Course: BSO 203 Instructor: S. Raychaudhuri

SESSION 2002-2003, SEMESTER I

QUIZ V

1. Given a normalised wavefunction in one dimension

$$\psi(x) = \sqrt{a}\; e^{-a|x|}$$

calculate the position uncertainty Δx and the momentum uncertainty Δp and find the uncertainty product $\Delta x\ \Delta p$. **(5)**

Answer: Since the (real) wavefunction is normalised,

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\ \psi(x)\ x\ \psi(x) = 0$$

since the integrand is odd. Again

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} dx\ \psi(x)\ x^2\ \psi(x) \\
&= 2a \int_0^{\infty} dx\ x^2\ e^{-2ax} \\
&= 2a \left(\frac{1}{2a}\right)^3 \int_0^{\infty} dz\ z^2\ e^{-z} \quad (z = 2ax) \\
&= \left(\frac{1}{2a}\right)^2 \Gamma(3) \\
&= \frac{1}{2a^2}
\end{aligned}$$

Hence

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \;=\; \frac{1}{a\sqrt{2}}$$

Likewise,

$$\langle p \rangle = \int_{-\infty}^{\infty} dx\ \psi(x)\ \left(-i\hbar\frac{\partial}{\partial x}\right)\ \psi(x) = 0$$

since the integrand is odd.

However, to calculate $\langle p^2 \rangle$, we need to use the Schrödinger equation, since $\hat{p}^2$ has a singularity at $x = 0$. Thus, we write

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} dx\ \psi(x)\ \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\ \psi(x) \\
&= 2m \int_{-\infty}^{\infty} dx\ \psi(x)\ [E - V(x)]\ \psi(x) \quad \text{(cf. tutorial)} \\
&= 2m \int_{-\infty}^{\infty} dx\ \psi(x)\ \left[\frac{\hbar^2 a}{m}\delta(x)\right]\ \psi(x) \\
&= 2m\,\frac{\hbar^2 a}{m}\ [\psi(0)]^2 \\
&= 2\hbar^2 a^2
\end{aligned}$$

Hence

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \quad = \quad \frac{1}{a\sqrt{2}}$$

And finally,

$$\Delta x \ \Delta p \quad = \quad \frac{1}{a\sqrt{2}} \ \hbar a\sqrt{2} \quad = \quad \hbar$$

If the singularity is not realised and one tries to calculate directly, one will get

$$\langle p^2\rangle = -\hbar^2 a \ ,$$

which is absurd.