Department of Physics, IIT-Kanpur

Instructors: TKG & DpC | PHY103A (2014-15, II-SEM) | Problem Set 2

1. Calculate the divergence for $\mathbf{V} = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi}$. Check the divergence theorem for the vector function over the volume of an inverted hemisphere of radius R, resting on the xy plane and centered at the origin.

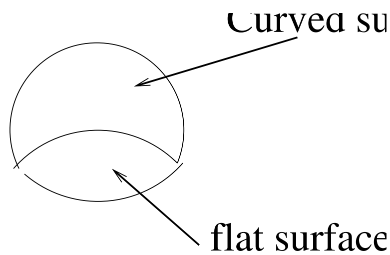

Figure 1:

2. Calculate the line integral of $\mathbf{V} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$ around the path shown in figure 2. The points are labelled by their Cartesian coordinates. Do it in spherical polar coordinates. Verify the Stokes' theorem.

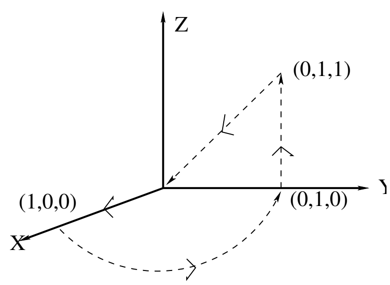

Figure 2:

3. Evaluate the integral
$$J = \int_V e^{-r}\Big[\boldsymbol{\nabla}\cdot\frac{\hat{r}}{r^2}\Big]d\tau,$$
where V is a sphere of radius R centered at the origin: a) using the divergence theorem and performing the resulting surface integral and b) using Dirac delta function.

4. Evaluate the divergence and curl everywhere for a piece-wise constant vector function (non-zero inside a cylindrical volume and zero outside):
$$\mathbf{A} = A_0\theta(s_0 - s)\theta(z)\theta(z_0 - z)\,\hat{z}.$$
Here, A_0, s_0, z_0 are constants and s and z are cylindrical variables.

1. Find the vector function $\mathbf{A}(x, y, z)$ whose curl is $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = B\,\hat{z}$.

2. Consider $\mathbf{V} = y^2\,\hat{\imath} + 2xy\,\hat{\jmath}$. Calculate $\int_a^b \mathbf{V} \cdot d\mathbf{r}$ with $a = (1,1)$ and $b = (2,2)$, using two different paths: i) (1,1) to (2,1) to (2,2) and ii) (1,1) to (2,2) (straight line). Evaluate $\oint \mathbf{V} \cdot d\mathbf{r}$ going from a to a via b anti-clockwise. Check if the vector $\mathbf{V}$ can be written as $\mathbf{V} = \boldsymbol{\nabla}\phi$.

3. a) A vector function is given by $\mathbf{A} = (-y\hat{\imath} + x\hat{\jmath})/2$. Verify the Stokes' theorem using northern/southern hemisphere whose base is on the xy plane with the origin at the center of the base. You will see that the result does not depend on the surface you are considering.

   b) Calculate the line integral $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{A} \cdot d\mathbf{r}$ and show that its value is the area of a triangle spanned by the vectors $\mathbf{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath}$ and $\mathbf{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath}$.

4. a) Evaluate curl of the vector function $\mathbf{B} = \hat{\phi}/s$ everywhere. Here ϕ and s are cylindrical variables.

   b) Evaluate the surface integral

$$J = \int_S e^{-s}\Big[\boldsymbol{\nabla} \times \frac{\hat{\phi}}{s}\Big] \cdot d\mathbf{a},$$

   where S is the surface of a circular disk of radius R, centered at origin: i) using the Stokes' theorem and performing the line integral and ii) using Dirac delta function.