Some Open Problems

1. A sequence of *Mersenne numbers* can be generated by the recursive function $a_{k+1} := 2a_k + 1$, for all $k \in \mathbb{N}$, with $a_0 = 0$. We obtain the sequence

$$1, 3, 7, 15, 31, 63, 127, \ldots.$$

Note that each term here is of the form $2^k - 1$, for $k \in \mathbb{N}$. Any prime in this sequence is called a *Mersenne prime*, i.e., a prime having the form $2^k - 1$ for some $k \in \mathbb{N}$. Prove or disprove that there are infinitely many Mersenne primes!

Observations: Note that $15 = 2^4 - 1$ and $63 = 2^6 - 1$ are not Mersenne primes. If k is composite then $2^k - 1$ is also composite (not prime). If k is composite then $k = \ell m$. Then

$$2^k - 1 = (2^\ell)^m - 1 = (2^\ell - 1)((2^\ell)^{m-1} + (2^\ell)^{m-2} + \ldots + 2^\ell + 1).$$

Thus, $2^\ell - 1$ divides $2^k - 1$. If $2^k - 1$ is a prime then either $2^\ell - 1 = 2^k - 1$ or $2^\ell - 1 = 1$. This implies that $k = \ell$ ($m = 1$) or $\ell = 1$ contradicting the fact that k is composite. Therefore, a necessary condition for a Mersenne number $2^k - 1$ to be prime is k should be prime. This is not a sufficient condition, for instance, 11 is prime but $2^{11} - 1 = 2047 = 23 \times 89$ is not a prime.

More generally, if $a^k - 1$ is prime then either $k = 1$ or $a = 2$. Since, by geometric sum,

$$a^k - 1 = (a - 1)(a^{k-1} + a^{k-2} + \ldots + a + 1)$$

we get $a^k - 1$ is divisible by $a - 1$. Since $a^k - 1$ is prime either $a - 1 = a^k - 1$ or $a - 1 = 1$. This implies either $a = a^k$ or $a = 2$. Since $a^k - 1$ is not prime for $a = 1$, we have either $k = 1$ or $a = 2$. Therefore primes can occur only as $a^k - 1$ only for $a = 2$.

2. A *perfect number* is a number which is the sum of all its factor excluding itself. For example, 6 is the first perfect number because $6 = 1 + 2 + 3$. The first four perfect numbers known since ancient times are $6, 28, 496, 8128$. Even with modern computers we know only 47 perfect numbers. There are two open questions on perfect numbers:

   1. Are there infinitely many perfect numbers?

   Observations: If the infinitude of Mersenne primes are proved then it would imply the infinitude of perfect numbers due to the following result known as far back as Euclid: If $2^k - 1$ is prime then $2^{k-1}(2^k - 1)$ is a perfect number.

   2. Are there odd perfect numbers?

Observations: All the known perfect numbers are even. Further, the question of existence of odd perfect numbers naturally appears after the Euler proved a necessary condition of even perfect number: If n is an even perfect number then $n = 2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime.

The result by Euler is a converse to Euclid's result stated above if there were no odd perfect numbers.

3. Pick any positive integer, say a_0. Construct the sequence $\{a_n\}$ as

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

For instance, if we choose $a_0 = 1$, we get the sequence

$$1, 4, 2, 1, \text{ it starts repeating.}$$

Thus, the sequence reaches 1 after which it starts repeating in the triplet $1, 4, 2$. The *Collatz conjecture* or $3x + 1$ conjecture is that if you start with any positive number a_0 the sequence will eventually take the value 1. Prove or disprove this statement!

Observations: For instance, if we choose $a_0 = 2^k$ for some $k \in \mathbb{N}$, then it can be easily seen that $a_k = 1$. Further, for any a_0, the sequence will attain 1 if and only if it attains the value 4^k, for some k.

4. A pair of primes is said to be *twin primes* if they differ by exactly two, i.e., the pair $\{p, p + 2\}$ where both are primes. Prove that there are infinitely many twin primes. More generally, prove that there are infinitely many pairs of prime $\{p, p + k\}$ for every *even* $k \in \mathbb{N}$.

Observations: In April 2013, Yitang Zhang proved that there is a $k < 7 \times 10^7$ such that the pair $\{p, p + k\}$ are infinitely many. Can k be as small as 2?

5. Prove or disprove the *Goldbach's conjecture* which states that every even integer greater than 2 can be expressed as the sum of two primes.

6. The *Fibonacci sequence* is given as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765 \ldots$$

where the n-th term $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$, is sum of the previous two terms. If we divide each F_n by 2 and write down the remainders, we get

$$0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots.$$

In other words, we have written the modulo 2 representaion of the Fibonacci sequence. Note that the resulting sequence is periodic (or repeating) of period three (after every third term). This period of the resulting periodic sequence is called *Pisano period.* For each n, the Pisano period of the Fibonacci sequence modulo n is

$$1, 3, 8, 6, 20, 24, 16, 12, 24, 60, 10, 24, 28, 48, 40, 24, 36, 24, 18, 60, 16, 30, 48, 24, 100, 84, 72, \ldots.$$

It is known that the resulting sequence of modulo n has Pisano period at most $n^2 - 1$, for $n \geq 2$. However, it is still an open problem to find a general formula for Pisano period in terms of n. A word of caution is that given a n it is *not* difficult to compute the Pisano period by modern means like cycle detection etc.

7. A prime number is said to be *Fibonacci prime* if it appears in the Fibonacci sequence. Prove or disprove that there are infinitely many Fibonacci primes.

8. An odd prime number p is called *regular* if p does not divide the numerator of the Bernoulli number B_k, for all even $k \leq p - 3$. Any odd prime which is not regular is called *irregular.* The odd primes $3, 5, 7, \ldots, 31$ are all regular primes. The first irregular prime is 37. It is known that there are infinitely many irregular primes. An unsolved problem is: are there infinitely many regular primes?

9. The *Riemann-zeta function* $\zeta : \mathbb{C} \to \mathbb{C}$ is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is trivial to note that the infinite series converges for all $s \in \mathbb{C}$ such that $\Re(s) > 1$. We extend the Riemann zeta function analytically to all of $\mathbb{C}$ with a simple pole at $s = 1$. It is known that $\zeta(s) = 0$ for all $s = -2k$ with $k \in \mathbb{N}$ which are called *trivial zeros.* Prove or disprove the Riemann hypothesis which states that the non-trivial zeros of $\zeta(s)$ are precisely the line $\Re(s) = 1/2$.

Observations: The Riemann zeta function is extended, for all s with non-positive real part, using the functional equation

$$\zeta(s) = 2^s(\pi)^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s).$$

with a simple pole at 1. The trivial zeroes are obtained by noting that $\zeta(-2k) = 0$, for all $k \in \mathbb{N}$, because $\sin(-2k\pi) = 0$. An alternate way to see trivial zeroes is from the relation $\zeta(-2k) = -\frac{B_{2k+1}}{2k+1}$, where B_{2k+1} are the odd indexed Bernoulli numbers which vanish.

10. The value of $\zeta(3)$ is called the *Apéry constant* and is an irrational. It is an open question whether Apéry constant is transcendental or not. Further, is $\zeta(2k+1)$, for all $k \in \mathbb{N}$, irrational? Are they transcendental too?

11. The Euler constant γ is defined as

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

It is an open question whether the Euler constant γ is irrational or rational. Is γ transcendental?

12. It is known that the set of Gaussian integers, $\mathbb{Z}[i]$, is a unique factorization domain (UFD). There are $m \in \mathbb{Z}$ such that $\mathbb{Z}[\sqrt{m}]$ is not a UFD. In fact, there are exactly nine negative integers, called *Heegner number* for which $\mathbb{Z}[\sqrt{m}]$ is a UFD. They are $-1, -2, -3, -7, -11, -19, -43, -67, -163$. It is not known whether there are finite or infinite positive integers m such that $\mathbb{Z}[\sqrt{m}]$ is a UFD.