Department of Physics, IIT-Kanpur

Instructors: TKG & DpC PHY103 (2014-15, II-SEM) Problem Set 1

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1. a) The plot of the following function looks like a hill on the xy plane:

$$h(x,y) = \exp[(2xy - 3x^2 - 4y^2 - 18x + 28y - 5)/60].$$

   i) Where is the top of the hill located? How high (in meter or in feet as you wish) is the hill? ii) In what direction is the slope steepest at the point (1,1)? iii) What is the rate of change of $h(x,y)$ at the point (1,1) in the direction $\mathbf{n} = (\hat{x}x + \hat{y}y)$?

   b) Given a vector $\mathbf{R} = \hat{x}(x - x') + \hat{y}(y - y') + \hat{z}(z - z')$, show that $\boldsymbol{\nabla}_{\mathbf{r}} R$ (gradient with respect to x, y and z coordinates) is a unit vector parallel to $\mathbf{R}$. Here, $R = |\mathbf{R}|$.

2. Find the scaler function $\phi(x,y,z)$ whose gradient is $\mathbf{A} = \boldsymbol{\nabla}\phi = (2xy + z^3)\hat{x} + x^2\hat{y} + 3xz^2\hat{z}$.

3. a) Evaluate $\boldsymbol{\nabla}\phi$ for $\phi = \ln|\mathbf{r}|$ and $\phi = 1/|\mathbf{r}|$. Calculate $\boldsymbol{\nabla} \cdot \mathbf{E}$ for $\mathbf{E} = \mathbf{r}/r^n$. Here $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $n = 0, \pm 1, \pm 2, ...$ Comment on $n = 3$ case.

   b) Repeat the same problem for two-dimensional case.

4. a) Suppose $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\boldsymbol{\nabla} \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$. Show that

$$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2} \text{ and } \quad \nabla^2 \mathbf{B} = \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

   b) Consider $\mathbf{A}(x,y,z) = (-y\hat{x} + x\hat{y})/2$ and the arbitrary scaler function $f(x,y,z)$. Show that

$$(\boldsymbol{\nabla} + \mathbf{A}) \times (\boldsymbol{\nabla} + \mathbf{A}) f = (\boldsymbol{\nabla} \times \mathbf{A}) f = f\hat{z}$$

   and

$$(\boldsymbol{\nabla} + \mathbf{A}) \cdot (\boldsymbol{\nabla} + \mathbf{A}) f = \nabla^2 f + \Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big) f + \frac{(x^2 + y^2)}{4} f.$$

Exercises

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1. Show that $\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f\boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} f$ for any scaler function $f(x,y,z)$.

2. Find a unit normal vector to the surface $2xy^2z - x^2yz^2 = 1$ at the point $(1,1,1)$.

3. a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, obtain the Cartesian components of $(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$.

   b) Evaluate $(\hat{r} \cdot \boldsymbol{\nabla})\hat{r}$, where

$$\hat{r} = \frac{\hat{x}x + \hat{y}y + \hat{z}z}{\sqrt{x^2 + y^2 + z^2}}.$$

   c) Obtain the Cartesian components of the operator $\mathbf{L} = \mathbf{r} \times \boldsymbol{\nabla}$.

4. Show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0$ for any scaler function $\phi(x,y,z)$ and $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$ for any vector function $\mathbf{A}(x,y,z)$.

5. Calculate the Laplacian of the following functions: i) $\phi(x,y,z) = \sin(ax)\sin(by)\sin(cz)$ with a, b and c are constants ii) $\phi(x,y,z) = e^{-5z}\sin(4y)\cos(3z)$.