The A_2 conjecture

The A_2 conjecture

w belongs to the A_p class, $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(y)\, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{-\frac{1}{p-1}}\, dy \right)^{p-1} < \infty$$

A **Calderón-Zygmund operator** with kernel K is a bounded operator in $L^2(\mathbb{R}^n)$ given by

$$T(f)(x) = \int K(x,y) f(y)\, dy \qquad x \notin \operatorname{supp}(f)$$

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \qquad x \neq y$$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C|x-x'|^\delta}{|x-y|^{n+\delta}}$$

for $|x - x'| \leq |x - y|/2$.

A_2 Conjecture and Theorem

Theorem

Let T a C-Z operator. There exists a constant $c(n,T)$ such that, for all weights $w \in A_2$,

$$\|T\|_{L^2(w)} \leq c(n,T)[w]_{A_2}.$$

- Beurling transform Petermichl and Volberg 2002
- Hilbert transform, Riesz transform Petermichl 2007-08
- Intermediate results : Lacey et al. Cruze Uribe et al. 2010, $\cdots$
- Solved by Hytönen 2010
- New proof by Andreï Lerner 2012

Buckley's Theorem for the maximal function

Theorem

For all weights $w \in A_p$,

$$\|M\|_{L^p(w)} \leq \frac{c(n)p}{p-1} [w]_{A_p}^{\frac{1}{p-1}}.$$

Step 1 : sufficient to do it for the dyadic analog.

Dyadic cubes and dyadic grids

Classical dyadic cubes : $Q = [2^{-k}j, 2^{-k}(j+1))^n$. We call $\ell(Q) = 2^{-k}$ its size (or scale), $\mathcal{D}$ the set of dyadic cubes.

- Dyadic cubes Q such that $\ell(Q) = 2^{-k}$ constitute a partition of $\mathbb{R}^n$.
- They are the disjoint union of 2^n dyadic cubes such that $\ell(Q) = 2^{-k-1}$.
- Let Q, Q' two dyadic cubes. Then $Q \subset Q'$ or $Q' \subset Q$, or Q and Q' are disjoint.

Lemma. *Let $\mathcal{L}$ a collection of dyadic cubes of bounded size. If $\mathcal{L}_{max}$ is the subcollection of maximal dyadic cubes, then each cube of $\mathcal{L}$ is contained in a cube of $\mathcal{L}_{max}$.*

Other dyadic grids

$\mathcal{D}_\alpha$, with $\alpha \in \{0, 1/3\}^n$, is the collection of $2^{-k}([0,1)^d + j + (-1)^k\alpha)$, with $k \in \mathbb{Z}$ et $j \in \mathbb{Z}^n$.

Lemma. *Each cube Q is contained in an interval Q' which belongs to one of the 2^n families $\mathcal{D}_\alpha$ and such that $\ell(Q') \leq 6\ell(Q)$.*

Maximal dyadic functions : Doob's Inequality

$$\lambda|\{M^{\mathcal{D}_\alpha} f > \lambda\}| \leq \int_{M^{\mathcal{D}_\alpha} f > \lambda} |f(x)| dx.$$

As a consequence, L^p inequality with constant $\frac{p}{p-1}$.
The proof is geometric and is valid for all Radon non negative measures.

Second step : dyadic inequalities

Let M_w the maximal (dyadic) operator for the measure wdx and M_σ for the measure σdx, with $\sigma = w^{-1}$. They are both bounded for the corresponding measure.

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q f &\leq [w]_{A_2}\frac{|Q|}{w(Q)}\Big(\frac{1}{\sigma(Q)}\int_Q f(x)\sigma^{-1}(x)\sigma(x)dx\Big) \\
&\leq [w]_{A_2}\frac{|Q|}{w(Q)}\inf_Q M_\sigma(f\sigma^{-1}) \\
&\leq [w]_{A_2}\frac{1}{w(Q)}\int_Q M_\sigma(f\sigma^{-1})dx \\
&\leq [w]_{A_2} M_w\left[w^{-1}\Big(M_\sigma(f\sigma^{-1})\Big)\right]
\end{aligned}
$$

. pause

Optimality of Buckley's Theorem

Luque, Pérez and Rela (to appear) :

Proposition

For fixed $p > 1$ let $\psi : [1, \infty) \to (0, \infty)$ an increasing function such that for all $w \in A_p$ and f positive,

$$\|Mf\|_{L^p(w)} \leq \psi([w]_{A_p})\|f\|_{L^p(w)}.$$

Then $\psi(t) \geq c(p)t^{\frac{1}{p-1}}$ for some constant $c(p) > 0$.

Proof for $p = 2$: we prove that $\psi(t) \geq t/4$ for $t \geq t_0$. For $1 < q < 2$ let $f \geq 0$ with norm 1 in L^q et telle que $\|Mf\|_q \geq 2\|M\|_q$.
We set

$$R(f) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(f)}{\|M\|_q^k}.$$

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$$f \leq R(f) \qquad \|R(f)\|_{L^q} \leq 2\,\|f\|_{L^q}, \qquad M(Rf) \leq 2Rf.$$

Take as a weight $w = (Rf)^{q-2}$.

$$\begin{aligned}
\|Mf\|_{L^q} &= \left(\int \left((Mf)(Rf)^{\frac{q}{2}-1} \right) \right)^q (Rf)^{q(1-\frac{q}{2})} dx \Big)^{\frac{1}{q}} \\
\|Mf\|_{L^q} &\leq \left(\int (Mf)^2 \, (Rf)^{q-2} dx \right)^{\frac{1}{2}} \left(\int (Rf)^q dx \right)^{1-\frac{q}{2}} \\
&\leq \psi([w]_{A_2}) \left(\int f^2 \, (Rf)^{q-2} dx \right)^{\frac{1}{2}} \left(\int (Rf)^q dx \right)^{1-\frac{q}{2}}.
\end{aligned}$$

So $\psi([w]_{A_2}) \geq 4\|M\|_q$.

Let us prove that $[w]_{A_2} \leq 2\|M\|_q$. Recall that $w = (Rf)^{q-2}$. We have, by Jensen inequality

$$
\begin{aligned}
[w]_{A_2} &= \sup_Q \Big(\frac{1}{|Q|}\int_Q (Rf)^{q-2}dx\Big)\Big(\frac{1}{|Q|}\int_Q (Rf)^{2-q}dx\Big) \\
&\leq \sup_Q \Big(\frac{1}{|Q|}\int_Q (Rf)^{-1}dx\Big)^{2-q}\Big(\frac{1}{|Q|}\int_Q (Rf)dx\Big)^{2-q} \\
&\leq [Rf]_{A_2}^{2-q} \leq [Rf]_{A_2} \leq 2\|M\|_q.
\end{aligned}
$$

Inverse Hölder Inequalities

Lerner, Ambrosi et Pérez (2008)

Proposition

Soit w be such that $M^{\mathcal{D}} w \leq [w]_{A_1} w$. If $r = 1 + 1/(2^{d+1}[w]_{A_1})$, then for every dyadic cube

$$\left(\frac{1}{|Q|} \int_Q w^r \right)^{1/r} dx \leq \frac{2}{|Q|} \int_Q w.$$

Proof for $d = 1$. Let

$$\{M_I^{\mathcal{D}} w > \lambda\} = \cup I_j \qquad \text{(disjoint union).}$$

Assume that $\lambda > \frac{1}{|I|}\int_I w dx$. Since $w(I_j) = \int_{I_j} w \leq 2\lambda |I_j|$, we have

$$\int_{M_I^{\mathcal{D}} w > \lambda} w dx = \sum w(I_j) \leq 2\lambda |M_I^{\mathcal{D}} w > \lambda|.$$

$$
\begin{aligned}
\int_I (M_I^{\mathcal{P}} w)^\delta w dx &= \delta \int_0^\infty \lambda^{\delta-1} \left(\int_{M_I^{\mathcal{P}} w > \lambda} w dx \right) d\lambda \\
&= \int_0^{w(I)/|I|} + \int_{w(I)/|I|}^\infty \\
&\leq \int_0^{w(I)/|I|} \lambda^{\delta-1} w(I) d\lambda + 2\delta \int_0^\infty \lambda^\delta |M_I^{\mathcal{P}} w > \lambda| d\lambda \\
&\leq \frac{1}{|I|^\delta} (\int_I w dx)^{\delta+1} + \frac{2\delta}{\delta+1} \int_I (M_I^{\mathcal{P}} w)^{\delta+1} dx.
\end{aligned}
$$

We now use the assumption $M^{\mathcal{D}}w \leq [w]_{A_1}w$.

$$\int_I (M_I^{\mathcal{D}}w)^\delta w dx \leq \frac{1}{|I|^\delta}(\int_I w dx)^{\delta+1} + \frac{2\delta[A_1]}{\delta+1}\int_I (M_I^{\mathcal{D}}w)^\delta w dx.$$

For δ small enough the second term of the right hand side can be substracted to the left hand side.

A flavor of the rest of the proof

The aim of Lerner : have new tools to analyze locally a function and replace

$$f_Q = \frac{1}{|Q|}\int_Q f dx \qquad\qquad a(f, Q) = \frac{1}{|Q|}\int_Q |f - f_Q| dx.$$

Let P_Q the probability $\frac{dx}{|Q|}$ on Q.

Use of a Median $m(f, Q)$ of f on Q and the (mean) Oscillation

$$\omega_\lambda(f, Q) = \inf\{\omega > 0; \text{there exists } c \text{ such that } P_Q(|f - c| > \omega) \leq \lambda\}.$$

$\omega_\lambda(f, Q) = \inf\{\omega > 0; \text{there exists } c \text{ such that} P_Q(|f - c| > \omega) \leq \lambda\}.$

Claim. $\omega_\lambda(f, Q) \leq \lambda^{-1} a(Q)$.

Take $c = f_Q$ and $\omega = \frac{a(Q)}{\lambda}$. Prove that $P_Q(|f - c| > \omega) \leq \lambda$.

Implies that $M^{\#}_\lambda f \leq \lambda^{-1} M^{\#} f$.

We have more :

Proposition. For T a C-Z operator, we have $M^{\#}_\lambda(Tf) \leq C_\lambda Mf$.

Proposition. For T a C-Z operator, we have $M^{\#}_{\lambda}(Tf) \leq C_{\lambda} Mf$.

Already done : there exists a constant c such that
$|Tf - T(f\chi_{\widetilde{Q}}) - c| \leq C \inf_Q Mf$.
Take this constant c and $\omega = K \inf_Q Mf$.
Since $|Tf - c| \leq |T(f\chi_{\widetilde{Q}}) + C \inf_Q Mf$,

$$
\begin{aligned}
P_Q\left(|Tf - c| \geq K \inf_Q Mf\right) &\leq P_Q\left(T(f\chi_{\widetilde{Q}}) \geq (K - C)\inf_Q Mf\right) \\
&\leq P_Q\left(T(f\chi_{\widetilde{Q}}) \geq \frac{(K - C)}{|\widetilde{Q}|}\int_{\widetilde{Q}} |f(x)|dx\right)
\end{aligned}
$$

But T is weak-type$(1,1)$, so that for some constant C'

$$
P_Q\left(T(f\chi_{\widetilde{Q}}) \geq s\right) \leq \frac{C'}{s|Q|}\int_{\widetilde{Q}} |f(x)|dx.
$$

Take $K \geq C + C'2^n\lambda^{-1}$.

Bochner-Riesz means.

Bochner-Riesz means via the restriction Theorem.

Bochner-Riesz means are given by

$$S_R f(x) = \int_{|\xi|\leq 1} \left(1 - \frac{|\xi|^2}{R^2}\right)^{\delta} \widehat{f}(\xi) e^{2\pi i \xi . x} d\xi.$$

Convergence in L^p reduces to L^p boundedness of the single operator S_1, that is, the fact that

$$m_\delta(\xi) = (1 - |\xi|^2)_+^{\delta}$$

is a Fourier multiplier of L^p. The convolution operator T_δ is given by

$$K_\delta(x) = c|x|^{-(n/2)-\delta} J_{n/2+\delta}(2\pi|x|).$$

Moreover

$$J_\nu(x) = c\frac{\cos(x - \frac{\nu\pi}{2} - \frac{\pi}{4})}{|x|^{1/2}} + O(|x|^{-3/2}).$$

So K_δ behaves like $|x|^{-\frac{n+1}{2}-\delta}$ at infinity.

Necessary condition. For T_δ to be bounded in L^p it is necessary that

$$\frac{n}{p} < \frac{n+1}{2} + \delta,$$

which we can rewrite as

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\delta + 1}{2n}.$$

Theorem. Assume that the restriction theorem holds for $(p, 2)$. Then the Bochner-Riesz conjecture holds for this value of p.

Sufficient to prove that

$$\|(\psi_k K_\delta) * f\|_p \leq C 2^{n(\frac{1}{p}-\frac{1}{2})-\delta-\frac{1}{2}} \|f\|_p.$$

Here $\psi_k(x) = \psi(x/2^k)$ is supported in $\{2^{k-1} \leq |x| \leq 2^{k+1}\}$.

First reductions.

Let $T_k f = (\psi_k K_\delta) * f$.
Claim. It is sufficient to prove that

$$I = \|(\psi_k K_\delta) * f\|_{L^p(B(0,2^{k+3}))} \leq C 2^{[n(\frac{1}{p}-\frac{1}{2})-\delta-\frac{1}{2}]k} \|f\|_{L^p(B(0,2^k))}$$

for f supported in $B(0, 2^k))$.

$$I \leq C 2^{nk(\frac{1}{p}-\frac{1}{2})} \|(\psi_k K_\delta) * f\|_2.$$

Claim. $|\widehat{\psi_k K_\delta}(\xi)| \leq C \frac{2^{-k\delta}}{(1+(2^k d(\xi,S))^N}$.

Claim. We conclude easily from this : no problem for $|\xi| < 1/2$ or $|\xi| > 3/2$. In between, we have

$$\int_{1/2}^{3/2} (1+2^k r)^{-N} \int_{S^{n-1}} |\widehat{f}(r\xi)|^2 d\sigma(\xi) \leq C 2^{-k} \|f\|_p^2,$$

which we wanted.

Claim. $|\widehat{\psi_k} * m_\delta(\xi)| \leq C \frac{2^{-k\delta/2}}{(1+(2^k d(\xi,S))^N}$.

We will prove this estimate when m_δ is replaced by n_δ, where $0 \leq n_\delta \leq 2^{-k\delta}$ on the annulus of thickness 2^{k+1} inside the unit sphere. Then

$$|\widehat{\psi_k} * n_\delta(\xi)| \leq 2^{-k\delta} \int_{1-r<2^{-k}} (|\psi_k| * d\sigma_r) dr.$$

But, for Tomas-Stein Theorem, seen that

$$|\psi_k| * d\sigma \leq 2^k (1 + (2^k d(\xi, S))^N.$$

To conclude it is sufficient to look at $\widehat{\psi_k} * \nu_\delta(\xi)$ where ν_δ is supported in the ball of radius $1 - 2^{-k}$. The key point here is to use the fact that ψ vanishes in a neighborhood of the origin and can be written as $|x|^{2N}\eta(x)$, so that

$$\widehat{\psi_k}(\xi) = c2^{-Nk}\Delta^N \phi_k(\xi), \qquad \phi_k(\xi) \leq C2^{kn}(1 + 2^k|\xi|)^{-N}.$$

We then take into account that when taking $\Delta^N \nu_\delta$, one has a rapid decay far from the unit sphere.

Relation with Prediction Theory

Let Y_n a sequence of Gaussian centred random variables with variance 1 such that $\mathbb{E}(Y_j Y_k) = r(j-k)$.
r is positive definite : for every finite sequence (ξ_j),

$$\sum_{j,k} r(j-k)\xi_j\xi_k \geq 0.$$

Bochner's Theorem (or Herglotz) : there exists a probability on $[0, 2\pi]$ such that

$$r(n) = \int_0^{2\pi} e^{-int} d\mu(t).$$

$$\mathbb{E}(Y_j Y_k) = \int_0^{2\pi} e^{-ijt} e^{ikt} d\mu(t)$$

Can one project on the past in the space of Gaussian r. v. ?
Equivalent to projection in $L^2(d\mu)$. Helson-Szegö 1965 $d\mu = wdx$ and $w = exp(u + Hv)$, with u and v bounded, $|u| < \pi/2$.

The Strichartz estimates

The Schrödinger equation

$$\begin{aligned} i\partial_t u - \Delta_x u &= h \\ u_{|t=0} &= f. \end{aligned}$$

For $h = 0$,

$$u(t,x) = \int_{\mathbb{R}^d} e^{2\pi i (x.\xi + 2\pi t|\xi|^2)} \widehat{f}(\xi) d\xi.$$

We recognize a variant of the extension operator $\mathcal{E}$ for the paraboloid Π, of the equation $\tau = 2\pi|\xi|^2$. We extend here the measure given on the paraboloid by $\widehat{f}(\xi)d\xi$.
The paraboloid has non vanishing curvature.If analogous to the sphere,

$$\|u\|_{L^q(\mathbb{R}^{d+1})} \leq C\|f\|_{L^2(\mathbb{R}^d)}$$

with $q = 2 + \frac{4}{d}$.

The proof.

For homogeneity reasons we can assume that $\widehat{f}$ is supported in the ball $B(0,1)$, and we consider the measure $d\mu(x) = \phi(x)dx$ on the paraboloid, with ϕ smooth cut-off function.

Claim. $|\widehat{\mu}(t,x)| \leq |t|^{-d/2}$.

The key point : $\widehat{\mu}(t,x) = \left(\frac{1}{2\pi t}\right)^{d/2} e^{-i|x|^2/(2\pi t)} *_x \phi$.

For g in $\mathcal{S}(\mathbb{R}^d)$ let $U(t)g = \widehat{\mu}(t,\cdot) *_x g$. Then

$$\|U(t)g\|_\infty \leq |t|^{-d/2}\|g\|_1, \qquad \|U(t)g\|_2 \leq \|g\|_2.$$

So

$$\|U(t)g\|_{p'} \leq |t|^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}\|g\|_p.$$

Next we use the trick that it is sufficient to prove an $(, L^{p'}, L^p)$ inequality for the convolution by $\widehat{\mu}$ in $\mathbb{R}^{d+1}$ (see S. Ray lectures).

$$\widehat{\mu} * f(t,\cdot) = \int_{\mathbb{R}} U(t-s) f(s) ds,$$

where we use the notation $f(t)$ for the function $x \mapsto f(t,x)$.

$$\begin{aligned} A(t) &= \|\widehat{\mu} * f(t,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq \int_{\mathbb{R}} \|U(t-s) f(s)\|_{L^{p'}(\mathbb{R}^d)} ds \\ &\leq C \int_{\mathbb{R}} |t-s|^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} \|f(s)\|_{L^p(\mathbb{R}^d)} ds. \end{aligned}$$

Use HLS to conclude.

Remark. This proof extends to all hypersurfaces with non vanishing Gaussian curvature. One always has the required estimate by the stationary phase method.

The whole equation.

Let us write $e^{it\Delta}$ for the operator given on $\mathbb{R}^d$ by

$$\widehat{e^{it\Delta}f} = e^{-4i\pi^2 t|\xi|^2}\widehat{f}.$$

Then the solution is given by

$$u(t) = e^{-it\Delta}f + \int_0^t e^{-i(t-s)\Delta}h(s)ds.$$

Theorem. Assume that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$. Then

$$\|u\|_{L_t^p(L_x^q)} \leq C\|f\|_{L_x^2} + \|h\|_{L_t^{p'}(L_x^{q'})}.$$

Proof for $p = q = 2 + \frac{4}{d}$.

$$\|u\|_p \leq C\|f\|_2 + \|h\|_{p'}.$$

We have the same inequalities for the propagator $e^{it\Delta}$ than for $U(t)$ and it is sufficient to assume that f is 0. Then the proof is the same.

The general case is also given by the HLS theorem.

Non linear equation.

$$
\begin{aligned}
i\partial_t - \Delta_x u &= \lambda |u|^{4/d} u \\
u_{|t=0} &= f.
\end{aligned}
$$

Assuming that λ is a real number, quantities $\|u(t)\|_2$ and $\|\nabla u(t)\|_2^2 - \lambda \|u(t)\|_{2+\frac{4}{d}}^{2+\frac{4}{d}}$ are invariant.
When $\lambda > 0$ it is called focusing. When $\lambda < 0$, it is defocusing.

$$
u(t) = e^{-it\Delta} f + \lambda \int_0^t e^{-i(t-s)\Delta} |u(s)|^{\frac{4}{d}} u(s) ds.
$$

Theorem. Let $\|f\|_{L^2_x} = 1$. If λ is sufficiently small, the NLS equation has a global solution such that $\|u\|_{2+\frac{4}{d}}$ is bounded. It is the unique solution having these properties and depends continuously of the data.

Fixed point in the metric space $X = \{v \in L^2_{2+\frac{4}{d}}\ ; \|v\|_{2+\frac{4}{d}} \leq C\}$ for the mapping that maps u to

$$(Tu)(t) = e^{-it\Delta}f + \lambda \int_0^t e^{-i(t-s)\Delta}|u(s)|^{\frac{4}{d}}u(s)ds.$$

For free : Tu is in X for λ small. Indeed,

$$\|v^{\frac{4}{d}}w\|_{p'} \leq \|v\|^{\frac{4}{d}}_{2+\frac{4}{d}}\|w\|_{2+\frac{4}{d}}.$$

Have to consider the $L^{2+\frac{4}{d}}$ norm of $Tu - Tv$ or, more precisely, of $|\lambda| \int_0^t e^{-i(t-s)\Delta}|u(s)|^{4/d}|u(s) - v(s)|ds$.

One can also prove finite existence in time for all λ.

Scattering.

$$\begin{aligned} i\partial_t - \Delta_x u &= \lambda |u|^{4/d} u \\ u_{|t=0} &= f. \end{aligned}$$

Define $f_+ = f + \lambda \int_0^\infty e^{it\Delta} |u(t)|^{\frac{4}{d}} u(t) dt$
Backward effect of the non linearity.
Theorem.

$$\|u(t) - e^{-it\Delta} f_+\|_{L^2_x} \to 0.$$

Proof. Have to look at the function of x given by $e^{-it\Delta} \int_t^\infty e^{is\Delta} |u(s)|^{\frac{4}{d}} u(s) ds$. Use the adjoint of the propagator seen as an operator from L^2_x to $L^{2+\frac{4}{d}}_{t,x}$.