\Gamma \]

\[
+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau_{SUPG} u^h \cdot \nabla w^h \left( \frac{\partial \phi^h}{\partial t} + u^h \cdot \nabla \phi^h - \nabla \cdot (\nu \nabla \phi^h) \right) d\Omega = 0.
\]

Here \( n_{el} \) is the number of elements, \( \Omega^e \) is the domain for element \( e \), and \( \tau_{SUPG} \) is the SUPG stabilization parameter.

With the notation \( b : \int_{\Omega^e} (\ldots) d\Omega : b \), denoting the element-level matrix \( b \) and element-level vector \( b \), corresponding to the element-level integral \( \int_{\Omega^e} (\ldots) d\Omega \), The element-level matrices and vectors are defined as follows:

\[
m = \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega : m_v,
\]

\[
c = \int_{\Omega^e} w^h u^h \cdot \nabla \phi^h d\Omega : c_v,
\]

\[
k = \int_{\Omega^e} \nabla w^h \cdot \nu \nabla \phi^h d\Omega : k_v,
\]

\[
\bar{k} : \int_{\Omega^e} u^h \cdot \nabla w^h u^h \cdot \nabla \phi^h d\Omega : \bar{k}_v,
\]

\[
\bar{c} = \int_{\Omega^e} u^h \cdot \nabla w^h \frac{\partial \phi^h}{\partial t} d\Omega : \bar{c}_v.
\]

From [8], the element-level Reynolds and Courant numbers can be written as

\[
Re = \frac{||u^h||^2 ||c||}{\nu ||k||},
\]

\[
Cr = \frac{\Delta t ||c||}{2 ||m||},
\]
where \( \| b \| \) is the norm of matrix \( b \). Also from [8], we write the components of the element-matrix-based \( \tau_{\text{SUPG}} \):

\[
\tau_{s_1} = \frac{\| e \|}{\| k \|},
\]

\[
\tau_{s_2} = \frac{\Delta t \| e \|}{2 \| k \|},
\]

\[
\tau_{s_3} = \tau_{s_1} Re = \left( \frac{\| e \|}{\| k \|} \right) Re,
\]

and the construction of \( \tau_{\text{SUPG}} \):

\[
\tau_{\text{SUPG}} = \left( \frac{1}{\tau_{s_1}} + \frac{1}{\tau_{s_2}} + \frac{1}{\tau_{s_3}} \right)^{-1}
\]

We note that \( \tau_{s_1}, \tau_{s_2} \) and \( \tau_{s_3} \) are the limiting values for, respectively, the advection-dominated, transient-dominated, and diffusion-dominated cases.

In [8], the element-vector-based \( \tau_{\text{SUPG}} \) is defined as

\[
(\tau_{\text{SUPG}}) \nu = \left( \frac{1}{\tau_{s_1}} + \frac{1}{\tau_{s_2}} + \frac{1}{\tau_{s_3}} \right)^{-1}^\nu
\]

where

\[
\tau_{s_{v1}} = \frac{\| e_v \|}{\| k_v \|},
\]

\[
\tau_{s_{v2}} = \frac{\| e_v \|}{\| e_v \|},
\]

\[
\tau_{s_{v3}} = \tau_{s_{v1}} Re = \left( \frac{\| e_v \|}{\| k_v \|} \right) Re.
\]

### 2.2 Navier-Stokes Equations of Incompressible Flows

The Navier-Stokes equations for incompressible flows can be written as

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u - f \right) - \nabla \cdot \sigma = 0 \quad \text{on } \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{on } \Omega,
\]

where \( \rho, u \) and \( f \) are the density, velocity and the external force, respectively. The stress tensor \( \sigma \) is defined as

\[
\sigma(p,u) = -pI + 2\mu \varepsilon(u).
\]
Here $p$ is the pressure, $I$ is the identity tensor, $\mu$ is the viscosity, $\nu$ is the kinematic viscosity, and $\varepsilon(u)$ is the strain-rate tensor:

$$\varepsilon(u) = \frac{1}{2}((\nabla u) + (\nabla u)^T).$$

The essential and natural boundary conditions associated with Eq. (21) are

$$u = g \text{ on } \Gamma_g, \quad n \cdot \sigma = h \text{ on } \Gamma_h,$$

where $g$ and $h$ are given functions. A divergence-free velocity field $u_0(x)$ is specified as the initial condition.

Given suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure, $S_u^h, V_u^h, S_p^h$ and $V_p^h = S_p^h$, the stabilized finite element formulation of Eqs. (21)-(22) can be written as follows: find $u^h \in S_u^h$ and $p^h \in S_p^h$ such that $\forall w^h \in V_u^h$ and $q^h \in V_p^h$:

$$\int_{\Omega} w^h \cdot \rho \left( \frac{\partial u^h}{\partial t} + u^h \cdot \nabla u^h - f \right) d\Omega + \int_{\Omega} \varepsilon(w^h) : \sigma(p^h, u^h) d\Omega - \int_{\Gamma_h} w^h \cdot h^h d\Gamma$$

$$+ \int_{\Omega} q^h \nabla \cdot u^h d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \frac{1}{\rho} \left[ \tau_{\text{PSPG}} u^h \cdot \nabla w^h + \tau_{\text{PSPG}} \nabla q^h \right] \left[ \rho \left( \frac{\partial u^h}{\partial t} + u^h \cdot \nabla u^h \right) - \nabla \cdot \sigma(p^h, u^h) - \rho f \right] d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau_{\text{LSIC}} \nabla \cdot w^h \rho \nabla \cdot u^h d\Omega = 0.$$

Here $\tau_{\text{PSPG}}$ and $\tau_{\text{LSIC}}$ are the PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters.

We now define the following element-level matrices and vectors:

$$m : \quad \int_{\Omega_e} w^h \cdot \rho \frac{\partial u^h}{\partial t} d\Omega \quad m_v,$$

$$c : \quad \int_{\Omega_e} w^h \cdot \rho (u^h \cdot \nabla u^h) d\Omega \quad c_v,$$

$$k : \quad \int_{\Omega_e} \varepsilon(w^h) : 2\mu \varepsilon(u^h) d\Omega \quad k_v,$$

$$g : \quad \int_{\Omega_e} (\nabla \cdot w^h) p^h d\Omega \quad g_v,$$

$$g^T : \quad \int_{\Omega_e} q^h (\nabla \cdot u^h) d\Omega \quad g_v^T,$$

$$\tilde{k} : \quad \int_{\Omega_e} (u^h \cdot \nabla w^h) \cdot \rho (u^h \cdot \nabla u^h) d\Omega \quad \tilde{k}_v,$$

$$\tilde{c} : \quad \int_{\Omega_e} (u^h \cdot \nabla w^h) \cdot \rho \frac{\partial u^h}{\partial t} d\Omega \quad \tilde{c}_v,$$

$$\tilde{g} : \quad \int_{\Omega_e} (u^h \cdot \nabla w^h) \cdot \nabla p^h d\Omega \quad \tilde{g}_v,$$

$$\beta : \quad \int_{\Omega_e} \nabla q^h \frac{\partial u^h}{\partial t} d\Omega \quad \beta_v.$$
The element-level Reynolds and Courant numbers are defined in the same way as they were defined before, given by Eqs. (9)-(12). The components of the element-matrix-based $\tau_{\text{SUPG}}$ are defined in the same way as they were defined before, given by Eqs. (13)-(15). $\tau_{\text{SUPG}}$ is constructed from its components in the same way as it was constructed before, given by Eq. (16). The components of the element-vector-based $\tau_{\text{SUPG}}$ are defined in the same way as they were defined before, given by Eqs. (18)-(20). The construction of $(\tau_{\text{SUPG}})_v$ is also the same as it was before, given by Eq. (17).

From [8], we write the element-matrix-based $\tau_{\text{SUPG}}$ as

$$\tau_{\text{SUPG}} = \left( \frac{1}{\tau_{\text{P1}}} + \frac{1}{\tau_{\text{P2}}} + \frac{1}{\tau_{\text{P3}}} \right)^{-\frac{1}{r}}$$

where

$$\tau_{\text{P1}} = \frac{\|g^T\|}{\|\gamma\|},$$

$$\tau_{\text{P2}} = \frac{\Delta t \|g^T\|}{2 \|\beta\|},$$

$$\tau_{\text{P3}} = \tau_{\text{P1}} Re = \left( \frac{\|g^T\|}{\|\gamma\|} \right) Re.$$

Also from [8], the element-vector-based $\tau_{\text{SUPG}}$ is written as

$$(\tau_{\text{SUPG}})_v = \left( \frac{1}{\tau_{\text{PV1}}} + \frac{1}{\tau_{\text{PV2}}} + \frac{1}{\tau_{\text{PV3}}} \right)^{-\frac{1}{r}}$$

where

$$\tau_{\text{PV1}} = \tau_{\text{P1}},$$

$$\tau_{\text{PV2}} = \tau_{\text{PV1}} \frac{\|\gamma_v\|}{\|\beta_v\|},$$

$$\tau_{\text{PV3}} = \tau_{\text{PV1}} Re.$$

Lastly from [8], the element-matrix-based $\tau_{\text{LSIC}}$ and the element-vector-based $\tau_{\text{LSIC}}$ are given as

$$\tau_{\text{LSIC}} = \frac{\|e\|}{\|e\|},$$

$$(\tau_{\text{LSIC}})_v = \tau_{\text{LSIC}}.$$

For the purpose of comparison, we also define here stabilization parameters that are based on an earlier definition of the length scale $h$ first introduced in [5]:

$$h_{\text{UGN}} = 2 \|u^h\| \left( \sum_{\alpha=1}^{n_{\text{en}}} |u^h \cdot \nabla N_\alpha| \right)^{-1}$$
where $N_a$ is the interpolation function associated with node $a$. The stabilization parameters are defined as

$$\tau_{\text{SUPG}} = \frac{h_{\text{UGN}}}{2\|u^h\|}$$  \hspace{1cm} (50)

$$\tau_{\text{PSPG}} = \frac{\Delta t}{2}$$  \hspace{1cm} (51)

$$h_{\text{UGN}} = \frac{h_{\text{UGN}}^2}{4\nu}$$  \hspace{1cm} (52)

$$\tau_{\text{LSIC}}_{\text{UGN}} = \frac{\tau_{\text{SUPG}}_{\text{UGN}}}{T_{\text{SUPG}}_{\text{UGN}}}$$  \hspace{1cm} (53)

$$(\tau_{\text{PSPG}})_{\text{UGN}} = \frac{\tau_{\text{SUPG}}_{\text{UGN}}}{T_{\text{SUPG}}_{\text{UGN}}}$$  \hspace{1cm} (54)

$$(\tau_{\text{LSIC}})_{\text{UGN}} = \frac{h_{\text{UGN}}}{2\|u^h\|} z.$$  \hspace{1cm} (55)

Here $z$ is given as follows:

$$z = \begin{cases} \left(\frac{Re_{\text{UGN}}}{3}\right) & \text{if } Re_{\text{UGN}} \leq 3, \\ 1 & \text{if } Re_{\text{UGN}} > 3, \end{cases}$$  \hspace{1cm} (56)

where $Re_{\text{UGN}} = \frac{\|u^h\| h_{\text{UGN}}}{2\nu}$.

**Remark 1** A Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was introduced in [9] as a potential alternative or complement to the LSIC stabilization. As part of the DCDD stabilization, a second element length scale that is based on the solution gradient was also introduced in [9].

**Remark 2** New definitions for the diffusion-dominated limits of the SUPG and PSPG stabilization parameters were introduced in [10]. These new definitions are closely related to the second element length scale that was first introduced in [9] and later employed in [10] as a diffusion length scale.

**Remark 3** For the advection-dominated limits of the SUPG and PSPG stabilization parameters, equivalent length scales can be defined by simply multiplying the stabilization parameter with $2\|u^h\|$.

For the comparative investigation we would like to carry out, we also provide here element length scales defined in other studies, based on the element shapes and advection field. For notational convenience, we first define the following unit vector:

$$\mathbf{s} = \frac{u^h}{\|u^h\|}.$$  \hspace{1cm} (57)

The element length given in [1] for a quadrilateral element can be written as

$$h_{\text{SDQ}} = \left| \frac{x_2 + x_3 - x_4 + x_1}{2} \right| \cdot \mathbf{s} + \left| \frac{x_3 + x_4 - x_1 + x_3}{2} \right| \cdot \mathbf{s}$$  \hspace{1cm} (58)

where $x_a$ is the nodal coordinate vector associated with node $a$. For triangular elements, we use the following expression from [11]:

$$h_{\text{SAI}} = \frac{1}{4} \left[ \|x_2 - x_1\| \cdot \mathbf{s} | + \|x_3 - x_2\| \cdot \mathbf{s} | + \|x_1 - x_3\| \cdot \mathbf{s} | \right]$$  \hspace{1cm} (59)
To write some of the other element lengths given in [11], we first define a special sign function:

\[ SSgn(y) = \begin{cases} -1 & y \leq 0 \\ +1 & y > 0 \end{cases} \]

and the stream-wise components of the nodal position vectors

\[ \delta_a = (x_a - x_o) \cdot s, \quad (61) \]

where

\[ x_o = \left( \sum_{a=1}^{\text{nen}} x_a \right) / \text{nen} \]

The number of upstream and downstream element nodes can be expressed as

\[ n_{uen} = \sum_{a=1}^{\text{nen}} \frac{1}{2} (1 - SSgn(\delta_a)) \quad n_{den} = \sum_{a=1}^{\text{nen}} \frac{1}{2} (1 + SSgn(\delta_a)) \quad (63) \]

Then one of the element lengths given in [11] can be written as

\[ h_{SA3} = \left( \sum_{a=1}^{\text{nen}} \frac{1}{2} (1 + SSgn(\delta_a)) \delta_a \right) / n_{den} \quad \left( \sum_{a=1}^{\text{nen}} \frac{1}{2} (1 - SSgn(\delta_a)) \delta_a \right) / n_{uen} \]

Another one of the element lengths given in [11] can be written as

\[ h_{SA3} = \max(\delta_1, \delta_2, \delta_{\text{nen}}) - \min(\delta_1, \delta_2, \delta_{\text{nen}}) \]

A third element length given in [11] is the node-based version of the one given by Eq. (64):

\[ (h_{SA4})_a = \delta_a - \left( \sum_{a=1}^{\text{nen}} \frac{1}{2} (1 + SSgn(\delta_a)) \delta_a \right) / n_{uen} \]

2.3 SUPG Stabilization and Smagorinsky Turbulence Viscosities

To compare the numerical viscosities generated by the SUPG stabilization with the eddy viscosity introduced by a Smagorinsky turbulence model, we first write an equivalent "viscosity" based on the SUPG stabilization parameter:

\[ \nu_{\text{SUPG}} = \tau_{\text{SUPG}} ||u^h||^2. \]

The eddy viscosity introduced by a Smagorinsky turbulence model that is based on the element length scales [13] is written as

\[ \nu_{\text{SMAG}} = (0.1 h_{\text{SMAG}})^2 (\varepsilon(u^h) : \varepsilon(u^h))^{1/2}, \]

where \( h_{\text{SMAG}} \) is the square-root (or cube-root) of the area (or volume) of the element.
3 Comparisons

3.1 Element Length Comparisons

We first inspect in 1D the functions \( (N_a) \) and \( (N_a + \tau_{\text{SUPG}} u^h \cdot \nabla N_a) \), which we will call, respectively, "Galerkin function" and "SUPG function". Figures 1-3 show, for linear, quadratic and cubic elements, these functions after they are assembled for a global node. While the Galerkin functions are continuous across element boundaries, the SUPG additions to them are not. For a linear element the SUPG addition is constant over an element, but for quadratic and cubic elements it is not. The same thing can be said for the element length \( h_{\text{UGN}} \) (see Figure 4). When averaged over an element, and normalized by the element length for a linear element, the normalized average values of \( h_{\text{UGN}} \) for quadratic and cubic elements are approximately 0.52 and 0.30. The normalized average values of the equivalent length scale computed from \( \tau_{\text{36}} \) (with the 1-norm of the element level matrices) for quadratic and cubic elements are approximately 1/4 and 1/6.

![Figure 1: For linear elements, Galerkin (broken line) and SUPG (solid line) functions, assembled for a global node A.](image1)

![Figure 2: For quadratic elements, Galerkin (broken line) and SUPG (solid line) functions, assembled for a global node A. For nodes at element boundaries (left) and interiors (right).](image2)

Stabilization parameters and element lengths based on different definitions, including those based on element-level matrices and vectors and \( (\tau_{\text{SUPG}})_{\text{UGN}} \) and \( (\tau_{\text{PSG}})_{\text{UGN}} \), were calculated and tested in [8]. The matrix norm used in [8] was the 1-norm. The tests were carried out for several shapes of bilinear quadrilateral and linear triangular elements, with \( \|u\| = 1.0 \) and \( \Delta t = 1.0 \), and as function of the advection direction. The test flow computations reported in [8] show that the definitions based on the element-level matrices and vectors perform well.

Here, element lengths are calculated and compared for linear, quadratic and cubic elements in 2D, based on four of the definitions given in this paper: the equivalent length scale computed from \( \tau_{\text{36}} \) (with the
J. Ed Akin, Tayfun Tezduyar, Mehmet Ungor, Sanjay Mittal

Figure 3: For cubic elements, Galerkin (broken line) and SUPG (solid line) functions, assembled for a global node A. For nodes at element boundaries (top left), upstream interiors (top right), and downstream interiors (bottom).

Figure 4: Variation of $h_{UGN}$ within linear and higher-order 1D elements.

The Frobenius norm of the element level matrices, $h_{UGN}$, $h_{SA1}$, and $h_{SA2}$. Definitions that depend on the location within an element are evaluated at the origin of the natural coordinate system for quadrilateral elements and at the centroid for triangular elements. Lagrangian and serendipity quadrilateral elements as well as triangular elements have been evaluated. The element lengths calculated based on the definitions listed above are shown in Figures 5-14. The element shape is indicated by a dashed line and the nodes are indicated by a circled cross. Each closed curve represents a different element length definition. For each advection direction, the element length is that of a line through the element center, parallel to the advection, bounded by its intersections with the closed curve. In other words, let us imagine a line passing through the center and find its two intersection points with the closed curve. Then the distance between those two points is the element length in that advection direction. Although the results displayed here for $\tau_{S1}$ are based on the Frobenius norm of the element level matrices, we see little difference between the $\tau_{S1}$'s calculated with different matrix norms. From Figures 5-14, we note that the difference between different element length definitions is more pronounced for higher-order elements. In general, the element length decreases with the increase in the order of the element. This observation is consistent with what we see for $h_{UGN}$ in 1D (see Figure 4).
3.2 Comparison of SUPG Stabilization and Smagorinsky Turbulence Viscosities

Flow past a cylinder at $Re = 3,000$ and $Re = 50,000$ are used as test problems to compare the numerical viscosities generated by the SUPG stabilization with the numerical viscosity introduced by a Smagorinsky turbulence model. The stabilization parameters are computed as given by Eqs. (49)-(56), but with the $\tau_{SUPG}$ component dropped. When calculating $Re_{UGN}$ used in Eq. (56), the kinematic viscosity $\nu$ is augmented with $\nu_{SMAG}$. Velocity and pressure are both interpolated with bilinear functions. A mesh with 14,960 nodes and 14,700 quadrilateral elements is employed. Close to the cylinder surface, the radial distance between the mesh points (normalized by the cylinder diameter) is $2.5 \times 10^{-4}$ at $Re = 3,000$ and $5 \times 10^{-5}$ at $Re = 50,000$. A close-up view of the mesh for the latter case is shown in Figure 15. In each case, the computations are carried out until a developed unsteady solution is obtained. Then, based on the velocity field at a given instant, $\nu_{SMAG}/\nu_{SUPG}$ is calculated.

Figure 16 shows the vorticity and $\nu_{SMAG}/\nu_{SUPG}$ for $Re = 3,000$. Shades of gray represent values of $\nu_{SMAG}/\nu_{SUPG}$ ranging from 0.00 (white) to 0.05, with black indicating 0.05 and higher values. Except for the regions in black, $\nu_{SMAG}/\nu_{SUPG}$ is less than 5%. Because Figure 16 shows pictures zoomed into a small part of the full domain, one can also infer that most of the full domain is marked in white, and therefore for those regions the ratio is essentially 0%. As additional information, we would like to note that when we inspect the overall data for $\nu_{SMAG}/\nu_{SUPG}$, we see that in most of the domain it is less than 1%. The turbulence model is active only in regions with significant vorticity. Except for a very few points in the near wake, $\nu_{SUPG}$ dominates $\nu_{SMAG}$. When a wall damping function is used with the turbulence model, $\nu_{SMAG}$ becomes even smaller. Similar observations can be made for $Re = 50,000$ (see Figure 17). It is important to remember that while $\nu_{SMAG}$ is an isotropic viscosity, $\nu_{SUPG}$ is the maximum value of a directional viscosity, with the maximum value attained in the advection direction. However,
in most of the domain $\nu_{\text{SMAG}}/\nu_{\text{SUPG}}$ is so small that, except for directions nearly perpendicular to the advection direction, $\nu_{\text{SMAG}}$ will still be substantially less than the direction-adjusted value of $\nu_{\text{SUPG}}$. It is also important to remember that $\nu_{\text{SUPG}}$ is generated by a residual-based formulation, while $\nu_{\text{SMAG}}$ is not.

4 Concluding Remarks

We investigated in this paper the element lengths (i.e. the local length scales) defined in different ways. These element lengths are closely related to the stabilization parameters used in the SUPG and PSPG formulations in finite element flow simulations. Our comparisons included parameters defined based on the element-level matrices and vectors, some earlier definitions of element lengths, extensions of these to higher-order elements, and calculations for quadrilateral and triangular elements with different shapes. We see from studies that the difference between different element length definitions is more pronounced for higher-order elements, and the element length decreases with the increase in the order of the element. We believe that the parameter definitions based on the element-level matrices and vectors provide a good, general framework that automatically takes into account the local length scales and the advection field. We also compared, based on some test flow computations, the numerical viscosities generated by the SUPG stabilization with the eddy viscosity associated with a Smagorinsky turbulence model. These test computations show that, in most of the flow domain, the SUPG viscosity, in terms of it maximum magnitude, which is attained in the flow direction, is much larger than the Smagorinsky viscosity. It is clear that better understanding is needed for the performance of the stabilized formulations with higher-order elements and also for the interaction between the stabilized formulations and the Smagorinsky turbulence model.
Figure 7: Element length, calculated with different definitions and as function of advection direction, for a rectangular Lagrangian element.

Acknowledgements

This work was supported by the Army Natick Soldier Center and NASA JSC.

References


Figure 8: Element length, calculated with different definitions and as function of advection direction, for a rectangular serendipity element.


Figure 9: Element length, calculated with different definitions and as function of advection direction, for a trapezoidal Lagrangian element.

Figure 10: Element length, calculated with different definitions and as function of advection direction, for a trapezoidal serendipity element.
Figure 11: Element length, calculated with different definitions and as function of advection direction, for a parallelogram Lagrangian element.

Figure 12: Element length, calculated with different definitions and as function of advection direction, for a parallelogram serendipity element.
Figure 13: Element length, calculated with different definitions and as function of advection direction, for an equilateral triangular element.

Figure 14: Element length, calculated with different definitions and as function of advection direction, for a right-angle triangular element.
Figure 15: Flow past a cylinder. A close-up view of the finite element mesh with 14,960 nodes and 14,700 elements.

Figure 16: Flow past a cylinder at Re=3,000. Vorticity (top) and $\nu_{SMAG}/\nu_{SUPG}$ with (middle) and without (bottom) the wall function in computation of $\nu_{SMAG}$. In displaying $\nu_{SMAG}/\nu_{SUPG}$, shades of gray represent the values ranging from 0.00 (white) to 0.05, with 0.05 and higher values indicated by black.
Figure 17: Flow past a cylinder at $Re=50,000$. Vorticity (top) and $\nu_{SMAG}/\nu_{SUPG}$ with (middle) and without (bottom) the wall function in computation of $\nu_{SMAG}$. In displaying $\nu_{SMAG}/\nu_{SUPG}$, shades of gray represent the values ranging from 0.00 (white) to 0.05, with 0.05 and higher values indicated by black.
Recent MEMS Preprints

With this preprint series the Department of Mechanical Engineering and Materials Science provides the opportunity for interested parties to keep abreast of the latest research carried out in the Department. The following is a list of the most recent preprints. To see the complete list, please refer to www.mems.rice.edu/preprints/index.html. To receive a copy of one of our preprints, please send a message to mems@rice.edu.

2002-026  K. Stein, T. Tezduyar, and R. Benney
Applications in Airdrop Systems: Fluid-Structure Interaction Modeling

2002-025  S. S. Collis, K. Ghayour, and M. Heinkenschloss
Optimal Control of Aeroacoustic Noise Generated by Cylinder Vortex Interaction

2002-024  S. Ramakrishnan, S. S. Collis
Variational Multiscale Modeling for Turbulence Control

2002-023  K. Ghayour, S. S. Collis, and M. Heinkenschloss
Optimal Control of Aeroacoustic Flows: Transpiration Boundary Control

2002-022  T. Tezduyar
Interface-Tracking and Interface-Capturing Techniques for Computation of Moving Boundaries and Interfaces

2002-021  T. Tezduyar
Stabilization Parameters and Local Length Scales in SUPG and PSPG Formulations

2002-020  K. Stein and T. Tezduyar
Advanced Mesh Update Techniques for Problems Involving Large Displacements

2002-019  J. Akin, T. Tezduyar, M. Ungor, and S. Mittal
Stabilization Parameters and Smagorinsky Turbulence Model

2002-018  K. Stein, T. Tezduyar, and R. Benney
Mesh Moving Techniques for Fluid-Structure Interactions with Large Displacements

2002-017  K. Stein, T. Tezduyar, V. Kumar, S. Sathe, R. Benney, E. Thornburg, C. Kyle, and T. Nonoshita
Aerodynamic Interactions Between Parachute Canopies

2002-016  P. D. Spanos, A. M. Chevallier, and N. P. Politis
Nonlinear Stochastic Drill-String Vibrations

2002-015  A. Ktena, D. R. Fotiadis, P. D. Spanos, A. Berger, and C. V. Massalas
Identification of 1D and 2D Preisach Models for Ferromagnets and Shape Memory Alloys

2002-014  P. D. Spanos, A. M. Chevallier, N. P. Politis, and M. L. Payne
Oil Well Drilling: A Vibrations Perspective

2002-013  L. Catabriga, A. Coutinho, and T. Tezduyar
Finite Element SUPG Parameters Computed from Local Matrices for Compressible Flows

2002-012  K. Stein, T. Tezduyar, and R. Benney
Mesh Update with Solid-Extension Mesh Moving Technique

2002-011  J. E. Akin, T. Tezduyar, M. Ungor, and S. Mittal
Stabilized Formulation Element Length Calculations for Higher-Order Elements

2002-010  T. Tezduyar
Computation of Moving Boundaries and Interfaces with Interface-Tracking and Interface-Capturing Techniques