I. INTRODUCTION

System response to parametric variations or perturbations is of great importance. It is a powerful experimental practice from which new information about a system can be extracted, which is not generally available by other means, especially for complex systems, i.e., disordered, interacting many-body, and/or simple chaotic systems. External parameters such as electromagnetic fields, temperatures, applied stresses, etc., are controllable and often suitable for this purpose. Some specific examples include conductance fluctuations in quantum dots [1], and variations in eigenmodes due to shape deformations of microwave cavities [2] and vibrating plates [3]. Intramolecular vibrational energy redistribution [4] in molecules is another example wherein the technique of parametric variations can lead to useful insights regarding the key perturbations responsible for strong mode mixings. Molecular spectroscopy in external fields is an area of current interest and the response of a molecular system to external fields can be usefully analyzed from the parametric perspectives [5].

In the extreme chaotic or disordered limit, there exist universalities in system response to perturbations [6], and the eigenstates respect the ergodic hypothesis [7], i.e., no phase space localization; there is only a scale to extract. On the other hand, the response of integrable or near-integrable systems is far richer being system dependent, and the Husimi distributions of the eigenstates are confined to classical tori [8,9]. A perturbation is likely to give rise to subsets of eigenlevels/states that respond very similarly as a group, and the parametric quantities thereby carry far more information. For mixed systems both regular and chaotic trajectories form phase space structures and the most obvious, naive initial starting point would be to attempt to analyze each structure separately. At this level of approximation, using the appropriate semiclassical theory for regular and chaotic regions involves the respective integrable and chaotic Hannay–Ozorio de Almeida sum rule expressions [10] weighted by the relative fractions of phase space volume. We find, instead, that the intermittent behavior of the unstable orbits prevents this simple picture from applying.

Tomsovic [11] introduced a measure that reveals phase space localization of quantum eigenstates. Originally introduced for chaotic systems, the measure correlates the changes in the eigenenergies due to a perturbation (termed “level velocities”) with the overlap intensities between the eigenstates and a probe state, which is often usefully chosen to be a coherent state. More recently, it has been identified as contributing to the scale associated with the fidelity in the weak perturbative regime [12]. For integrable systems phase space localization is the norm, not the exception. There and in the near-integrable regime, the localization is associated with tori, resonance zones, and stable periodic orbits. The correlation measure has been shown to highlight different features of classical phase space depending upon the perturbation [13]. Several years ago Weissman and Jortner [14] performed a similar study involving the Husimi distributions and parametric changes in eigenenergy levels.

This paper is organized as follows: we begin by developing the semiclassical theory of the correlations for near-integrable systems where we also remark on the more general theory for the level velocities in mixed systems. The correlations have been previously studied in highly excited rovibrational states in molecules where multiresonant Hamiltonians are applicable [13]. Section III gives the semiclassical theories of the level velocities, strength functions, and overlap intensity–level velocity correlation coefficients. The semiclassical theory is compared with various results from integrable billiards and the standard map which has the entire range of dynamics from integrable to fully chaotic. We finish with a discussion of the similarities and distinctions between near-integrable and chaotic systems and some comments about mixed systems.
II. PRELIMINARIES

Consider a near-integrable quantum system governed by a smoothly parameter-dependent Hamiltonian $\hat{H}(\lambda)$ with classical analog $H(\mathbf{I}, \theta; \lambda)$, where

$$H(\mathbf{I}, \theta; \lambda) = H_0(\mathbf{I}) + \lambda H_1(\mathbf{I}, \theta).$$

Without loss of generality, the phase space volume of the energy surface, $V$, is taken to be a constant as a function of $\lambda$. This ensures that the eigenvalues do not collectively drift in some direction in energy, but rather wander locally. The parametrized strength function is given by

$$S_{\alpha}(E, \lambda) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{iE \delta t / \hbar} \langle \alpha | e^{-i\hat{H}(\lambda)t / \hbar} | \alpha \rangle dt$$

$$= \sum_a p_{\alpha a}(\lambda) \delta [E - E_a(\lambda)],$$

where $p_{\alpha a}(\lambda) = |\langle \alpha | E_a(\lambda) \rangle|^2$. $S_{\alpha}(E, \lambda)$ is the Fourier transform of the autocorrelation function of a normalized initial state $|\alpha\rangle$. In the following $\tilde{S}_{\alpha}(E, \lambda)$ will denote the smooth part resulting from the Fourier transform of just the extremely rapid initial decay due to the shortest time scale of the dynamics (zero-length trajectories). We will take $|\alpha\rangle$ to be a Gaussian wave packet because of its ability to probe "quantum phase space," but other choices may be useful depending on the circumstances. In one spectroscopic application, we found it to be advantageous to take $|\alpha\rangle$ as a momentum state [13].

The overlap intensity–level velocity correlation coefficient is defined as

$$C_{\alpha}(\lambda) = \left( \frac{p_{\alpha a} \partial E_{\alpha}(\lambda)}{\sigma_\alpha \sigma_E} \right)_E,$$

where $\sigma_\alpha^2$ and $\sigma_E^2$ are the local variances of intensities and level velocities, respectively. The brackets denote a local energy average in the neighborhood of $E$. It weights most the level velocities whose associated eigenstates possibly share common localization characteristics and measures the tendency of these levels to move in a common direction. In this expression, the phase space volume remains constant so that the level velocities are zero centered (otherwise the mean must be subtracted), and $C_{\alpha}(\lambda)$ is rescaled to a unitless quantity with unit variance making it a true correlation coefficient. The set of states included in the local energy averaging can be left flexible except for a few constraints. Only energies where $\tilde{S}_{\alpha}(E, \lambda)$ is roughly a constant can be used or some intensity unfolding must be applied. The energy range must be small so that the classical dynamics are essentially the same throughout the range, but it must also be broad enough to include enough eigenstates for statistical purposes.

III. SEMICLASSICAL DYNAMICS

We develop a theory based upon semiclassical dynamics for the overlap correlation coefficient of regular states by examining its individual components, the level velocities, and intensities. The theory for the variance of the level velocities involves the trace formula for near-integrable systems developed by Ullmo, Grinberg, and Tomsovic [15]. The near-integrable trace formula is used rather than the simpler integrable trace formula of Berry and Tabor [16] since it correctly describes the contributions of short, unstable periodic orbits that occur after a system breaks integrability. The level velocities are derivatives, and thus are far more sensitive to the perturbation than the intensities. Hence, it is more important to accurately describe the level velocities. The intensity variances for near-integrable systems are given by the integrable result as long as the coherent state is not near a resonance zone.

It turns out that a general result for the level velocity variance in mixed systems cannot be obtained simply by applying the appropriate Hannay–Ozorio de Almeida sum rule for the effectively regular and chaotic trajectories separately. The "effective" trajectories are not easily defined since they do not merely correspond to the trajectories within the regular islands, i.e., Kolmogorov-Arnold-Moser (KAM) islands, and the chaotic sea, respectively. Although, it is observed that the near-integrable result for the level velocities is in good agreement with the calculated results before the breakup of the last KAM torus.

A. Level velocities

By adapting a method employed by Berry and Keating [17] for classical chaotic systems with the topology of a ring threaded by quantum flux and later generalized for all continuous time chaotic systems [18,19], the variance of the level velocities can be obtained. The details for near-integrable systems are in Appendix A where the variance of the level velocities is found to be

$$\sigma_E^2 = \frac{2(2\pi)^{d+1}}{V \hbar} \sum_{M} A_{\mathbf{M}}^{2 \frac{\delta S_{\mathbf{M}}}{\delta \lambda}} \left[ J_0^2(s_{\lambda}) + \frac{a^2}{4} J_1(s_{\lambda}) \right]$$

$$\times \exp \left( -2 \frac{\epsilon E_{\mathbf{M}}}{\hbar} \right),$$

where the $\epsilon \rightarrow 0$ limit is understood, and the amplitude factor is determined by the integrable system, $H_0(\mathbf{I})$,

$$A_{\mathbf{M}} = \frac{1}{2 \pi |\mathbf{M}|^{(d+1)/2}} \sqrt{K_{\mathbf{M}}},$$

The prime on the sum excludes the $\mathbf{M} = 0$ term. $\mathbf{M}$ labels the rational tori and is a $d$-dimensional vector with positive integer components whose classical paths are those which at time $t$ have returned to the same point on their torus after making $M_1$ circuits of coordinate $\theta_1$, $M_2$ circuits of coordi-
In the case $k$ on the resonant torus. Thus, excluding mixed systems, integrable systems, but this is not true for mixed phase space tion changes hold equally well for integrable and near- numerical calculations discussed in the following, ballistic ac-
change of chaotic trajectories. The variance is therefore pro-
"ballistic" action change in contrast to the diffusive action 
tional to the period of motion. We can think of this as a
For rational tori, the change of the classical action is propor-
with $s_\lambda = \Delta S_M(\lambda)/\hbar$ and
$$
\tilde{a}(\lambda) = \frac{\kappa - 1}{\kappa + 1},
$$
where
$$
\kappa = \left( -\frac{\det(M_s - 1)}{\det(M_s - 1)} \right)^{1/2}.
$$
In the case $\kappa \to 1$ then $\tilde{a}(\lambda) \to 0$ and the spectral staircase reduces to Ozorio de Almeida’s result [20].

The summation over rational tori can be accomplished by applying the Hannay–Ozorio de Almeida sum rule for integrable systems [10],
$$
\sum_M A_M^2 \cdots \to \frac{V}{(2\pi)^{d+1}} \int \frac{dT}{T^2} \cdots.
$$
Thus,
$$
\sigma_E^2 \approx \frac{2\epsilon}{\hbar} \int \frac{1}{T^2} \left\{ \left[ \left( \frac{\partial S_M}{\partial \lambda} \right)^2 \right] J_0^2(s_\lambda) + \tilde{a}^2 J_1^2(s_\lambda) \right\} \left( \left[ \left( \frac{\partial \Delta S_M}{\partial \lambda} \right)^2 \right] \frac{\tilde{a}^2}{4} [J_0(s_\lambda) - J_2(s_\lambda)]^2 + J_1^2(s_\lambda) \right\}_M \times \exp \left\{ -\frac{2\epsilon T}{\hbar} \right\} dT.
$$
For rational tori, the change of the classical action is propor-
tional to the period of motion. We can think of this as a “ballistic” action change in contrast to the diffusive action changes of chaotic trajectories. The variance is therefore proportional to the square of the period. According to the numerical calculations discussed in the following, ballistic action changes hold equally well for integrable and near-integrable systems, but this is not true for mixed phase space systems. Thus, excluding mixed systems,
where $f$ is the fraction of ballistically behaving orbits and $g$ is a symmetry factor for the system. The quantities $\zeta(E, \lambda; \hbar)$ and $K(E, \lambda)$ would be determined by the ballistic and diffusive versions of Eq. (11), respectively. The section on the kicked rotor will address this issue.

B. Overlap Intensities

Next, the overlap intensities are investigated and a semiclassical derivation of their variance is obtained. For an integrable system the overlap intensities are best expressed in action-angle coordinates; details are in Appendix B and the result is

$$
\sigma_a^2 = \frac{\prod \sigma_j^2}{\pi^d} \left( \exp \left[ \frac{-2(\Delta I)^2}{\hbar^2} \right] \right),
$$

(14)

where in the exponential argument of the wave packet, each component is $(\Delta I) = \sigma_j (1 - I_a)$. The variance is a Gaussian weighted average of the action of the wave packet over the rational tori.

The Gaussian weighting can be evaluated by replacing the $M$ averging by an average of the action variables over the energy surface,

$$
\left\langle \exp \left[ \frac{-2(\Delta I)^2}{\hbar^2} \right] \right\rangle_M \approx \frac{(2\pi)^d}{V} \int \exp[-2(\Delta I)^2/\hbar^2] \delta(E - H_0(I)) dI.
$$

(15)

The Hamiltonian in the numerator is expanded about $I_a$ giving

$$
H_0(I) = H_0(I_a) + \omega \cdot (I - I_a) + \cdots,
$$

(16)

where $H_0(I_a) = E$ and $\omega_a = \partial H_0(I_a)/\partial I$ Only linear terms are necessary in the semiclassical limit as long as each width component is chosen such that $\sigma_j/\hbar \to \infty$ as $\hbar \to 0$; we take $\sigma_j \approx \hbar^{1/2}$ so that uncertainty in $I$ and $\theta$ shrink similarly as $\hbar$ shrinks. After performing the integration of the weighting over the energy shell, the result is

$$
\left\langle \exp \left[ \frac{-2(\Delta I)^2}{\hbar^2} \right] \right\rangle_M \approx \left( \frac{\pi \hbar^2}{2} \right)^{(d-1)/2} \prod \sigma_j \frac{1}{\sqrt{\sum (\omega_{a_j}/\sigma_j)^2}}.
$$

(17)

which has an $\hbar$ scaling of $\hbar^{(d-1)/2}$. Therefore, the leading order $\hbar$ scaling for the intensity variance for near-integrable systems is

$$
\sigma_a^2 = \frac{\prod \sigma_j}{V \sqrt{\sum (\omega_{a_j}/\sigma_j)^2}} \propto \hbar^{(d-1)/2}.
$$

(18)

C. Correlations

Finally, the semiclassical theories for the overlap intensities and the level velocities outlined above can now be used to construct the correlation function. Beginning with the covariance, it can be shown that

$$
\text{Cov}_a(\lambda) = \left( \rho_{sa}(\lambda) \frac{\partial E_s(\lambda)}{\partial \lambda} \right)_n
$$

$$
= \frac{2 \pi \hbar^d e}{V} \left( S_a(E; \lambda) \frac{\partial N(E; \lambda)}{\partial \lambda} \right)_E.
$$

(19)

The semiclassical expressions for the strength function and the parametric derivative of the staircase function give

$$
\text{Cov}_a(\lambda) = \frac{-e^2(2d+1)^2 \pi^{1/2} \prod \sigma_j}{V \hbar} \left( \sum M^\prime |A_M| D_{M^\prime}\right) \left[ \frac{\partial \tilde{S}_M}{\partial \lambda} \right]
$$

$$
\times \left[ J_0(s_\lambda) - i \tilde{a}(\lambda) J_1(s_\lambda) \right] + \left[ \frac{i}{2} \frac{\partial \tilde{S}_M}{\partial \lambda} \right]
$$

$$
\times \left[ J_1(s_\lambda) + \frac{i}{2} \tilde{a}(\lambda) \right] \left[ J_0(s_\lambda) - J_2(s_\lambda) \right]
$$

$$
\times \exp \left[ \frac{-(\Delta I)^2}{\hbar^2} - 2 e T_M/\hbar \right]_E.
$$

(20)

Again, the diagonal approximation has been made in the above expression. Following arguments parallel to those in deriving the variances leads to the expression

$$
\text{Cov}_a(\lambda) = \frac{-2 \prod \sigma_j}{\pi^{d/2} \tau_H} \int \frac{dT}{T} e^{-2T/\tau_H}
$$

$$
\times \left[ \frac{\partial \tilde{S}_M}{\partial \lambda} \exp \left[ \frac{-(\Delta I)^2}{\hbar^2} \right] \right]_M.
$$

(21)

for integrable systems, where $\tau_H = \hbar/\epsilon$ is the Heisenberg time.

In order to make further progress in understanding the correlation function, it is necessary to know the time dependence of the $M$-averaged expression. It is important to note that the Gaussian weighting factor will not decouple from the parametric action derivative at long times and thus, the local average of the derivatives of the action is not necessarily zero. The weighted action derivatives can be approximated by the same method as used in the calculation of the variance of the level velocities, so for integrable systems
\[
\left( \frac{\partial S_M}{\partial \lambda} \right) \exp \left[ -\frac{(\Delta I)^2}{\hbar^2} \right] \bigg|_M
\]
\[
= \int (\partial S_M/\partial \lambda) \exp[-(\Delta I)^2/\hbar^2] \delta(T-T_M) dM
\]
\[
= T \xi(\lambda).
\]

Note that the above average is linear with the period. The \( \hbar \) scaling of \( \xi(\lambda) \) is the same as for the average Gaussi weighting in the preceding section, which is \( \hbar^{(d-1)/2} \).

The final result of the covariance is
\[
\text{Cov}_{\sigma}(\lambda) \approx \frac{-\xi(\lambda) \prod \sigma_j}{\xi^{1/2}(E,\lambda) \exp[-2(\Delta I)^2/\hbar^2] \bigg|_M}.
\]

where the expectation value is evaluated in Eq. (17). For near-integrable systems in the semiclassical limit, the above result implies that \( \text{Cov}_{\sigma}(\lambda) \approx \hbar^{d(d-1)/4} \), whereas the correlation \( \text{Cov}_{\sigma}(\lambda) \approx \hbar^{(d-1)/4} \) to leading order.

IV. RESULTS

A. Rectangular and box billiards

First, we consider billiards that are totally separable in Cartesian coordinates and thus completely integrable. These types of billiards have been used in the study of spectral rigidity [21]. In particular, we examine a rectangular billiard with sides \( a, b \) and a box billiard with sides \( a, a, b \); the side length \( a \) is varied. The density of states is kept constant by not changing the area \( A \) of the rectangle or the volume \( V \) of the box. The action/angle variables are proportional to the momentum/position coordinates in each direction,
\[
I_j = \ell_j, p_j / \pi \quad \text{and} \quad \theta_j = \pi q_j / \ell_j,
\]
where \( \ell_j \) is the side length of the \( j \)th side. The Hamiltonians are given by
\[
H(I) = \frac{\pi^2}{2m} \sum_j \left( \frac{I_j^2}{\ell_j^2} \right).
\]

The classical action of the rectangle for a closed orbit on a given resonant torus \( M \) is
\[
S_M = \sqrt{8mE} [M_1^2a^2 + M_2^2(Aa)^2]^{1/2}
\]
the period is
\[
T_M = \sqrt{2m} [M_1^2a^2 + M_2^2(Aa)^2]/E
\]

and the derivative of the action along the orbit is
\[
\frac{\partial S_M}{\partial a} = \frac{8mE}{a} \frac{M_1^2a^2 - M_2^2(Aa)^2}{[M_1^2a^2 + M_2^2(Aa)^2]^{1/2}}
\]

Similarly, the classical action of the box is
\[
S_M = \sqrt{8mE} [M_1^2a^2 + M_2^2(Aa)^2]^{1/2},
\]
the period is
\[
T_M = \sqrt{2m} [M_1^2a^2 + M_2^2(Aa)^2]^{1/2}/E,
\]
and the derivative of the action is
\[
\frac{\partial S_M}{\partial a} = \frac{8mE}{a} \frac{M_1^2 + M_2^2a^2 - 2M_2(Aa)^2}{[(M_1^2 + M_2^2a^2 + 2M_2(Aa)^2)]^{1/2}}
\]

Using the above quantities and performing the integration in Eq. (11), the variances of the action derivatives are \( 2E^2T^2a^2 \) and \( 16E^2T^2/5a^2 \) for the rectangle and the box, respectively. Hence, the respective values of \( \xi(E, a) \) are \( 2E^2/5a^2 \) and \( 16E^2/15a^2 \), which approach the variances of the level velocities in the semiclassical limit. Figure 1 demonstrates how well \( \xi(E, a) \) approximates the level velocities especially as \( \hbar \to 0 \). Note that the level velocities are virtually independent of \( \hbar \) as predicted.

The \( \hbar \) scaling of the intensity variance is predicted to be \( \hbar^{5/2} \) and \( \hbar^4 \), respectively, for the rectangular and box billiards. Using Eq. (17) with \( \omega_{a, j} = \pi^2 I_{a, j}/m \ell_j^2 \) the semiclassical theory and quantum intensity variances are compared in Fig. 2. The agreement is quite good as \( \hbar \to 0 \).

The last two quantities, the covariance and correlation, are shown in Figs. 3 and 4, respectively, along with the results from the semiclassical theory. The values of \( \xi(E, a) \) are obtained from Eq. (22). For the rectangle
\[
\xi(E, a) = \frac{2\pi^{3/2}h}{a m \sigma \sqrt{2mE}} \left( \frac{I_{a, 1}}{a^2} - \frac{I_{a, 2}^2}{(Aa)^2} \right),
\]
and for the box
\[
\xi(E, a) = \frac{\pi^2 h^2}{a m^2 E \sigma} \left( \frac{I_{a, 1}^2}{a^2} + \frac{I_{a, 2}^2}{a^2} - \frac{2I_{a, 3}^2}{(V/a^2)^2} \right).
\]

The agreement is excellent in both cases. Note that we have used no fitting parameters in any of the figures in this section.

B. Standard map

In this section we compare the semiclassical theories to the numerical results for the standard map. The classical standard map is defined by
\[
q_{i+1} = q_i + p_{i+1} \mod (1),
\]

\[
056205-5
\]
Integrable at

The resonant tori $\mathbf{M}$ including repetitions for the standard map are given by a constant momentum equal to the rational fraction $m/n$, where $m = 0, \ldots, n - 1$ and $n$ is the period of the torus. The derivative of the action with respect to the kicking strength is

$$\frac{\partial S_{\mathbf{M}}}{\partial k} = \frac{1}{(2\pi)^2} \sum_{i=1}^{n} \cos(2\pi q_i).$$

The smoothed density of states for the eigenangles [23] is

$$d_{\phi}(\phi) = \frac{N}{2\pi} + \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp(in\phi) \text{Tr} U^n \exp(-n\epsilon),$$

where the sum excludes $n=0$ and the trace of the propagator in the integrable regime is

$$\text{Tr} U^n = \sum_{\mathbf{M}} \sqrt{\frac{N}{in}} \exp(2\pi iNS_{\mathbf{M}}).$$

Integrating over the angle, the smoothed spectral staircase is

$$N_{\phi}(\phi) = \frac{N\phi}{2\pi} - \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{M}} \frac{1}{n} \sqrt{\frac{iN}{n}} \times \exp(in\phi) \exp(2\pi iNS_{\mathbf{M}}) \exp(-n\epsilon).$$

The upper panel is for the rectangle and the lower panel is for the box. The solid line represents the quantum results while the dashed line is the semiclassical theory.
For the integrable case \((k = 0)\), the periodic points are \(q_i = q_0 + m \cdot i/n \mod(1)\) and hence, \(S_M/k = 0\) for all resonant tori. Thus, by Eqs. (11) and (12) the variance of the level velocities is zero.

In the near-integrable regime, the variance of the eigenangles is approximately given by the result

\[
\sigma^2_\phi \approx 4 \pi^2 \zeta N^2.
\]

(42)

Note that the eigenangles are divided by an effective \(\hbar\) making the variance of the eigenangles quadratic in \(N\). We do not have an analytic formula for \(\zeta\), but it can be obtained numerically from the periodic orbit dependence of the variance of the action derivatives averaged over the tori. For \(k = 0\), it is straightforward to calculate analytically the positions of the stable and unstable periodic orbits that arise as soon as integrability is broken. Curiously, using these points as initial conditions for the periodic orbits throughout the near-integrable regime, roughly \(k \in [0, 2]\), gives results within a few percent of that found by locating the true periodic orbits. In order to make long time calculations, this compromise is indispensable and does not harm the approximation noticeably. In Fig. 5, the near-integrable theory is shown to give excellent agreement with the quantum eigenangle velocity variance even though there exists a significant proportion of unstable orbits for the larger values of \(k\) shown that are being treated as though they were stable.

In the mixed regime, the corresponding proposition of Eq. (13) is

\[
\sigma^2_\phi \approx 4 \pi^2 \zeta(k) f N^2 + g \frac{D(k)}{4 \pi^2 (1 - f)} N,
\]

(43)

where \(g = 4\) and \(D(k)\) is the classical action diffusion constant. It was shown previously [24] that a direct method based on classical orbits gave a good correspondence between the theory and quantum values for
where $V(q) = \cos(2\pi q)$, but no understanding was given of the two coefficients. Equation (43) gives one plausible, but naive, interpretation.

In order to test numerically the separation of diffusive and ballistic contributions in the mixed regime, it is impossible to work with the periodic orbits. They proliferate exponentially and must be found numerically. The time over which this can be done by brute force calculation is too short for our purposes. Instead, a Monte Carlo technique is needed. We have devised a method that begins with a uniform sampling of initial conditions in the phase space, and relies on the properties of finite time Lyapunov exponents [25]. Although, a stable periodic orbit has a vanishing Lyapunov exponent, at a partial period of its motion, the trace $\mathcal{Q}$ of its stability matrix can grow as a power of the propagation time. This can give a stable orbit the appearance of being unstable depending on circumstances. For example, constructing a surface of section based on whether $|\mathcal{Q}|<2$ fails miserably. If, instead, one seeks the least squares deviation of the time development of $\mathcal{Q}$ from the form

$$\mathcal{Q} = a T^\gamma \exp \mu T,$$

then the estimate of the Lyapunov exponent can be used to judge whether an orbit is in the stable or unstable part of the phase space; i.e., the construction of the correct surface of section is recovered in this way.

Figure 6 gives a histogram of the finite time Lyapunov exponents $\mu$, calculated using Eq. (45) for the orbits in this regime. The orbits are separated into three rough categories depending upon their stabilities as mentioned in the preceding section. The broadened $\delta$-function feature near zero represents the stable orbits that have action derivatives with a quadratic dependence upon time. The set of most highly unstable or chaotic trajectories give an approximate log-normal distribution [25]. In between these two features lie the weakly unstable orbits. It turns out that the coefficient $\xi(k)$ converges well if averaged over the stable orbits and up to a similar number of the weakly unstable orbits. However, it also turns out that when compared to the coefficient $c_2(k)$ of Eq. (44), the fraction $f$ deduced by their comparison is inconsistent with the relative fraction used to derive $\xi(k)$ for $k$ values in the range $[1,5,4,0]$; see Appendix C.

A second inconsistency with the simplistic separation problem is that the remaining strongly unstable orbits do not show a purely diffusive behavior. They show both a quadratic and linear dependence in their action derivative variances; i.e., the numerical evaluation of $D(k)$ is suspect. It could also be evaluated according to its asymptotic expression [24], but again requires a large value of $f$ to be consistent with the coefficient $c_1(k)$ unless $k$ exceeds roughly 5 or so. At that point $f \approx 0$ and the system is basically in the fully chaotic regime. We attempted several schemes of separating out orbits according to ballistic or diffusive behaviors, none of which worked and are outlined in Appendix C.

In spite of the lack of apparent separability, the simplistic expression, Eq. (44), works quite well. We recalculate the results of Ref. [24] taking into account the Heisenberg time associated with the mean eigenangle spacing in Fig. 7. As a function of kicking strength $k$ the agreement between the semiclassical theory and the variance of the eigenangle velocities is quite good throughout the entire transition from integrability to full chaos.

Next we turn to the overlap intensities. Since the energy is not conserved for maps, Eq. (17) does not apply. The variance is obtained directly from Eq. (14) where the average is over all resonant tori; it becomes just the Gaussian normalization factor. The width of the Gaussian wave packet in Cartesian coordinates is taken to be proportional to the

![Figure 6](image1.png)

**FIG. 6.** Histogram of finite time Lyapunov exponents in the standard map in the mixed regime. The parameters are $k=2.7$ and $n=500$, where $n$ is the time interval between successive kicks. The Lyapunov exponents are in units of $1/n$. The values of $\mu$ were obtained by a least squares fit of the natural logarithm of Eq. (45).

![Figure 7](image2.png)

**FIG. 7.** The variance of the level velocities for the standard map ($N=500$). The variance and $k$ are dimensionless. The solid line is for the quantum results while the dashed line is for the classical phase space average of Eq. (44). The time $n$ is taken to be $N/g = 125$. 
square root of $\hbar$. $\sigma = \beta \sqrt{\hbar} = \beta \sqrt{2 \pi N}$. Thus, for $k=0$ the variance of the intensities is

$$\sigma_n^2 \approx \frac{\beta}{N^2}$$  \hfill (46)$$

This expression does not take into account the reflection, $(q,p) \rightarrow (1-q,1-p)$ and time reversal symmetries of the kicked rotor [26]. Thus, we symmetrize the wave packet accordingly.

Figure 8 shows the good agreement between the semiclassical theory and the intensity variance for the integrable regime at $k=0.01$. The only exceptional cases are the tori at $p=0$ and $p=0.5$, which happen to be self-symmetric and on the lowest order resonances. Whereas for $k=0$, they would lead to double the intensity variance and be accurately accounted for once the symmetry was fully incorporated, at $k=0.01$ they are strongly affected by their location on low order resonances. As we do not have the generalizations necessary for those cases, we avoid the resonance zones instead.

Finally, we turn to the correlations. The covariance between the intensities and level velocities is given by Eq. (23) divided by $\hbar = 1/2 \pi N$ since the level velocities in the standard map are actually eigenangle velocities. After symmetrizing the wave packet, Fig. 9 compares the semiclassical prediction with the quantum results. Also, the correlation is compared in Fig. 10. In both cases the agreement between the semiclassical theory and the quantum results is good.

In the near-integrable regime, the covariance and correlation is strongest for resonance zones; see Fig. 11 and Ref. [13]. The quantizing tori are deforming at a greater rate inside the resonance while the tori outside the resonance are only slightly perturbed. Thus, the associated level velocities are larger for this area of phase space. Also, note that there is a dip in the correlation near the stable periodic orbit at the center of the resonance zone. The dip is only present in the correlation and not in the covariance. The correlation is divided by the variance of the intensities, which is large near the center of the resonance. The intensity variance is divided by the winding numbers, Eq. (17), which are small near the center of the resonance making the intensity variance large.

V. DISCUSSION

We have derived semiclassical expressions of the correlation of level velocities and overlap intensities in near-integrable systems. Their $\hbar$ scaling is very different than previously derived in the chaotic limit [19] where the level velocity variance is
The oscillating part of the spectral staircase is a sum over rational tori with topology $\mathbf{M}$ of the unperturbed Hamiltonian $H_0(\mathbf{I})$ [15],

$$N_{osc}(E,\lambda) = \frac{2}{\hbar^{d-1}} \sum_{\mathbf{M}} A_M \text{Re} \left\{ \exp \left( \frac{i \bar{S}_M(\lambda)}{\hbar} \right) \right\}$$

$$- i \frac{\eta_M \pi}{2} + i \frac{\beta_M \pi}{4} \left[ J_0(\Delta S_M(\lambda)/\hbar) \right]$$

$$- i \bar{a}(\lambda) J_1(\Delta S_M(\lambda)/\hbar) \exp \left( -\frac{\epsilon T_M(\lambda)}{\hbar} \right) \right\}.$$  

(A6)

The phase $\eta_M = \mathbf{M} \cdot \eta$, where $\eta$ are the Maslov indices and $\beta_M$ is equal to the sign of the determinant of the curvature matrix.
The variance of the level velocities is obtained by taking the mean square of the counting function derivatives,

$$
\left\langle \left( \frac{\partial N_e(E, \lambda)}{\partial \lambda} \right)^2 \right\rangle_E = \sum_n \sum_m \frac{\partial E_n(\lambda)}{\partial \lambda} \frac{\partial E_m(\lambda)}{\partial \lambda} \times \delta_\lambda[E - E_n(\lambda)] \delta_\lambda[E - E_m(\lambda)] \right\rangle_E.
$$  \hspace{1cm} (A7)

For a nondegenerate spectrum, the summation is nonzero only if \( n = m \) because of the product of the two \( \delta \) functions. Since Lorentzian smoothing is applied, then

$$\delta^2_e(x) = \frac{1}{2\pi \epsilon} \delta_{e2}(x) \hspace{1cm} (A8)$$

for \( \epsilon \ll \tilde{d}^{-1} \). Thus we have

$$\left\langle \left( \frac{\partial N_e(E, \lambda)}{\partial \lambda} \right)^2 \right\rangle_E = \frac{\bar{d}}{2\pi \epsilon} \left( \frac{\partial E_n(\lambda)}{\partial \lambda} \right)^2 \right\rangle_n. \hspace{1cm} (A9)$$

The summation over the rational tori is most sensitive to the changing actions and periods because of the associated rapidly oscillating phases, i.e., the division by \( \hbar \) in the exponential. Since the energy smoothing term \( \epsilon \) is taken smaller than a mean level spacing, it scales at least by \( \hbar^{-d} \) and the derivatives of the period vanish as \( \hbar \to 0 \). Thus, only the derivatives of the actions are considered, and the oscillating part of the staircase yields

$$\frac{\partial N_{osc}(E)}{\partial \lambda} \approx \frac{2}{\hbar^{(d+1)/2}} \sum'_M A_M \text{Im} \left\{ -\exp \left( \frac{iS_M(\lambda)}{\hbar} \right) \right\}
- \frac{i \eta_M \pi}{2} + \frac{i \beta_M \pi}{4} \left( \frac{\partial S_M}{\partial \lambda} \right)
\times \left[ J_0(s_{\lambda}) - i\tilde{a}(\lambda) J_1(s_{\lambda}) + i \left( \frac{\partial S_M}{\partial \lambda} \right) \right]
\times \left[ J_1(s_{\lambda}) + \frac{i \tilde{a}(\lambda)}{2} \left( J_0(s_{\lambda}) - J_2(s_{\lambda}) \right) \right]
\times \exp \left( -\frac{\epsilon T_M}{\hbar} \right). \hspace{1cm} (A10)$$

After making the diagonal approximation and energy averaging, the variance of the level velocities for a near-integrable system is

$$\sigma^2_e \approx \frac{2(2\pi)^{d+1}\epsilon}{\hbar} \sum_M \left\{ A_M^2 \left( \frac{\partial S_M}{\partial \lambda} \right)^2 \left[ J_0^2(s_{\lambda}) + \frac{\tilde{a}^2}{2} J_1^2(s_{\lambda}) \right]
+ \frac{1}{2} \left( \frac{\partial S_M}{\partial \lambda} \right) \left( \frac{\partial S_M}{\partial \lambda} \right) \left( \frac{\partial J_0(s_{\lambda})}{\partial \lambda} \right) \left( \frac{\partial J_0(s_{\lambda})}{\partial \lambda} \right)
+ \frac{\tilde{a} J_1^2(s_{\lambda})}{2} + \left( \frac{\partial S_M}{\partial \lambda} \right)^2 \left[ \frac{\tilde{a}^2}{4} \left( \frac{\partial J_0(s_{\lambda})}{\partial \lambda} \right) \left( \frac{\partial J_0(s_{\lambda})}{\partial \lambda} \right)
+ J_1^2(s_{\lambda}) \right] \right\} \exp \left( -\frac{2\epsilon T_M}{\hbar} \right). \hspace{1cm} (A11)$$

The middle term vanishes since \( (\partial S_M/\partial \lambda) \) and \( (\partial S_M/\partial \lambda) \) are uncorrelated and both average to zero.

**APPENDIX B: OVERLAP INTENSITIES**

The variance of the intensities is derived by examining the oscillating part of the strength function

$$S_{\alpha, osc}(E, \lambda) = -\frac{1}{\pi} \text{Im} \left\langle \alpha | \theta G(\theta, \theta'; E) | \alpha \rangle d\theta d\theta' \right\rangle, \hspace{1cm} (B1)$$

where

$$G(\theta, \theta'; E) = \frac{1}{i\hbar (2\pi \hbar)^{(d-1)/2}} \times \sum_j |D_j|^2 \exp[iS_j(\theta, \theta'; E)/\hbar - i\nu_j \pi/2] \hspace{1cm} (B2)$$

is the semiclassical energy Green’s function. The above sum is over all paths that connect \( \theta \) to \( \theta' \) on a given energy surface \( E \). \( D_j \) is a determinant involving second derivatives of the actions,

$$D_s = \begin{vmatrix} \frac{\partial^2 S}{\partial \theta \partial \theta'} & \frac{\partial^2 S}{\partial \theta \partial E} \\ \frac{\partial^2 S}{\partial E \partial \theta'} & \frac{\partial^2 S}{\partial E^2} \end{vmatrix}. \hspace{1cm} (B3)$$

We assume that the Gaussian wave packet in Cartesian coordinates can be define locally as a Gaussian in action-angle variables using the Wigner function.

**TABLE I.** Fixed Lyapunov exponent cutoff. \( f \) is the fraction of orbits that have a Lyapunov exponent less than \( \mu = 0.5 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \xi(k) )</th>
<th>( f )</th>
<th>( f_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>( 9.62 \times 10^{-5} )</td>
<td>0.982</td>
<td>1.000</td>
</tr>
<tr>
<td>2.0</td>
<td>( 1.11 \times 10^{-4} )</td>
<td>0.856</td>
<td>0.798</td>
</tr>
<tr>
<td>2.5</td>
<td>( 1.76 \times 10^{-4} )</td>
<td>0.358</td>
<td>0.336</td>
</tr>
<tr>
<td>3.0</td>
<td>( 3.66 \times 10^{-4} )</td>
<td>0.161</td>
<td>0.142</td>
</tr>
<tr>
<td>3.5</td>
<td>( 3.58 \times 10^{-4} )</td>
<td>0.166</td>
<td>0.146</td>
</tr>
<tr>
<td>4.0</td>
<td>( 3.67 \times 10^{-4} )</td>
<td>0.118</td>
<td>0.107</td>
</tr>
</tbody>
</table>
where we have used the diagonal approximation for the sum over $M$. Since the trace of Green’s function is the density of states given by Berry and Tabor, and the density of states is the energy derivative of the spectral staircase function, then $|D_s| = T^2 \sum' \sigma^2_M / (2 \pi)^{d-1}$. Hence, given the Hannay-Ozorio sum rule for $\sum' \sigma^2_M$, Eq. (9), then

\[ \sum' |D_s| \cdots \rightarrow \frac{V}{(2 \pi)^{d-1}} \int dT \cdots . \]  

Since the Gaussian weighting of the action variables is independent of the period, it follows that

\[ \sigma^2_a \approx \frac{\prod' \sigma^2_M}{\pi^d} \left\{ \exp \left[ - \frac{2(\Delta I)^2}{h^2} \right] \right\}_M . \]  

### APPENDIX C: BALLISTIC VERSUS DIFFUSIVE BEHAVIOR

We tried three different methods for separating out the ballistic and diffusive orbits. The fraction of ballistic orbits $f$ was deduced by

\[ f = \frac{c_2(k)}{4 \pi^2 \zeta(k)} , \]  

where $c_2(k)$ was determined by Eq. (44). The first method was to divide the orbits by a fixed value of the Lyapunov exponent. It turned out that a Lyapunov exponent of 0.5 gave fairly good result, but there is no explanation of this value. Table I summarizes these results.

The next method was to take twice the fraction of stable orbits as the number of ballistic orbits. This is summarized in Table II.

For the final method we took the greatest number of orbits up to some Lyapunov exponent that did not change its coefficient significantly. $\mu_3$ is the corresponding Lyapunov exponent.

### Table II. Twice the regular orbits, $f_2$ is the number of regular orbits and $\mu_2$ is the cutoff of the Lyapunov exponent associated with $f_2.$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\zeta(k)$</th>
<th>$f$</th>
<th>$f_2$</th>
<th>$\mu_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>$1.17 \times 10^{-4}$</td>
<td>0.812</td>
<td>0.687</td>
<td>0.318</td>
</tr>
<tr>
<td>2.0</td>
<td>$1.49 \times 10^{-4}$</td>
<td>0.638</td>
<td>0.526</td>
<td>0.443</td>
</tr>
<tr>
<td>2.5</td>
<td>$2.19 \times 10^{-4}$</td>
<td>0.288</td>
<td>0.256</td>
<td>0.420</td>
</tr>
<tr>
<td>3.0</td>
<td>$2.19 \times 10^{-4}$</td>
<td>0.269</td>
<td>0.241</td>
<td>0.590</td>
</tr>
<tr>
<td>3.5</td>
<td>$2.21 \times 10^{-4}$</td>
<td>0.268</td>
<td>0.246</td>
<td>0.702</td>
</tr>
<tr>
<td>4.0</td>
<td>$2.29 \times 10^{-4}$</td>
<td>0.189</td>
<td>0.179</td>
<td>0.713</td>
</tr>
</tbody>
</table>

### Table III. Stable quadratic, $f_3$ is the number of orbits included in the ballistic term which does not change its coefficient significantly, $\mu_3$ is the corresponding Lyapunov exponent.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\zeta(k)$</th>
<th>$f$</th>
<th>$f_3$</th>
<th>$\mu_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>$1.84 \times 10^{-4}$</td>
<td>0.516</td>
<td>0.412</td>
<td>0.2</td>
</tr>
<tr>
<td>2.0</td>
<td>$2.58 \times 10^{-4}$</td>
<td>0.368</td>
<td>0.302</td>
<td>0.3</td>
</tr>
<tr>
<td>2.5</td>
<td>$3.30 \times 10^{-4}$</td>
<td>0.191</td>
<td>0.165</td>
<td>0.2</td>
</tr>
<tr>
<td>3.0</td>
<td>$4.26 \times 10^{-4}$</td>
<td>0.139</td>
<td>0.124</td>
<td>0.4</td>
</tr>
<tr>
<td>3.5</td>
<td>$4.01 \times 10^{-4}$</td>
<td>0.148</td>
<td>0.137</td>
<td>0.4</td>
</tr>
<tr>
<td>4.0</td>
<td>$4.31 \times 10^{-4}$</td>
<td>0.105</td>
<td>0.100</td>
<td>0.4</td>
</tr>
</tbody>
</table>
EXPLORING CLASSICAL PHASE SPACE STRUCTURES . . .


[27] This scaling of the variance of the intensities differs from our previous publication (Ref. [19]), which contained an error in the derivation of the average intensities and sum rule.