Time: 1 hour | Course: BSO 203 | Marks: 40

SESSION 2002-2003, SEMESTER I

MID-SEMESTER EXAMINATION I

1. An improved version of Wien's distribution formula has the form

$$e_b(\lambda, T) = \frac{A}{\lambda^5}\left[e^{-\frac{B}{\lambda T}} + e^{-\frac{2B}{\lambda T}}\right] .$$

(a) Show that this distribution obeys Wien's displacement law and calculate Wien's constant b in terms of B. (You need to know that the solution of the transcendental equation $(5-x)e^x + (5-2x) = 0$ is $x \simeq 4.9656$).

(b) Show that the total emissivity obeys the Stefan-Boltzmann law and calculate Stefan's constant in terms of A and B.

(c) Use the experimental values of b and σ to calculate the constants A and B.

4 + 4 + 2

Answer: This is very similar to Problem 3 of Problem Set 1, except that there are now *two* exponential terms. In fact, such a form folllows if the denominator of the Planck distribution is expanded keeping the first two terms.

(a) To find the peak, set

$$\frac{\partial e_b}{\partial \lambda} = 0$$

which simplifies to the transcendental equation

$$(5-x)e^x + (5-2x) = 0 \qquad x = \frac{B}{\lambda_m T} .$$

The solution of this is given as $x = 4.9656$, whence it follows that

$$\lambda_m T = \frac{B}{4.9656} \equiv b$$

(b) The total emissivity is obtained by integrating over all wavelengths:

$$\begin{aligned} e_b &= \int_0^\lambda d\lambda\; e_B(\lambda, T) \\ &= A\left[J_1(B) + J_2(B)\right] \end{aligned}$$

where

$$J_1(B) = \int_0^\lambda d\lambda\; \lambda^{-5}\; e^{-B/\lambda T} \qquad J_2(B) = J_1(2B) .$$

The integral J_1 is easily performed by substituting $x = B/\lambda T$, and the result is

$$J_1(B) = \frac{T^4}{B^4}\,\Gamma(4) = \frac{6T^4}{B^4} ,$$

from which it follows that

$$J_2(B) = \frac{3T^4}{8B^4} \ ,$$

and hence,

$$e_b = \frac{6A}{B^4}\left(1 + \frac{1}{16}\right)\ T^4 \equiv \sigma T^4 \ ,$$

which is the Stafan-Boltzmann law with

$$\sigma = \frac{51A}{8B^4} \ .$$

(c) Solving the equations for b and σ we get

$$B = 4.9656b \ , \qquad A = \frac{8}{51}(4.9656b)^4\sigma \ .$$

Putting in numerical values

$$b = 2.898 \times 10^{-3} \text{ mK} \ , \qquad \sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4} \ ,$$

we get

$$A = 3.814 \times 10^{-16} \text{ Wm}^3 \ , \qquad B = 1.439 \times 10{-}2 \text{ mK} \ .$$

(Interested students may like to think about what happens when more terms in the expansion of the Planckian denominator are considered ...)

2. (a) A beam of photons of wavelength λ is incident head-on with a beam of electrons of kinetic energy 30 keV. Find the maximum wavelength change of the scattered radiation.

(b) The $\lambda = 4686$ Å line of He^+ arises from the transition $n = 4 \to n = 3$. Assuming that these lines are split according to Sommerfeld's formula, determine how many lines are expected in the fine structure. Apply the selection rule $\Delta\ell = \pm 1$ and determine how many lines survive.

6 + 4

Answer:

(a) This is a problem of Compton scattering where the electron is not at rest. The qualitative feature of this was discussed in Quiz 2, Q. 2. In this question the problem must be tackled quantitatively. The simple way to do this is to make a Lorentz transformation to the rest frame (denoted by star $*$) of the electron. In this frame, one gets the usual formula derived in the lectures:

$$\lambda_S^* - \lambda_0^* = \lambda_C(1 - \cos\theta_S^*)$$

where the Compton wavelength is $\lambda_C = h/m_e c$. Clearly the maximum of this is when $\theta^* = \pi$, *i.e.* the photon is reflected back in the direction in which it came. In this case, the maximum wavelength change is

$$\lambda^*_{SM} - \lambda^*_0 = 2\lambda_C \ .$$

This will hold true in the laboratory frame as well.

To transform to the laboratory frame, we must use the formulae for the relativistic Doppler effect (since the particles in question are photons, moving at c). First we need to find the velocity of the electron. Its momentum is

$$cp_e = \sqrt{T_e(T_e + 2m_e c^2)} = 177.65 \text{ keV}$$

using $T_e = 30$ keV and $m_e c^2 = 511$ keV. Now the velocity is determined from the momentum using

$$p_e = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \Rightarrow \frac{v}{c} = 0.328$$

(This comparatively large value shows that it is incorrect to calculate the electron velocity from the non-relativistic formula $T_e = \frac{1}{2}mv^2$.)

In the laboratory frame, the incident photon is approaching the electron. Hence its Doppler correction will be a blue-shift:

$$\nu_0 = \nu^*_0 \sqrt{\frac{c+v}{c-v}} \Rightarrow \lambda_0 = \lambda^*_0 \sqrt{\frac{c-v}{c+v}} = 0.714\lambda_0 \ .$$

The scattered photon will be receding from the electron, *i.e.* it will be red-shifted:

$$\nu_S = \nu^*_S \sqrt{\frac{c-v}{c+v}} \Rightarrow \lambda_S = \lambda^*_S \sqrt{\frac{c+v}{c-v}} = 1.40\lambda_S \ .$$

Putting in these corrections, the maximum shift in wavelength comes out to be

$$\lambda^*_{SM} - \lambda^*_0 = 1.428\lambda_C + 0.686\lambda_0 \ .$$

(b) Sommerfeld's formula for the energy levels of hydrogen is

$$E_{n,\ell} = -R_H \left[\frac{Z^2}{n^2} + \frac{Z^4\alpha^2}{n^3} \left(\frac{1}{\ell+1} - \frac{3}{4n} \right) \right]$$

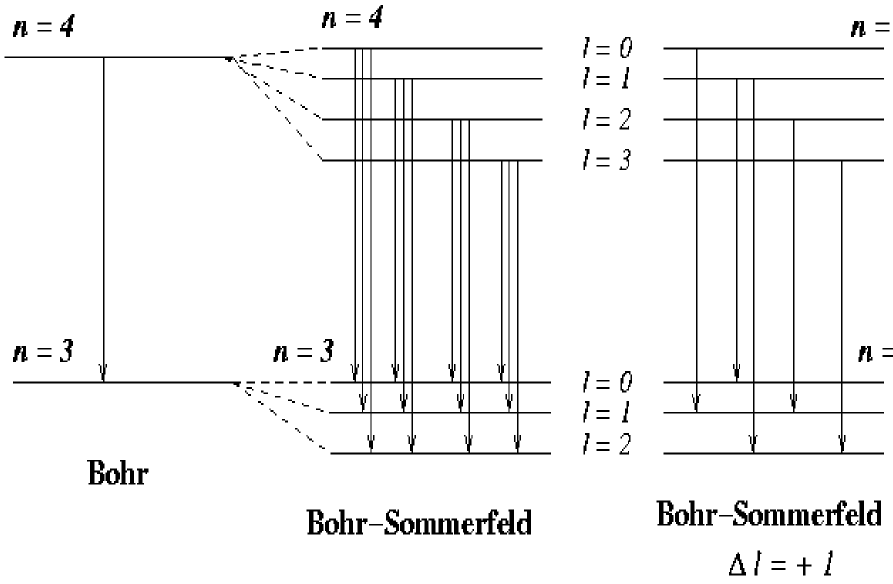
where $\alpha = e^2/\hbar c \simeq 1/137.035$ and $\ell = 0, 1, \ldots, (n-1)$.

For He^+, $Z = 2$, but this is unimportant for a qualitative analysis. For $n = 3$, $\ell = 0, 1, 2$, *i.e.* the Bohr energy level will be split into 3 sub-levels. For $n = 4$, $\ell = 0, 1, 2, 3$, *i.e.* the Bohr energy level will be split into 4 sub-levels. If all transitions between these levels are allowed, there will be $3 \times 4 = 12$ possible transitions and hence 12 lines in the fine structure.

If we apply the selection rule $\Delta\ell = \pm 1$, the only allowed transitions will be:

$$n = 4, \ell = 0 \longrightarrow n = 3, \ell = 1$$
$$n = 4, \ell = 1 \longrightarrow n = 3, \ell = 0$$
$$n = 4, \ell = 1 \longrightarrow n = 3, \ell = 1$$
$$n = 4, \ell = 2 \longrightarrow n = 3, \ell = 1$$
$$n = 4, \ell = 3 \longrightarrow n = 3, \ell = 2$$

which are 5 in number. The figure makes this quite clear.



3. Re-work Bohr's theory of the hydrogen atom assuming that the force between the electron and the nucleus is purely gravitational.
   (a) Calculate in this case the 'Bohr radius', the ionisation potential of hydrogen, and the 'Rydberg constant' for hydrogen.
   (b) What should the mass of the proton be if these values are to equal their observed values? What is the physical significance of this result?

6 + 4

Answer: In this problem, the Bohr theory is required to be worked out replacing the electrostatic force by the gravitation force *i.e.*

$$F_e = \frac{e^2}{4\pi\epsilon_0 r^2} \longrightarrow F_G = \frac{G_N m_e m_p}{r^2} \ .$$

The Bohr quantisation condition remains the same:

$$mvr = n\hbar$$

for circular orbits.

As usual, we write the force balance equation

$$\frac{mv^2}{r} = \frac{G_N m_e m_p}{r^2}$$

and combine it with the Bohr quantisation condition to get

$$r_n = \frac{\hbar^2}{G_N m_e^2 m_p}\, n^2 \ .$$

(a) To get the 'Bohr radius', we simply set $n = 1$ and get

$$a_0 = \frac{\hbar^2}{G_N m_e^2 m_p} = 1.2 \times 10^{29} \text{ m}.$$

This is an enormous value, comparable to the size of the observable Universe, indicating the fact that the gravitational force between the electron and the proton is a very weak binding indeed.

(b) The energy of an electron in a Bohr orbit

$$E = \frac{1}{2}mv^2 - \frac{G_N m_e m_p}{r} = -\frac{G_N m_e m_p}{2r}$$

which leads to energy levels

$$E_n = -\frac{G_N^2 m_e^3 m_p^2}{2\hbar^2}\frac{1}{n^2} \ .$$

To get the ionisation potential, set $n = 1$ to get

$$|E_1| = \frac{G_N^2 m_e^3 m_p^2}{2\hbar^2} = 2.63 \times 10^{-78} \text{ eV} \ .$$

This is a very small value, indicating that the electron is very loosely bound by gravitational force.

The 'Rydberg constant' is

$$R_H = \frac{c}{h}\, E_1 = 1.9 \times 10^{-57}\ cm^{-1}$$

indicating that any radiation emitted by such a hydrogen atom will have wavelengths longer than the size of the observable Universe.

(c) The only change involved in taking the gravitational force versus the electric force is in replacing

$$\frac{e^2}{4\pi\epsilon_0} \longrightarrow G_N m_e m_p$$

If these are to be equated by increasing the proton mass, then we must have

$$m_p = frace^2 4\pi\epsilon_0 G_N m_e = 3.79 \times 10^{12}\ \mathrm{Kg}\ .$$

This makes it very clear that the gravitational force is indeed very weak, and negligible in atomic interactions. If the proton were indeed to have a comparable mass, confined within a nuclear size of order (1 fm)^3, then it would certainly form a tiny black hole.

4. (a) A Gaussian wave-profile $\psi(x,0) = (2\pi\sigma)^{-1/2} e^{-x^2/2\sigma}$ is allowed to evolve with time according to the de Broglie dispersion relation $w(k) = \hbar k^2/2m$. Find its oscillation frequency.

(b) Derive an equation of continuity from the one-dimensional Schrödinger equation when the potential has the form $V(x) = V_0 \exp(-i\Gamma/\hbar)$.

6 + 4

Answer:

(a) This problem was essentially worked out in the lecture on matter waves. However, only the modulus of the wave-group was worked out and not the oscillation frequency. In this question, students are required to find the oscillation frequency. A Gaussian wave-profile $\psi(x,0) = (2\pi\sigma)^{-1/2} e^{-x^2/2\sigma}$ has a Fourier transform

$$\phi(k) = \frac{1}{\sqrt{8\pi}}\ e^{-\frac{1}{2}\sigma k^2}\ .$$

Using this Fourier transform together with the time-dependent inverse Fourier transform, we get a time-dependent wave-group

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk\ \phi(k) e^{i(kx-\omega t)}$$

where $\omega = \hbar k^2/2m$. This is a Gaussian integration, leading to a solution

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\sigma'}} e^{-x^2/2\sigma'}\ , \qquad \sigma' = \sigma + \frac{i\hbar t}{m}\ .$$

Being a complex number this can be written in modulus-amplitude form, with the amplitude giving rise to oscillations. The exponential term can be written

$$e^{-x^2/2\sigma'} = e^{-\frac{x^2}{2}\frac{\sigma'*}{|\sigma'|^2}} = e^{-\frac{x^2}{2}\frac{\mathrm{Re}(\sigma')}{|\sigma'|^2}} e^{-\frac{x^2}{2} - i\frac{\mathrm{Im}(\sigma')}{|\sigma'|^2}}$$

and the pre-factor can be written as

$$\frac{1}{\sqrt{2\pi\sigma'}} \propto e^{-i\theta/2} \,, \qquad \theta = \tan^{-1}\frac{\hbar t}{m\sigma} \,.$$

Thus the oscillatory part has argument

$$\frac{1}{2}\tan^{-1}\frac{\hbar t}{m\sigma} + \frac{\mathrm{Im}(\sigma')}{|\sigma'|^2} = \frac{1}{2}\tan^{-1}\frac{\hbar t}{m\sigma} + \frac{\hbar t/m}{\sigma^2 + \hbar^2 t^2/m^2}$$

and for one time period $t = T$, this argument must be 2π. Thus, writing $\nu = 1/T$, we get ν on solving the transcendental equation

$$\frac{1}{2}\tan^{-1}\frac{\hbar}{m\sigma\nu} + \frac{\hbar/m\nu}{|\sigma^2 + \hbar^2/m^2\nu^2} = 2\pi \,.$$

(b) This problem was worked out in the lecture with $\Gamma = 0$, i.e. a real potential rather than a complex one. In the presence of a complex potential $V(x) = V_0 e^{-i\Gamma/\hbar}$, where Γ is a real constant, the Schrödinger equation assumes the form

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V_0\, e^{-i\Gamma/\hbar} = i\hbar\frac{\partial\psi}{\partial t} \,.$$

Taking this equation and its complex comjugate, and multiplying the first by ψ^* and the second by ψ, and subtracting, we get the equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{-2V_0}{\hbar}\sin\frac{\Gamma}{\hbar}\rho$$

where $\rho = |\psi|^2$ and $J = \frac{\hbar}{m}\mathrm{Im}\psi^*\frac{\partial\psi}{\partial x}$. The algebraic manipulations are exactly as done in the lecture.

This is the usual equation of continuity if $\Gamma = 0$, indicating that the probability is conserved. For $\Gamma \neq 0$, we have a source/sink term, depending on the sign, indicating that particles are being created or are decaying. If $\Gamma = \gamma t$, we recover the radiactive decay law, with a mean life $\tau = \hbar/\gamma$.