STRONGLY PROXIMAL SUBSPACES OF FINITE CODIMENSION IN C(K)

BY

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Abstract. We characterize strongly proximinal subspaces of finite codimension in C(K) spaces. We give two applications of our results. First, we show that the metric projection on a strongly proximinal subspace of finite codimension in C(K) is Hausdorff metric continuous. Second, strong proximinality is a transitive relation for finite-codimensional subspaces of C(K).

1. Introduction. Let X be a Banach space and Y a closed subspace of X. The metric projection of X onto Y is the set-valued map defined by

\[ P_Y(x) = \{ y \in Y : \|x - y\| = \text{dist}(x, Y) \} \text{ for } x \in X. \]

If for every \( x \in X \), \( P_Y(x) \neq \emptyset \), we say that Y is a proximinal subspace of X.

For a Banach space X, we denote the closed unit ball and the unit sphere by \( B_X \) and \( S_X \) respectively. We restrict ourselves to real scalars. All subspaces we consider are assumed to be closed.

In [7] and [8], G. Godefroy, V. Indumathi and F. Lust-Piquard studied the following stronger version of proximinality.

**Definition 1.1.** Let Y be a closed subspace in a Banach space X and \( x \in X \). For \( \delta > 0 \), consider the set

\[ P_Y(x, \delta) = \{ y \in Y : \|x - y\| < d(x, Y) + \delta \}. \]

A proximinal subspace Y is said to be strongly proximinal at \( x \in X \) if given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y. \]

Necessary and sufficient conditions for strong proximinality of a finite-codimensional subspace Y in a Banach space X are given in [7]. To describe those results we need the notions of SSD-points and QP-points.

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Definition 1.2.

(a) Let \(X\) be a Banach space. The norm \(\|\cdot\|\) is said to be \textit{strongly subdifferentiable} (for short SSD) at \(x \in X\) if the one-sided limit

\[
\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}
\]

exists uniformly for \(h \in S_X\).

We say that \(x\) is an \textit{SSD-point} of \(X\) if the norm is SSD at \(x\). Recall that the \textit{duality map} \(J_X^*\) of \(X\) is defined as

\[
J_X^*(x) = \{g \in B(X^*) : g(x) = \|x\|\} \quad \text{for } x \in X.
\]

In [4], it was shown that \(x\) is an SSD-point if and only if the duality map \(J_X^*\) is (norm-norm) upper semicontinuous at \(x\), that is, for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[
J_X^*(z) \subseteq J_X^*(x) + \varepsilon B_{X^*} \quad \text{if } \|z - x\| < \delta, \|z\| = \|x\|.
\]

(b) We say that \(x\) is a \textit{QP-point} of \(X\) if there exists \(\delta > 0\) such that

\[
J_X^*(z) \subseteq J_X^*(x) \quad \text{if } \|z - x\| < \delta, \|z\| = \|x\|.
\]

It was shown in [7, Lemma 3.3] that QP-points are SSD-points but the converse is not true.

The following two propositions describe the connections between strongly proximinal subspaces of finite codimension and QP- and SSD-points.

**Proposition 1.3 ([7]).** Let \(Y\) be a finite-codimensional subspace of a Banach space \(X\). If \(Y\) is strongly proximinal then \(Y^\perp\) is contained in the set of SSD-points of \(X^*\).

It remains an open question if the converse of Proposition 1.3 is true. However, we have the following result. Recall that a finite-dimensional Banach space \(E\) is called \textit{polyhedral} if \(B_E\) has finitely many extreme points.

**Proposition 1.4 ([7]).** Let \(Y\) be a finite-codimensional subspace of a Banach space \(X\) such that \(Y^\perp\) is contained in the set of QP-points of \(X^*\). Then \(Y^\perp\) is polyhedral and \(Y\) is strongly proximinal.

In Theorem 2.1 of Section 2 we show that in \(C(K)^*\), \(K\) a compact Hausdorff space, SSD-points and QP-points coincide and they are precisely the finitely supported measures on \(K\). As a corollary, a finite-codimensional subspace \(Y\) of \(C(K)^*\) is strongly proximinal if and only if \(Y^\perp\) is contained in the set of SSD-points of \(C(K)^*\).

Section 3 contains two applications of our results from Section 2. The first one is a continuity property of the metric projection \(P_Y\), where \(Y\) is a strongly proximinal subspace of finite codimension in \(C(K)\). We will need the following definitions.
Definition 1.5. Suppose $Y$ is a proximinal subspace of a Banach space $X$.

(a) $P_Y$ is called *lower semicontinuous* at $x \in X$ if given $\varepsilon > 0$ and $y_0 \in P_Y(x)$, there exists $\delta > 0$ such that for $z$ satisfying $\|z - x\| < \delta$, one can find $y \in P_Y(z)$ such that $\|y - y_0\| < \varepsilon$.

(b) $P_Y$ is called *lower Hausdorff semicontinuous* (henceforth lHsc) at $x$ if it is lower semicontinuous at $x$ and the $\delta$ in (a) above can be chosen independent of $y_0 \in P_Y(x)$. Equivalently, $P_Y$ is lHsc at $x$ if for any $x_n \to x$,

$$\sup\{d(y, P_Y(x_n)) : y \in P_Y(x)\} \to 0.$$ 

(c) $P_Y$ is called *upper Hausdorff semicontinuous* (henceforth uHsc) at $x$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $z$ satisfying $\|z - x\| < \delta$ we have $P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y$.

(d) $P_Y$ is called *Hausdorff metric continuous* at $x$ if it is continuous as a single-valued map from $X$ to $2^Y$ with respect to the Hausdorff metric defined by

$$h(A, B) = \max\{\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A)\}, \quad A, B \in 2^Y.$$ 

Remark 1.6.

(a) It is a simple consequence of the definition that if $Y$ is a strongly proximinal subspace then $P_Y$ is uHsc.

(b) If $Y$ is proximinal in $X$, then $P_Y$ is Hausdorff metric continuous if and only if $P_Y$ is both lHsc and uHsc.

(c) Sometimes in the literature (see [2, 11]) lower Hausdorff semicontinuity is referred to as strong lower semicontinuity.

In [11] it was shown that if $X \subseteq c_0$ and $Y \subseteq X$ is a strongly proximinal subspace of finite codimension in $X$, then $P_Y$ is Hausdorff metric continuous. A more general result was obtained in [2], where the authors showed that if $X$ is a Banach space with property ($*$) (see [2, 3] for the definition) and $Y \subseteq X$ is a proximinal subspace of finite codimension, then $P_Y$ is lHsc. By [2], every separable polyhedral space has a renorming with property ($*$). In particular, if $1 \leq \alpha < \omega_1$ is a countable ordinal then the space $C(\omega^\alpha)$ is an $\ell_1$-predual and hence isomorphically polyhedral space. Thus $C(\omega^\alpha)$ has a renorming with property ($*$) (see [5]).

As a first application of our results, we show in Section 3 that if $Y$ is a strongly proximinal subspace of finite codimension in $C(K)$, then $P_Y$ is Hausdorff metric continuous.

A second application is to show that the relation of being a strongly proximinal subspace is transitive for finite-codimensional subspaces in $C(K)$. In particular, we show that if $Y$ and $M$ are finite-codimensional subspaces
in $C(K)$ such that $Y \subseteq M \subseteq X$, $Y$ is strongly proximinal in $M$ and $M$ is strongly proximinal in $C(K)$, then $Y$ is strongly proximinal in $C(K)$. A similar result for proximinal subspaces of finite codimension in subspaces of $c_0$ was established in [10]. However, in [1], the authors constructed an example to show that the transitivity of proximinal subspaces of finite codimension in $C(K)$ fails in general.

2. Strongly proximinal subspaces of $C(K)$. The following theorem is our main result in this section.

**Theorem 2.1.** Let $\mu \in C(K)^*$ with $\|\mu\| = 1$. Then the following assertions are equivalent.

(a) $\mu$ is finitely supported.
(b) $\mu$ is a QP-point.
(c) $\mu$ is an SSD-point.

**Proof.** (a)$\Rightarrow$(b). We write $\mu = \sum_{i=1}^{n} \alpha_i \delta_{k_i}$ where $k_i \in K$ and $\sum_{i=1}^{n} |\alpha_i| = 1$. If $F \in J_{C(K)^*}(\mu)$ then $F(\mu) = \sum_{i=1}^{n} \alpha_i \delta_{k_i}(F) = 1 = \sum_{i=1}^{n} |\alpha_i|$. It follows that $F \in J_{C(K)^*}(\mu)$ if and only if $\delta_{k_i}(F) = \text{sign}(\alpha_i), i = 1, \ldots, n$.

Let $\nu \in S_{C(K)}$, be such that $\|\mu - \nu\| < \varepsilon$ where $\varepsilon = \min\{|\alpha_i| : 1 \leq i \leq n\}$. Then $k_1, \ldots, k_n$ are atoms of $\nu$ and $\text{sign}(\alpha_i) = \text{sign}(\nu(k_i))$.

Now let $G \in J_{C(K)^*}(\nu)$. We claim $\delta_{k_i}(G) = \text{sign}(\nu(k_i)) = \text{sign}(\alpha_i), i = 1, \ldots, n$. Indeed, assume $\delta_{k_i}(G) \neq \text{sign}(\nu(k_i))$ for some $i$. We can write $\nu = \sum_{i=1}^{n} \nu(k_i)\delta_{k_i} + \nu_1$ where $\nu_1 = \nu_{|K\setminus\{k_1, \ldots, k_n\}}$. But then $G(\nu) = \sum_{i=1}^{n} \nu(k_i)|\delta_{k_i}(G)| + G(\nu_1) < \sum_{i=1}^{n} |\nu(k_i)||\delta_{k_i}(G)| + |G(\nu_1)| = 1$. This contradicts $G \in J_{C(K)^*}(\nu)$. Thus $\delta_{k_i}(G) = \text{sign}(\alpha_i), i = 1, \ldots, n$, and hence $G \in J_{C(K)^*}(\mu)$. This proves $\mu$ is a QP-point.

(b)$\Rightarrow$(c). Follows from [7, Lemma 3.3].

(c)$\Rightarrow$(a). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of $\mu$. Then $|\mu| = \mu^+ + \mu^-$ is a probability measure and $\text{supp}(|\mu|) = \text{supp}(\mu)$. It is now straightforward to verify that $|\mu|$ is an SSD-point if and only if $\mu$ is. Thus without loss of generality, we may assume that $\mu$ is a probability measure.

We first note that if $f \in J_{C(K)}(\mu)$ then $f = 1$ on $\text{supp}(\mu)$.

Suppose $\text{supp}(\mu)$ is not finite. By regularity of $\mu$ we can find a decreasing sequence $\{V_n\}_{n \geq 1}$ of open sets in $K$ such that $\mu(V_n) > 0$ and $\lim_n \mu(V_n) = 0$. Let $\mu_n = \mu_{K\setminus V_n}/\|\mu_{K\setminus V_n}\|$. Then $\|\mu_n\| = 1$ and $\|\mu - \mu_n\| = \mu(V_n)/\|\mu_{K\setminus V_n}\| \to 0$.

Fix $x_n \in V_n \cap \text{supp}(\mu)$. Let $f_n : K \to [0,1]$ be such that $f_n(x_n) = 0$ and $f_n = 1$ on $K \setminus V_n$. Then $f_n \in J_{C(K)}(\mu_n)$ but $\|f_n - f\| = 1$. This contradicts $\mu$ being an SSD-point. ■
Remark 2.2.

(a) Recall that $x^* \in S_{X^*}$ is called a point of (norm-weak) upper semicontinuity of the preduality map of $X^*$ if given any weak neighborhood $V$ of the origin in $B_{X^*}$, there exists $\delta > 0$ such that if $y^* \in S_{X^*}$ satisfies $\|x^* - y^*\| < \delta$, then $J_X(y^*) \subseteq J_X(x^*) + V$. It was proved in [6] that an SSD-point of a dual space attains its norm and is a point of (norm-weak) upper semicontinuity of the preduality map. Now if $\mu \in C(K)^*$ with $\text{supp}(\mu)$ uncountable then we can actually show that $\mu$ is not even point of (norm-weak) upper semicontinuity of the preduality map of $C(K)^*$. To see this, first note that $\mu$ is not a purely atomic measure. Thus there exists $k \in \text{supp}(\mu)$ such that $\mu(k) = 0$. Let $f_n \in S_{C(K)}$ be such that $f_n(k) = 0$ for all $n$ and $\mu(f_n) \to 1$. Set $V = \{g \in B_{C(K)} : g(k) < 1/2\}$. We have $f_n \notin J_{C(K)}(\mu) + V$ for all $n$. By [6, Lemma 2.1], the preduality map is not (norm-weak) upper semicontinuous at $\mu$.

(b) If $K$ is countably compact, it is not true in general that if $\mu$ is a point of (norm-weak) upper semicontinuity of the preduality map of $C(K)^*$ then $\mu$ is a finitely supported measure. For example, consider the space of convergent sequences $c$ and $\mu = (1/2^n)_{n=1}^{\infty} \in \ell_1$. Then $J_c(\mu) = J_{\ell_\infty}(\mu) = (1, 1, \ldots)$ and by [6, Theorem 2.3], $\mu$ is a point of (norm-weak) upper semicontinuity of the preduality map of $\ell_1$.

We can now describe strongly proximinal subspaces of finite codimension in $C(K)$.

Corollary 2.3. Let $Y$ be a finite-codimensional subspace in $C(K)$. Then the following assertions are equivalent.

(a) $Y$ is strongly proximinal.

(b) Every closed subspace $Z$ of finite codimension with $Y \subseteq Z \subseteq X$ is strongly proximinal.

(c) Every hyperplane containing $Y$ is strongly proximinal.

(d) $Y^\perp \subseteq \{f \in X^* : f$ is an SSD-point of $X^*\} = \{f \in X^* : f$ is an QP-point of $X^*\}$.

Proof. (a)$\Rightarrow$(b). Since $Y$ is strongly proximinal, $Y^\perp \subseteq$ SSD-points of $X^*$ by Proposition 1.3. By the equivalence of (b) and (c) in Theorem 2.1, $Y^\perp \subseteq$ QP-points of $X^*$. Since $Z^\perp \subseteq Y^\perp$ the result follows from Proposition 1.4.

(b)$\Rightarrow$(c). Follows trivially.

(c)$\Rightarrow$(d). Follows from Proposition 1.3 and the equivalence of (b) and (c) in Theorem 2.1.

(d)$\Rightarrow$(a). Follows from Proposition 1.4. \qed
3. Applications. Our first application is the following result on continuity of the metric projection.

**Theorem 3.1.** Let $Y$ be a strongly proximinal subspace of finite codimension in $C(K)$. Then $P_Y$ is Hausdorff metric continuous.

We need to fix some notation for which we closely follow [11].

Let $Y$ be a proximinal subspace of codimension $n$ in a Banach space $X$. For $x \in X$, set $Q_Y(x) = x - P_Y(x)$. For $\{f_1,\ldots, f_k\} \subseteq Y^\perp$ with $1 \leq k \leq n$ we define

$$Q_{f_1,\ldots, f_k}(x) = \bigcap_{i=1}^k \{z \in B_X : f_i(z) = f_i(x)\}.$$

Note that $Q_Y(x) \subseteq Q_{f_1,\ldots, f_k}(x)$, and if $\{f_1,\ldots, f_n\}$ is a basis of $Y^\perp$ then $Q_Y(x) = Q_{f_1,\ldots, f_n}(x)$.

Let $E$ be an $n$-dimensional polyhedral space and $\Phi \in S_E$. Consider the sets

$$A_\Phi = \{f \in B_{E^\ast} : f(\Phi) = 1\}, \quad C_\Phi = \{f \in \text{ext } B_{E^\ast} : f(\Phi) = 1\}.$$ 

Then $C_\Phi$ is a finite set and $\bigcap_{f \in A_\Phi} J_E(f) = \bigcap_{f \in C_\Phi} J_E(f)$. Let $\{f_1,\ldots, f_k\}$, $1 \leq k \leq n$, be a maximal linearly independent subset of $C_\Phi$. Then $\bigcap_{i=1}^k J_E(f_i)$ is a minimal face of $B_E$ containing $x$.

Let $D(Y) = \{x \in X : \text{dist}(x, Y) = 1\}$.

**Definition 3.2.** Suppose $x \in D(Y)$ and there exists a maximal independent set $\{f_1,\ldots, f_k\} \subseteq S_Y^\perp$, $1 \leq k \leq n$, such that

$$Q_{f_1,\ldots, f_k}(x) = \bigcap_{i=1}^k J_X(f_i).$$

Then we say $x$ is a $k$-corner point with respect to $\{f_1,\ldots, f_k\}$.

We summarize the above discussion in the following lemma. Note that if $Y^\perp$ is polyhedral, then so is $X/Y$.

**Lemma 3.3.** Let $Y$ be a proximinal subspace of codimension $n$ in $X$. Suppose $Y^\perp$ is polyhedral and $x \in D(Y)$. Then there exists a maximal independent set $\{f_1,\ldots, f_k\} \subseteq S_Y^\perp$, $1 \leq k \leq n$, such that $x$ is a $k$-corner point with respect to $\{f_1,\ldots, f_k\}$.

The following result was proved in [11]. Though it is stated for Hausdorff metric continuity of $P_Y$, it is evident from the proof given there that it is valid for both $lHsc$ and $uHsc$.

**Theorem 3.4 ([11, Theorem 3.10]).** Let $X$ be a Banach space and $Y$ a proximinal subspace of codimension $n$ in $X$ with $Y^\perp$ polyhedral. Assume that whenever $x \in D(Y)$ is a $k$-corner point with respect to a set of linearly...
independent functionals \( \{f_1, \ldots, f_k\} \) in \( Y^\perp \) for some positive integer \( 1 \leq k \leq n \), then the map \( Q_{f_1, \ldots, f_k} \) is Hausdorff metric continuous at \( x \). Then the metric projection \( P_Y \) is Hausdorff metric continuous on \( X \).

We are now ready to prove Theorem 3.1. We separate out the following simple lemma from the proof of Theorem 2.1.

**Lemma 3.5.** Suppose \( \mu \in S_{C(K)^*} \) is given by \( \mu = \sum_{i=1}^n \alpha_i \delta_{k_i} \).

(a) If \( f \in JC(K)(\mu) \) then \( f(k_i) = \text{sign}(\alpha_i), \ i = 1, \ldots, n \).

(b) If \( f_n \in BC(K) \) are such that \( \mu(f_n) \to 1 \) then \( f_n(k_i) \to \text{sign}(\alpha_i), \ i = 1, \ldots, n \).

**Proof of Theorem 3.1.** By Remark 1.6(a), \( P_Y \) is uHsc. Thus by Remark 1.6(b) we only need to show that \( P_Y \) is lHsc. Also, scaling by the norm of \( z \in X \), it is evident that \( P_Y \) is lHsc at \( z \) if and only if \( P_Y \) is lHsc at every \( x \in D(Y) \).

Let \( x \in D(Y) \). Since \( Y \) is strongly proximinal it follows from Corollary 2.3 that \( Y^\perp \subseteq \text{QP-points of } C(K)^* \). By Proposition 1.4, \( Y^\perp \) is polyhedral. Thus Lemma 3.3 implies there exists a maximal linearly independent set \( \{\mu_1, \ldots, \mu_m\} \subseteq SY^\perp, \ 1 \leq m \leq n \), such that \( x \) is an \( m \)-corner point with respect to \( \{\mu_1, \ldots, \mu_m\} \). By Theorem 3.4, it is enough to prove that \( Q_{\mu_1, \ldots, \mu_m} \) is lHsc at \( x \).

By Theorem 2.1, for each \( j = 1, \ldots, m \), \( \text{supp}(\mu_j) \) is a finite set, say \( \bigcup_{j=1}^m \text{supp}(\mu_j) = \{k_1, \ldots, k_I\} \). For each \( i = 1, \ldots, I \), we choose a neighborhood \( \theta_i \) of \( k_i \) such that \( \theta_i \cap \theta_j = \emptyset, \ i \neq j \).

Let \( \varepsilon > 0 \) be given and \( x_n \in D(Y) \) with \( x_n \to x \). Suppose \( y \in Q_{\mu_1, \ldots, \mu_m}(x) \). We need to produce an \( n_0 \) such that for \( n \geq n_0 \), there exists \( v_n \in Q_{\mu_1, \ldots, \mu_m}(x_n) \) such that \( \|v_n - y\| < \varepsilon \).

Since \( Q_{\mu_1, \ldots, \mu_m}(x) = \bigcap_{i=1}^m J_{C(K)}(x) \), it follows from Lemma 3.5 that if \( k_i \in \text{supp}(\mu_j) \) for some \( j = 1, \ldots, m \) then \( y(k_i) = \text{sign}(\mu_j(k_i)) \).

Fix \( z_n \in Q_{\mu_1, \ldots, \mu_m}(x_n) \). Since \( x_n \to x \), we have \( \mu_j(z_n) \to 1, \ j = 1, \ldots, m \). Thus by Lemma 3.5, there exists \( n_0 \) such that \( |z_n(k_i) - \text{sign}(\mu_j(k_i))| < \varepsilon/2 \) whenever \( k_i \in \text{supp}(\mu_j) \).

We define further neighborhoods \( B_i \) of \( k_i \) as follows:

\[
B_i = \begin{cases} 
\theta_i \cap \{s \in K : y(s) > 1 - \varepsilon\} & \text{if } y(k_i) = 1, \\
\theta_i \cap \{s \in K : y(s) < -1 + \varepsilon\} & \text{if } y(k_i) = -1.
\end{cases}
\]

Let \( v'_n \in BC(K) \) be such that

\[
v'_n(k) = \begin{cases} 
z_n(k) & \text{if } k \in \{k_1, \ldots, k_I\}, \\
y(k) & \text{if } k \in K \setminus \bigcup_{i=1}^n B_i.
\end{cases}
\]

Then set

\[
v''_n = v'_n \wedge (y + \varepsilon) \quad \text{and} \quad v_n = v''_n \vee (y - \varepsilon).
\]
Then \( v_n(k_i) = x_n(k_i) \) and by construction \( v_n \in Q_{\mu_1, \ldots, \mu_n}(x_n) \). Also, it is straightforward to verify that \( \|v_n - y\| < \varepsilon \) for \( n > n_0 \). This completes the proof.

**Remark 3.6.** Recall that a proximinal subspace \( Y \) of \( X \) is called a Chebyshev subspace if \( P_Y(x) \) is single-valued for each \( x \in X \). Clearly, if \( Y \) is a Chebyshev subspace of \( X \) then the Hausdorff metric continuity of \( P_Y \) amounts to the continuity of \( P_Y \) in the usual sense. R. R. Phelps in [13] constructed a Chebyshev subspace \( Y \) in \( C(K) \) with \( \text{codim} Y = 2 \) for some extremally disconnected compact Hausdorff space \( K \). However, a result of P. D. Morris (see [12, Theorem 4]) states that if \( Y \) is a Chebyshev subspace of finite codimension greater than one in \( C(K) \), then \( P_Y \) cannot be continuous. Thus the strong proximinality condition in Theorem 3.1 cannot be replaced by \( Y \) being proximinal (even Chebyshev) in \( C(K) \).

Our next application of Theorem 2.1 is to show that strong proximinality is transitive for finite-codimensional subspaces of \( C(K) \). We begin by the following lemma which is a simple consequence of the definition of an SSD-point in terms of one-sided differentiability mentioned in the introduction.

**Lemma 3.7.** Let \( Y \) and \( Z \) be two closed subspaces of a Banach space \( X \) such that \( X = Y \oplus \ell_1 Z \) and \( x \in X \). Let \( x = y + z \) where \( y \in Y, z \in Z \). Then \( x \) is an SSD-point in \( X \) if and only if both \( y \) and \( z \) are SSD-points in the respective subspaces.

**Proposition 3.8.** Let \( Y \) be a strongly proximinal subspace of finite codimension in \( C(K) \). Suppose \( \mu \in C(K)^* \) attains its norm on \( Y \) and \( \mu|_Y = F \) is an SSD-point of \( Y^* \). Then \( \mu \) is an SSD-point of \( C(K)^* \).

**Proof.** Suppose the codimension of \( Y \) is \( n \) and \( \mu_1, \ldots, \mu_n \) span \( Y^\perp \). By Theorem 2.1, \( \text{supp}(\mu_i) \) is a finite set for every \( i, 1 \leq i \leq n \). So \( D = \bigcup_{i=1}^n \text{supp}(\mu_i) \) is finite.

Let \( J = \{ h \in C(K) : h|_D = 0 \} \). Then \( J \) is an \( M \)-ideal of finite codimension in \( C(K) \). Observe that \( J \subseteq Y \subseteq C(K) \) and by [9, Corollary I.1.19], \( J \) is an \( M \)-ideal in \( Y \) as well.

By [9, Example I.1.4(a)], there exists a subspace \( N \subseteq C(K)^* \) isometric to \( J^* \) such that \( C(K)^* = J^\perp \oplus \ell_1 N \). Similarly, we can write \( Y^* = J^\perp_1 \oplus \ell_1 M \) where \( J^\perp_1 \) is \( J^\perp \) for \( J \) considered as a subspace of \( Y \), and \( M \subseteq Y^* \) is isometric to \( J^* \).

We write \( F = F_1 + F_2 \), where \( F_1 \in J^\perp_1, F_2 \in M \). Since \( F \) is an SSD-point of \( Y^* \), by Lemma 3.7, \( F_1, F_2 \) are SSD-points in the respective summands. Since SSD-points of a dual Banach space are norm attaining (see [6]), it follows that \( F_2 \) attains its norm on \( J \).
Now write \( \mu = \mu_1 + \mu_2 \) where \( \mu_1 \in J^\perp \), \( \mu_2 \in N \). The support of \( \mu_1 \) is contained in \( D \), and thus \( \mu_1 \) is finitely supported. Hence by Theorem 2.1, \( \mu_1 \) is an SSD-point of \( J^\perp \). It remains to show that \( \mu_2 \) is an SSD-point in \( N \). Without loss of generality we assume \( \| \mu_2 \| = 1 \). Since \( \mu_2 |_J = F_2 |_J \), \( \mu_2 |_J \) attains its norm on \( J \).

If \( \mu_2 \) is not an SSD-point of \( N \), then there exist \( \varepsilon > 0 \) and \( \nu_n \in C(K)^* \) with \( \| \nu_n \| = \| \nu_n |_J \| = 1 \) and \( h_n \in J \) such that \( \| \nu_n - \mu_2 \| \to 0 \), \( \nu_n(h_n) = 1 \) but \( \text{dist}(h_n, \{ x \in J : \mu_2(x) = 1 \}) \geq \varepsilon \) for all \( n \).

But \( \mu_2(x) = F_2(x) \) for all \( x \in J \) and \( \| \nu_n |_Y - F_2 \| \to 0 \). Thus \( \text{dist}(h_n, \{ x \in J : F_2(x) = 1 \}) \geq \varepsilon \) for all \( n \). This contradicts \( F_2 \) being an SSD-point in \( M \).

**Corollary 3.9.** Let \( Y \subseteq C(K) \) be a subspace of finite codimension and \( M \) a subspace of \( C(K) \) such that \( Y \subseteq M \subseteq C(K) \). If \( Y \) is strongly proximinal in \( M \) and \( M \) is strongly proximinal in \( C(K) \), then \( Y \) is strongly proximinal in \( C(K) \). In other words, strong proximinality is transitive for finite-codimensional subspaces of \( C(K) \).

**Proof.** Considering \( Y \) as a subspace of \( M \), by Proposition 1.3, we know that \( Y^\perp \) is contained in the set of SSD-points of \( M^* \). Since \( M \) is strongly proximinal, by Proposition 3.8, \( Y^\perp \) is contained in the set of SSD-points of \( C(K)^* \). The conclusion follows from Corollary 2.3.

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**REFERENCES**


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