Course: BSO 203 Instructor: S. Raychaudhuri

Session 2002-2003, Semester I

PROBLEM SET IV

Wave-Particle Duality and the Schrödinger equation

1. (a) Using Planck's law, the de Broglie relation and the relativistic transformations of energy and momentum, determine the formulae for the relativistic Doppler effect.

   (b) Show that the same formula can be obtained by considering the emission of a photon from an atom with conservation of energy and momentum.

   (c) Obtain Bragg's formula by treating the diffraction of X-rays as a scattering phenomenon for photons.

   [*Historical note:* These three demonstrations of wave-particle duality were discovered by Emden (1921), Schrödinger (1922) and Duane (1923) respectively.]

2. (a) Write down the wave-profile of a particle whose position is known exactly at time $t = 0$. Work out its Fourier transform and comment on the momentum distribution.

   (b) Discuss the time evolution of such a wave-group given the following dispersion relations

   - the ultra-relativistic one: $(\hbar\omega) = c(\hbar k)$;
   - the relativistic one: $(\hbar\omega)^2 = c^2(\hbar k)^2 + (mc^2)^2$;
   - the non-relativistic one: $(\hbar\omega) = (\hbar k)^2/2m$.

3. (a) Establish that the space derivative $\psi'(x)$ of the Schrödinger wave-function

   - is continuous at $x = a$ if $V(x)$ is finite at $x = a$;
   - has a finite discontinuity at $x = a$ if $V(x) = \Omega\delta(x - a)$.

   (b) Using the time-independent Schrödinger equation, find the potential energy corresponding to the wavefunction

   $$\psi(x) = \psi(0)\ e^{-a|x|}\ .$$

4. (a) Find the Schrödinger probability current corresponding to the wave function

   $$\psi(x) = \frac{1}{\sqrt{\pi\sigma}} e^{-x^2/\sigma} e^{i(kx-wt)}\ .$$

   Interpret this result physically.

   (b) Show that the Schrödinger equation is invariant under a Galilean transformation. (This is the actual proof that the Schrödinger equation is non-relativistic.)

5. (a) Use the operator equivalences for energy and momentum in the relativistic dispersion relation for a de Broglie wave to derive an equation of motion.

   (b) Prove that this equation is invariant under a Lorentz transformation and find the transformation law for the wave-function.

   (c) Derive an equation of continuity from this relativistic wave equation.

   (d) Is this equation of continuity physically interpretable?

   (*Historical note:* This equation, in fact, is the original one written down by Schrödinger in 1926. However, precisely because of the difficulties associated with part (d) of this problem, he abandoned it in favour of the nonrelativistic equation which now bears his name. The equation was revived two years later by Oskar Klein and Charles Gordon and is now known as the Klein-Gordon equation. The true physical interpretation in terms of a quantum field theory was given by Pauli and Weisskopf in 1933. Today, we know that it describes spinless bosons like the π^0 meson.)