Course: BSO 203 Instructor: S. Raychaudhuri

SESSION 2002-2003, SEMESTER I

PROBLEM SET VII

One-dimensional Scattering and Discussions

BRIEF SOLUTIONS

1. Attached sheet

2. Attached sheet

3. Discuss the following propositions from a physics viewpoint.

   (a) Quantum mechanics is applicable only to microscopic systems.

   Quantum mechanics is universally true. It is a deeper theory than classical mechanics because it takes into account the fact that there cannot be an ideal observer who knows about a physical system without disturbing it at all. In fact, classical mechanics is only the limiting case of quantum mechanics when the disturbance can be neglected. It is usually in microscopic systems, however, that quantum mechanical effects are large.

   (b) We cannot get rid of the disturbance induced by a measurement by using smaller and smaller probes.

   It may be argued that if we take a vanishingly small probe then we should make a vanishingly small disturbance. This is analogous to the process by which we consider an electrostatic field to be measured by a vanishingly small test charge, though in practice, we can never have a charge smaller than e. However, this is where the Uncertainty Principle comes in. It tells us that if making a probe small reduces the disturbance induced in the measurement of a certain dynamical variable, then it must <u>increase</u> the disturbance in the measurement of its canonical conjugate variable. This can be seen by reducing the energy/momentum of the probe, which gives a smaller uncertainty in the momentum of the system it measures, but increases the position uncertainty since the probe now has a longer de Broglie wavelength. Similar arguments can be used for most systems. However, the Uncertainty Principle cannot be proved by argument or example; it is a fundamental law of physics.

   (c) The wavefunction is a measure of the probability of existence of the state it describes.

   This statement is made very often and it is loosely true. Indeed the wavefunction, which can be a function of many variables, including space and time,

$$\Psi = \Psi(\vec{x}, t, q_1, q_2, \ldots)$$

   tells us the probability density that at the position $\vec{x}$, at time t and for the values $q_1, q_2, \ldots$, the probability of having the system in the neighbourhood is given by

$$\mathcal{P} = |\Psi|^2 \; d^3\vec{x} \; dq_1 dq_2 \ldots \; .$$

Note that there is no factor dt. If some internal variables q_i is discrete, there is no factor dq_i. We can also see that if the wavefunction vanishes for some values of the arguments, it is equivalent to saying that the system cannot take up those configurations.

(d) The collapse hypothesis is consistent with special relativity, but not with micro-causality.

The EPR paradox shows that on measurement, the wavefunction collapses *instantly* to an eigenfunction of the variable being measured, irrespective of the distance between the observer and the measured object. This means that at the microscopic level, the signal for collapse to occur can propagate with infinite velocity. This does not conflict with the principle of special relativity, which merely states that the laws of physics should be the same in all inertial frames. However, it does indicate that, at the microscopic level, we can have violations of causality, which is equivalent to going backwards in time. Time reversal is perfectly consistent with the microscopic equations of motion; it is only when we consider an ensemble of microscopic systems that the second law of thermodynamics comes into play, indicating that the flow of time in unidirectional. There is thus no real paradox, since the EPR paradox disappears for ensembles.

(e) In the limit $\hbar \to 0$, the classical picture is restored.

Once can argue as follows: in the limit $\hbar \to 0$, the right side of *all* commutation relations vanishes, and hence all dynamical variables have simultaneous eigenstates. Cayley's theorem then tells us that we may as well work with the eigenvalues, which is precisely the classical picture. However, there do exist unusual systems, where quantum effects can build up to give macroscopic effects. The most striking example is superconductivity.

4. Discuss the following propositions from a physics viewpoint.

(a) There is no time-energy uncertainty relation.

In quantum mechanics, time is not a dynamical variable. It is just an evolution parameter. Thus, there is no time operator and no factor dt in the definition of probability. However, in textbooks, one sees an expression of the form $\Delta E\ \Delta t \sim \hbar$ which is often mistakenly called the time-energy uncertainty relation. In fact, this is a mere re-statement of the usual position-momentum uncertainty for a photon emitted from an atomic system. If we assume the photon momentum to have a spread Δp and the wavefunction to have a spread Δx, then clearly the time taken for the emission process will be $\Delta t \sim \frac{1}{c}\Delta x$. Moreover, the momentum spread will induce an energy spread $\Delta E \sim c\Delta p$. Multiplying the two, we get

$$\Delta E\ \Delta t \sim c\Delta p\ \frac{1}{c}\Delta x \sim \hbar\ .$$

This is a specific statement with regard to photons emitted from atoms and no generalisation can be made. In general, it is quite okay to measure energy exactly at a given instant of time.

(b) The Schrödinger wavefunction must, in general, be complex.

Since there is a factor i on the right side of the Schrödinger equation, we must admit the possibility of complex wavefunctions. The real question is whether wavefunctions which are purely real can satisfy al the physical requirements. The answer is easily obtained. The Schrödinger equation can be re-written as an equation of continuity with

$$\vec{J} = \frac{\hbar}{m} \mathrm{Im} \Psi^* \vec{\nabla} \Psi \ .$$

If Ψ is real, it immediately follows that $\vec{J} = \vec{0}$, and hence

$$\frac{\partial}{\partial t} |\Psi(\vec{x}, t)|^2 = 0 \ ,$$

which means that the probabiliy distruibution for the particle is fixed for all time. Clearly, this is inadequate to describe systems in motion. Hence we must have complex Schrödinger wavefunctions.

(c) For a non-zero potential, the eigenstates of energy are different from the eigenstates of momentum.

The Hamiltonian of a one-dimensional system is given by

$$\hat{H} = \frac{p^2}{2m} + V(\hat{x}) \ .$$

The commutation relation of the Hamiltonian with the momentum is

$$[\hat{p}, \hat{H}] = [\hat{p}, V(\hat{x})] = i\hbar \frac{\partial}{\partial \hat{x}} V(\hat{x})$$

(prove this by expanding $V(\hat{x})$ in a power series in $\hat{x}$). Thus, unless V is a constant (which can be set to zero), the Hamiltonian does not commute with momentum, and hence will not have simultaneous eigenstates.

(d) Stationary states correspond to probability distributions that are independent of time.

If the potential $V(\vec{x})$ is independent of time, the Schrödinger equation has the solution

$$\Psi(\vec{x}, t) = \psi(\vec{x}) \ e^{-iEt/\hbar} \ .$$

The probability distribution is given by

$$|\Psi(\vec{x}, t)|^2 = |\psi(\vec{x})|^2 \ ,$$

which is independent of time.

(e) The larger the energy the greater is the number of nodes in the wavefunction.

We have seen this for a particle in a box or a square well. This is a generic feature of well problems, though it cannot be proved for the general case. Usually, to every state a step higher there corresponds one more node. This can be seen in several ways. For piecewise continuous potentials, we have sine and cosine functions of kx, which have more and more nodes for higher values of k. For continuous potentials, we have polynomial solutions, which have more and more nodes as we take polynomials of higher order. A qualitative argument can be given in the following way: consider real wavefunctions and a ground state wavefunction which is positive everywhere. The first excited state must be orthogonal to this, i.e. the overlap integral must vanish. For this, the wavefunction must be positive somewhere and negative somewhere, else the cancellation will not occur. This means there must be at least one node. If we consider the next excited state, it must have even more regions with positive and negative values, in order to be orthogonal to both the lower states, i.e. it must have more nodes. This argument can be extended to any order.

5. Discuss the following propositions from a physics viewpoint.

   (a) Scattering phenomena are the best probes of the microsopic world.

   Since the microscopic world cannot be seen, our information about it comes from three sources:

   - Indirect evidence: we look at macroscopic phenomena and obtain information about the microscopic world by inference. Canonical examples are the discovery of atoms from the laws of stoichiometry and the laws of valency from the proportions in which elements combine to form compounds. However, the information which can be obtained in this way is severely limited.
   - Evidence from unstable systems: some microscopic systems are unstable and decay into products which can be studied to infer something about the system itself. Canonical examples are radioactive nuclei and atoms in excited states.
   - Scattering of a microscopic probe from a microscopic system (or an ensemble of such). This is the only way in which we can learn about stable systems. It is also susceptible of control by the experimentalist because we can vary the parameters (such as energy) of the probe. In fact, normally we excite atoms by the use ofa probe, so that radiation by excited atoms may be viewed as an inelastic scattering phenomenon.

   Clearly, scattering phenomena are the best way to obtain information about mucroscopic systems.

   (b) The quantum mechanical problem of scattering is analogous to the optical phenomena of reflection and refraction.

   The analogies are clear: a potential reduces the wave velocity and hence behaves as an optically dense medium. We therefore have reflection and transmission of de Broglie waves exactly as in optics. In fact, we can reverse the argument and say that optical phenomena are nothing but quantum scattering phenomena of the de Broglie waves associated with the photon, which we call electromagnetic radiation.

   (c) Barrier penetration is a consequence of the Uncertainty principle.

   Barrier penetration occurs because a wavefunction cannot suddenly drop to zero as soon as it encounters a barrier, but must be attentuated continuously. This is because the wavefunction must be continuous and smooth, otherwise the momentum will be singular at the discontinuity. This would violate the Uncertainty principle, which forces the momentum spread to be finite if the position spread is also finite (which it is for a wave incident on a barrier). Thus the Uncertainty principle is ultimately responsible for the boundary conditions on the wavefunction and hence for barrier penetration.

   (d) Tunneling through a barrier is a purely quantum mechanical phenomenon.

   Strictly speaking this is true, since all natural phenomena are quantum phenomena and classical physics is only a limit. However, there are cases even in classical physics where tunnelling is seen. For example, if we consider a 45^0 prism with total internal reflection. The light wave does not really get sharply reflected at the slanting surface, but actually penetrates a few wavelengths into the air beyond the surface. The air byond the prism then acts as a potential step. If we now bring a plate of glass very close (a few wavelengths) to the slanting surface, a weak transitted beam will be observed. This beam has actually tunnelled through the air gap, which is like a barrier. The experiment can be done in the laboratory, provided two well-machined glass surfaces are available and a sensitive detector is placed to observe the transmitted beam.

(e) Resonances enable us to study bound states of a system from scattering data, especially when the bound states are unstable.

As seen in the square well, we obtain maximum transmission coefficients ($\mathcal{T} = 1$) at the resonant values of energy, i.e. when the energy matches with the bound states of the potential well. If the bound system is unstable, we cannot observe the bound states from transitions, because, quite often, the system breaks up before a transition can occur. However, the bound state energy levels can be still read-off from the resonances in the potential. In fact, many such resonances in the scattering of protons and neutrons were observed in the 1940's and 1950's, leading to the eventual discovery in 1964 that protons and neutrons are just bound states of smaller particles called quarks.