Introduction to Γ-convergence

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1 Motivation

Let Ω be an open bounded subset of \( \mathbb{R}^n \) and let \( \partial \Omega \) denote the boundary of Ω. For any given 0 < a < b, let \( \mathcal{M} = \mathcal{M}(a,b,\Omega) \) denote the class of all \( n \times n \) matrices, \( A = A(x) \), with \( L^\infty(\Omega) \) entries such that,

\[
a \|\xi\|^2 \leq A(x)\xi,\xi \leq b \|\xi\|^2 \quad \text{a.e. } x \quad \forall \xi \in \mathbb{R}^n.
\]

Recall the following result on variational inequality on a Hilbert space. Refer [6] for a complete theory on variational inequality.

**Theorem 1.1.** Let \( a(x,y) \) be a coercive bilinear form on \( H \), \( K \subset H \) be a closed and convex subset of \( H \) and \( f \in H' \). Then there exists a unique solution \( x \in K \) to

\[
a(x,y-x) \geq \langle f, y-x \rangle, \quad \forall y \in K.
\]

(1.1)
The case \( K = H \) in the above result is popularly known as Lax-Milgram result. In fact, by choosing \( y = x + z \) and \( y = x - z \) for any \( z \in H \) in (1.1), we have \( a(x, z) = \langle f, z \rangle \) for all \( z \in H \) and for every given \( f \in H' \).

The Lax-Milgram result implies the existence of a weak solution to the following second order elliptic equation with Dirichlet boundary condition,

\[
\begin{aligned}
- \text{div}(A \nabla u) &= f \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega, 
\end{aligned}
\tag{1.2}
\]

where \( A \in \mathcal{M}(a, b, \Omega) \) and let \( f \in H^{-1}(\Omega) \). In fact, one also has the estimate

\[
\|u\|_{H^1_0(\Omega)} \leq \frac{1}{a} \|f\|_{H^{-1}(\Omega)}. \tag{1.3}
\]

The bounded elliptic operator \( A = -\text{div}(A \nabla) \) from \( H^1_0(\Omega) \) into \( H^{-1}(\Omega) \) is an isomorphism and the norm of \( A^{-1} \) is not larger than \( a^{-1} \) (cf. (1.3)).

Moreover, the weak solution \( u \) of (1.2) can also be characterized as the minimizer in \( H^1_0(\Omega) \) of the functional

\[
J(v) = \frac{1}{2} \int_\Omega A \nabla v \cdot \nabla v \, dx - \langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)},
\]

i.e.,

\[
J(u) = \min_{v \in H^1_0(\Omega)} J(v).
\]

Thus, the problem of studying the asymptotic behaviour of the second order elliptic problem

\[
\begin{aligned}
- \text{div}(A_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in } \Omega \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega, 
\end{aligned}
\tag{1.4}
\]

with \( \{A_\varepsilon\} \subset \mathcal{M} \) is equivalent to finding a functional \( J \) on \( H^1_0(\Omega) \) whose minimum is the solution of the homogenized elliptic equation such that both the minimizers and minima of \( J_\varepsilon \) converge to the minimizers and minima of \( J \). Thus, we need to study the convergence of functionals such that the minimizers and minima converge.

2 Direct Method of Calculus of Variation

**Definition 2.1.** Let \( X \) be a topological space. A function \( F : X \to \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} \) is said to be lower semicontinuous (lsc) at a point \( x \in X \) if

\[
F(x) = \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} F(y).
\]
$F$ is lower semicontinuous on $X$ if $F$ is lower semicontinuous at each point $x \in X$.

**Remark 2.1.** Let $X$ be a topological space satisfying first axiom of countability. Then a function $F : X \to \mathbb{R}$ is lower semicontinuous at $x \in X$ iff

$$F(x) \leq \liminf_{n \to \infty} F(x_n)$$

for every sequence $\{x_n\}$ converging to $x \in X$.

**Exercise 1.** Show that if $F$ is lower semicontinuous then the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.

**Definition 2.2.** A function $F : X \to \mathbb{R}$ is coercive on $X$ if the closure of the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is compact in $X$ for every $\alpha \in \mathbb{R}$.

**Exercise 2.** Show that if $F$ is a coercive functional on $X$ and $G \geq F$, then $G$ is coercive.

**Exercise 3.** If $F$ is coercive then there is a non-empty compact set $K$ such that

$$\inf_{x \in X} F(x) = \inf_{x \in K} F(x).$$

**Theorem 2.1.** Let $X$ be a topological space. Assume that the function $F : X \to \mathbb{R}$ is coercive and lower semicontinuous. Then $F$ has a minimizer in $X$.

**Proof.** If $F$ is identically $+\infty$ or $-\infty$, then every point of $X$ is a minimum point for $F$. If $F$ takes the value $-\infty$, then all those points are minimizers of $F$. Suppose now that $F$ is not identically $+\infty$ and $F > -\infty$. Let $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n \to \infty} F(x_n) = \inf_{y \in X} F(y) := d.$$ 

The existence of such a sequence is clear. Without loss of generality, we can assume that $F(x_n) < +\infty$ for all $n$. Let $\alpha := \sup_n F(x_n) < +\infty$. Moreover, since $F$ is coercive, the sublevel set $\{F \leq \alpha\}$ is compact and hence there is a subsequence $\{x_k\}$ of $\{x_n\}$ which converges to a point $x \in X$. Since $F$ is lsc we obtain

$$d = \inf_{y \in X} F(y) \leq F(x) \leq \liminf_{k \to \infty} F(x_k) = d.$$
Thus, $F(x) = d$ and hence is the minimizer of $F$ in $X$. which proves our theorem.

**Definition 2.3.** A family of functionals $\{F_n\}$ on $X$ is said to be equicoercive, if for every $\alpha \in \mathbb{R}$, there is a compact set $K_\alpha$ of $X$ such that the sublevel sets $\{F_n \leq \alpha\} \subseteq K_\alpha$ for all $n$.

**Exercise 4.** If $\{F_n\}$ is a family of equi-coercive, then there is a non-empty compact $K$ (independent of $n$) such that

$$\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x).$$

**Proposition 2.1.** A family of functions $F_n$ on $X$ is equi-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \to \mathbb{R}$ such that $F_n \geq \Psi$ on $X$, for every $n$.

**Proof.** Let $\Psi : X \to \mathbb{R}$ be a lower semicontinuous coercive function such that $F_n \geq \Psi$ on $X$, for every $n$. Set $K_\alpha := \{\Psi \leq \alpha\}$. $K_\alpha$ is closed and compact because of the lsc and coercivity of $\Psi$, respectively. Moreover, $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all $n$. Thus, $F_n$ are equi-coercive.

Conversely, let $F_n$ be equi-coercive. Then, for each $\alpha \in \mathbb{R}$, there is a compact set $K_\alpha$ such that $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all $n$. We shall now define $\Psi : X \to \mathbb{R}$ as

$$\Psi(x) = \begin{cases} +\infty, & \text{if } x \notin K_\alpha, \forall \alpha \in \mathbb{R} \\ \inf\{\alpha \mid x \in K_\beta \text{ for all } \beta > \alpha\}. & \end{cases}$$

We now show that $\Psi \leq F_n$ for all $n$. Let $x \in X$. If $F_n(x) = +\infty$, for all $n$, then by definition, $\Psi(x) = F_n(x) = +\infty$. Otherwise, let $F_k$ be a subfamily such that $F_k(x) = \beta_k < \infty$. Thus, $x \in K_{\beta_k}$ for all $k$ and hence $\Psi(x) = \inf_k \{\beta_k\} \leq F_n(x)$. Thus, $\Psi(x) \leq F(x)$, for every $x \in X$. It now remains to show that $\Psi$ is lsc and coercive. Note that any $x \in \{\Psi \leq \alpha\}$ implies $x \in K_\beta$ for all $\beta > \alpha$. Therefore, the sublevel

$$\{\Psi \leq \alpha\} = \cap_{\beta > \alpha} K_\beta$$

is an arbitrary intersection compact sets and hence is closed and compact. □

**Definition 2.4.** Let $X$ be a vector space. We say a function $F : X \to \mathbb{R}$ is convex if

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$$

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for every $t \in (0,1)$ and for every $x, y \in X$ such that $F(x) < +\infty$ and $F(y) < +\infty$. We say a function $F: X \to \mathbb{R}$ is strictly convex if $F$ is not identically $+\infty$ and

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ and for every $x, y \in X$ such that $x \neq y$, $F(x) < +\infty$ and $F(y) < +\infty$.

**Remark 2.2 (Jensen Inequality).** Let $X$ be a real vector space and let $f: X \to \mathbb{R}$ be a convex function. Then for any given $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$f \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i f(x_i). \quad (2.1)$$

Furthermore, if $f$ is strictly convex then equality holds in (2.1) iff $x_1 = x_2 = \ldots = x_n$. In fact, more generally, if $X$ is a Banach space, $(E, \mu)$ is a probability measure space, $f: X \to [0, +\infty]$ is a lsc, convex function, then

$$f \left( \int_{E} g \, d\mu \right) \leq \int_{E} f \circ g \, d\mu,$$

for all $\mu$-integrable $g: E \to X$.

**Proposition 2.2.** Let $X$ be a vector space. Let $F: X \to \mathbb{R}$ be a strictly convex function. Then $F$ has at most one minimizer in $X$. \hfill $\square$

**Proof.** If $x$ and $y$ are two minimizers of $F$ in $X$, then

$$F(x) = F(y) = d := \min_{z \in X} F(z) < +\infty.$$

If $x \neq y$, by strict convexity we have

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y) = d, \quad \forall t \in (0,1).$$

This contradicts the fact that $d$ is a minimum of $F$. Therefore $x = y$. \hfill $\square$

Thus, combining Theorem 2.1 and Proposition 2.2, we have that on a topological vector space $X$, if $F$ is a lower semicontinuous, coercive and strictly convex function, then $F$ has a unique minimizer. We end this section with a definition from convex analysis.
**Definition 2.5** (Convex Conjugate). Let $X$ be a topological vector space and let $X^*$ be its topological dual. If $F : X \to \mathbb{R}$, its convex conjugate $F^* : X^* \to \mathbb{R}$ is defined as

$$F^*(x^*) = \sup_{x \in X} \{x^*(x) - F(x)\}. \tag{2.5}$$

**Exercise 5.** If $F$ is convex and lower semicontinuous then $F = (F^*)^*$.

**Exercise 6.** Let $A$ be a $n \times n$ symmetric, positive definite matrix and $F : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$F(x) = \frac{1}{2} \langle x, Ax \rangle. \tag{2.6}$$

Show that

$$F^*(x^*) = \frac{1}{2} \langle x^*, A^{-1}x^* \rangle. \tag{2.7}$$

3 **Γ-Convergence**

The notion of Γ-convergence was introduced by Ennio De Giorgi in a sequence of papers (cf. [5, 3, 4]). An excellent account of this concept is the book of Dal Maso [2] and A. Braides [1].

**Definition 3.1.** A function $F$ is said to be the Γ-limit of $F_n$ (denoted as $F_n \Gamma \rightharpoonup F$) w.r.t the topology of $X$, if $F = F^+ = F^-$, where

(i)

$$F^-(x) = \sup_{U \in N(x)} \liminf_{n \to \infty} \inf_{y \in U} F_n(y). \tag{3.1}$$

(ii)

$$F^+(x) = \sup_{U \in N(x)} \limsup_{n \to \infty} \inf_{y \in U} F_n(y). \tag{3.2}$$

We say $F^-$ is the Γ-lower limit and $F^+$ is the Γ-upper limit.

**Remark 3.1.** If $X$ is a topological space satisfying first axiom of countability, the Γ-limit can be characterised as satisfying the following two conditions:

(i) For every $x \in X$ and for every sequence $\{x_n\}$ converging to $x$ in $X$, we have

$$\liminf_{n \to \infty} F_n(x_n) \geq F(x). \tag{3.3}$$

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(ii) For every $x \in X$, there exists a sequence $\{x_n\}$ converging to $x$ in $X$ (called the $\Gamma$-realising sequence) such that

$$\lim_{n \to \infty} F_n(x_n) = F(x).$$

Exercise 7. Show that if $F_n \Gamma \rightarrow F$, $G_n \Gamma \rightarrow G$ and $F_n \leq G_n$, for each $n$, then $F \leq G$.

Exercise 8. Show that if $F_n \Gamma$-converges to $F$, then $F$ is lower semicontinuous.

Exercise 9. Let $X$ be a topological vector space. Show that if $F_n : X \rightarrow \mathbb{R}$ is convex for each $n$, then $\Gamma$-$\lim\sup_n F_n$ is convex. Also show that the $\Gamma$-$\lim\inf_n F_n$ is, in general, not convex.

Exercise 10. Compute the $\Gamma$-limit of a constant sequence $F_n = F$ on $X$.

Theorem 3.1. Let $X$ be a topological space and $F_n$ be a family functions on $X$.

1. If $U$ is an open subset of $X$, then

$$\inf_{x \in U} F^+(x) \geq \limsup_n \inf_{x \in U} F_n(x).$$

2. If $K$ is a compact subset of $X$, then

$$\inf_{x \in K} F^-(x) \leq \liminf_n \inf_{x \in K} F_n(x).$$

Proof. 1. Let $x \in U$. Then, from the definition of $\Gamma$-upper limit which says $F(x)$ is sup over all neighbourhoods of $x$, we have

$$F^+(x) \geq \limsup_{n \to \infty} \inf_{y \in U} F_n(y).$$

Therefore,

$$\inf_{x \in U} F^+(x) \geq \limsup_{n \to \infty} \inf_{y \in U} F_n(y).$$

2. Since $F^-$ is lsc and by the compactness of $K$, $F^-$ attains its minimum on $K$ (cf. Theorem 2.1). Set $d := \liminf_n \inf_{x \in K} F_n(x)$ and let $x_n$ be a sequence (extracting subsequence, if necessary) in $K$ such that $\lim_n F_n(x_n) = d$. Thus, there is a subsequence $x_k$ which converges to
some \( x \in K \). Therefore, for every neighbourhood \( U \) of \( x \), \( \inf_{y \in U} F_k(y) \leq F_k(x_k) \) for infinitely many \( k \). Now, taking \( \lim \inf \) both sides,
\[
\lim \inf \inf_{y \in U} F_k(y) \leq \lim \inf F_k(x_k) = d
\]
and taking supremum over all neighbourhoods \( U \) of \( x \), we still have
\[
F^-(x) = \sup_U \lim \inf \inf_{y \in U} F_k(y) \leq d.
\]

Now, since \( x \in K \), \( \inf_{x \in K} F^-(x) \leq d \).

**Theorem 3.2** (Fundamental Theorem of \( \Gamma \)-convergence). Let \( X \) be a topological space. Let \( \{F_n\} \) be an equi-coercive family of functions and let \( F_n \) \( \Gamma \)-converges to \( F \) in \( X \), then

(i) \( F \) is coercive.

(ii) \( \lim_{n \to \infty} d_n = d \), where \( d_n = \inf_{x \in X} F_n(x) \) and \( d = \inf_{x \in X} F(x) \). That is, the minima converges.

(iii) The minimizers of \( F_n \) converge to a minimizer of \( F \).

**Proof.** Since \( \{F_n\} \) are equi-coercive, by Proposition 2.1, there is a lsc, coercive function \( \Psi \) on \( X \) such that \( F_n \geq \Psi \). Now, by Exercise 7, \( F \geq \Psi \) and by Exercise 2 \( F \) is coercive.

Now, by putting \( U = X \) in Theorem 3.1, we get \( d \geq \limsup_n d_n \). We now need to show that \( d \leq \liminf_n d_n \). If \( F_n \) are all not identically \( +\infty \), then \( \liminf_n d_n < +\infty \). Set \( \liminf_n d_n = \alpha \). By the equi-coercivity of \( F_n \), there is a compact set \( K_\alpha \) such that \( \{F_n \leq \alpha\} \subseteq K_\alpha \), for all \( n \). Consider,
\[
d \leq \inf_{y \in K_\alpha} F(y) \leq \liminf_n \inf_{y \in K_\alpha} F_n(y) = \liminf_n \inf_{y \in X} F_n(y) = \liminf_n d_n.
\]
Thus, \( \limsup_n d_n \leq d \leq \liminf_n d_n \) and hence, \( \lim_n d_n = d \).

Since \( F \) is coercive and lsc (\( \Gamma \)-limit is always lsc), then by Theorem 2.1, \( F \) attains its minimum. Let \( x^*_n \) be a minimizer of \( F_n \), then since \( F_n \) are equi-coercive \( x^*_n \) belong to a compact set \( K \) of \( X \) and hence converges up to
a subsequence. Let $x^*_n \to x^*$ in $X$. We need to show that $F(x^*) = d$. By $\Gamma$-lower limit,
\[ F(x^*) \leq \liminf_n F_n(x^*_n) = \liminf_n d_n = d. \]
But, $d \leq F(x^*)$. Hence $d = F(x^*)$. \hfill \Box

**Theorem 3.3** (Compactness). *If $X$ is a topological space satisfying second axiom of countability then any sequence of functionals $F_n : X \to \mathbb{R}$ has a $\Gamma$-convergent subsequence.*

**Proof.** Let $\{U_k\}_{k \in \mathbb{N}}$ be a countable base for the topology of $X$. For each $k$, let $d^m_k = \inf_{y \in U_k} F_n(y)$. Thus, $\{d^m_k\}_n$ is a sequence in $\mathbb{R}$ which is compact, hence has a subsequence $\{d^m_k\}_m$ whose limit as $m \to \infty$ exists in $\mathbb{R}$. Thus, for each $k$, we have subsequence $\{d^m_k\}_m$ whose limit as $m \to \infty$ exists in $\mathbb{R}$. Choose the diagonal sequence $d^k_k$ whose limit exists in $\mathbb{R}$ as $k \to \infty$. In other words, we have chosen a subsequence $F_k$ of $F_n$ such that
\[ \lim_{k \to \infty} d^k_k = \lim_{k \to \infty} \inf_{y \in U_k} F_k(y). \]

Now, define $F(x) = \sup_{U \in \mathbb{N}(x)} \lim_{k \to \infty} \inf_{y \in U_k} F_k(y)$ and we have by definition $F_k \Gamma$-converges to $F$. \hfill \Box

**Example 3.1.** Let $A_\varepsilon \overset{H}{\to} A_0$ then we wish to show that $J_\varepsilon \overset{\Gamma}{\to} J$ in the weak topology of $H^1_0(\Omega)$ where
\[ J_\varepsilon(u) = \int_\Omega A_\varepsilon \nabla u \cdot \nabla u \, dx \]
and
\[ J(u) = \int_\Omega A_0 \nabla u \cdot \nabla u \, dx. \]

Let $u \in H^1_0(\Omega)$. We need to find a sequence $\{u_\varepsilon\}$ in $H^1_0(\Omega)$ such that $u_\varepsilon$ converges to $u$ weakly in $H^1_0(\Omega)$ and $\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = J(u)$. Let $u_\varepsilon \in H^1_0(\Omega)$ be the solution of
\[ -\text{div}(A_\varepsilon \nabla u_\varepsilon) = -\text{div}(A_0 \nabla u). \quad (3.1) \]

Then, it follows from $H$-convergence that $u_\varepsilon \rightharpoonup u$ weakly in $H^1_0(\Omega)$ and $\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u \cdot \nabla u \, dx$. Thus, we have shown the existence of a sequence $\{u_\varepsilon\}$ converging weakly to $u$ in $H^1_0(\Omega)$ such that
\[ \lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = J(u). \]
Now, let \( w_\varepsilon \in H^1_0(\Omega) \) be a sequence such that \( w_\varepsilon \rightharpoonup u \) weakly in \( H^1_0(\Omega) \). Then, the solution \( u_\varepsilon \) obtained in (3.1) minimizes the functional

\[
\frac{1}{2} J_\varepsilon(v) - \int_\Omega A_0 \nabla u \cdot \nabla v \, dx.
\]

Hence, in particular, we have

\[
\frac{1}{2} \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx - \int_\Omega A_0 \nabla u \cdot \nabla w_\varepsilon \, dx \geq \frac{1}{2} \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx
\]

and taking \( \liminf \) on both sides of above inequality we have

\[
\liminf_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon) \geq J(u).
\]

Hence \( J_\varepsilon \rightharpoonup J \) in the weak topology of \( H^1_0(\Omega) \).

In the above example, we assume the \( H \)-convergence of the matrix coefficients to describe the \( \Gamma \)-limit. A general question of interest is the following: If for any sequence of functionals, by compactness, there is a \( \Gamma \)-limit, then under what conditions one can get an integral representation of \( \Gamma \)-limit. In the next section, we describe the situation in one-dimension.

4 Integral Representation (One-Dimension)

For any given \( 1 < p < \infty \) and \( c_1, c_2, c_3 > 0 \), let \( \mathcal{F} = \mathcal{F}(p, c_1, c_2, c_3) \) be the class of all functionals \( F : W^{1,p}(\Omega) \to [0, +\infty) \) such that

\[
F(u) = \int_\Omega f(x, \nabla u(x)) \, dx
\]

where \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \)

\( \mathbf{H} 1. \) is a Borel function such that \( \xi \mapsto f(x, \xi) \) is convex for all \( x \in \Omega \),

\( \mathbf{H} 2. \) and satisfies the growth conditions of order \( p \)

\[
c_1 |\xi|^p - c_2 \leq f(x, \xi) \leq c_3 (1 + |\xi|^p), \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.
\]
Exercise 11. If $f$ satisfies $H_1$ and $H_2$, then $f$ satisfies the local Lipschitz condition

$$|f(x, \xi) - f(x, \zeta)| \leq k(1 + |\xi|^{p-1} + |\zeta|^{p-1})|\xi - \zeta| \quad \forall \xi, \zeta \in \mathbb{R}^n.$$ 

The constant $k$ depends only on $c_3$ and $p$.

We take $n = 1$ in the dimension of Euclidean space and set $\Omega = (a, b)$. Observe that any functional in $\mathcal{F}$ is invariant by addition of a constant $c$, i.e., $F(u + c) = F(u)$. Thus, it is sufficient to characterize in the space

$$X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$$

equipped with $L^p$ norm instead of $W^{1,p}(\Omega)$. Since $X$ is embedded in $L^\infty(a, b)$, $L^1(a, b) \subset X^*$.

**Proposition 4.1.** Let $X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$ equipped with $L^p$ norm. Let $F \in \mathcal{F}$ and consider its integrand $f$ as a function on $X$, then $F^* : X^* \to \mathbb{R}$ is given as

$$F^*(\phi) = \int_a^b f^*\left(x, -\int_a^x \phi(t) \, dt\right) \, dx, \quad \forall \phi \in L^1(a, b).$$

**Proof.** Let us assume $f(x, \cdot) \in C^1(\mathbb{R})$ for all $x \in (a, b)$. Due to the growth conditions and continuity of $f$,

$$f^*(x, \xi^*) = \sup_{\xi \in \mathbb{R}}\{\xi^* \cdot \xi - f(x, \xi)\} = \max_{\xi \in \mathbb{R}}\{\xi^* \cdot \xi - f(x, \xi)\}.$$ 

Thus, if $\zeta$ is the point at which maximum is attained, then

$$f^*(x, \zeta^*) = \zeta^* \cdot \zeta - f(x, \zeta) \quad \text{if and only if} \quad \zeta^* - \frac{\partial f}{\partial \zeta}(x, \zeta) = 0. \quad (4.1)$$

Let $\phi \in L^1(a, b)$, define $\Phi \in W^{1,1}(a, b)$ as,

$$\Phi(x) = -\int_a^x \phi(t) \, dt.$$
Note that $\Phi' = -\phi$ and $\Phi(a) = 0$. Thus, the convex conjugate of $F$ is given as

$$F^* (\phi) = \sup_{v \in X} \left\{ \int_a^b \left( \phi(x)v(x) - f(x, v'(x)) \right) dx \right\}$$

$$= \sup_{v \in X} \left\{ \int_a^b \left( \Phi(x)v'(x) - f(x, v'(x)) \right) dx \right\} \quad \text{(using integration by parts)}$$

$$= \max_{v \in X} \left\{ \int_a^b \left( \Phi(x)v'(x) - f(x, v'(x)) \right) dx \right\}$$

$$= \int_a^b \left( \Phi(x)u'(x) - f(x, u'(x)) \right) dx.$$

By computing Euler equations, we have $\Phi - \frac{\partial f}{\partial u}(x, u') = c$, for some constant $c$. But $\Phi(a) = 0$ and $\frac{\partial f}{\partial u}(a, u'(a)) = 0$, implies that $c = 0$ and thus, $\Phi = \frac{\partial f}{\partial u}(x, u')$ a.e. on $(a, b)$. By choosing $\xi^* = \Phi(x)\\text{ and } \xi = u'(x)$ in $(4.1)$, we have

$$\Phi(x) = \frac{\partial f}{\partial u}(x, u'(x)) \quad \text{if and only if } f^*(x, \Phi(x)) = \Phi(x)u'(x) - f(x, u'(x)).$$

Hence,

$$F^*(\phi) = \int_a^b \left( \Phi(x)u'(x) - f(x, u'(x)) \right) dx$$

$$= \int_a^b f^*(x, \Phi(x)) dx$$

$$= \int_a^b f^* \left( x, -\int_a^x \phi(t) dt \right) dx$$

Now, for a general $f$ satisfying hypotheses $H_1$ and $H_2$, we define $f_\varepsilon(x, \xi) = \int_a^b \rho_\varepsilon(x - y)f(y, \xi) dy$, where $\rho_\varepsilon$ are the sequence of mollifiers. Observe that $f_\varepsilon$ are convex in the second variable and, by Jensen’s inequality, $f_\varepsilon \geq f$. Also, observe that $\lim_\varepsilon f_\varepsilon^*(x, \xi^*) = f^*(x, \xi^*)$ for all $x \in (a, b)$ and $\xi^* \in \mathbb{R}$. We have, for each $\varepsilon$,

$$F_\varepsilon^*(\phi) = \int_a^b f_\varepsilon^*(x, \phi(t)) dt \quad \forall \phi \in L^1(a, b).$$

Now, by dominated convergence theorem and $F^* \geq F_\varepsilon^*$, we get

$$F^*(\phi) \geq \lim_k F_k^*(\phi) = \int_a^b f^* \left( x, -\int_a^x \phi(t) dt \right) dx.$$
Also, by the convex conjugate definition, \( f^*(x, \xi^*) \geq \xi^* \xi - f(x, \xi) \) for all \( x, \xi, \xi^* \). Now, choose \( \xi^* = \Phi(x), \xi = v' \), where \( v \in X \) and integrate both sides of above inequality,

\[
\int_a^b f^*(x, \Phi(x)) \, dx \geq \int_a^b (\Phi(x)v'(x) - f(x, v'(x))) \, dx = \int_a^b (\phi(x)v(x) - f(x, v'(x))) \, dx.
\]

Taking supremum over \( v \in V \), we have \( F^*(\phi) \leq \int_a^b f^*(x, \Phi(x)) \, dx \). ☐

**Proposition 4.2.** Let \( g_n : \Omega \times \mathbb{R}^n \to [0, +\infty) \) satisfy hypotheses \( H1 \) and \( H2 \), for all \( n \). If \( g_n(\cdot, \xi) \) weak* converges to \( g(\cdot, \xi) \) for all \( \xi \in \mathbb{R} \), then \( g_n(\cdot, v(\cdot)) \) weak* converges to \( g(\cdot, v(\cdot)) \), for all \( v \in C([a, b]) \).

**Proof.** Let \( v \in C([a, b]) \) and \( \phi \in L^1(a, b) \). Also, let \( (x_{i-1}, x_i) \) be \( k \) number of partitions of \( (a, b) \) for \( i = 1, 2, \ldots, k \) such that \( x_0 = a \) and \( x_k = b \). Consider,

\[
\left| \int_a^b (g_n(x, v) - g(x, v)) \phi \, dx \right| \leq \sum_{i=1}^k \left| \int_{(x_{i-1}, x_i)} (g_n(x, v(x)) - g_n(x, v(x_i))) \phi \, dx \right|
\]

\[
+ \sum_{i=1}^k \left| \int_{(x_{i-1}, x_i)} (g_n(x, v(x_i)) - g(x, v(x_i))) \phi \, dx \right|
\]

\[
+ \sum_{i=1}^k \left| \int_{(x_{i-1}, x_i)} (g(x, v(x_i)) - g(x, v(x))) \phi \, dx \right|
\]

The second term converges to zero, by hypothesis, and by uniform local Lipschitz continuity (cf. Exercise 11 of \( g_n \) and \( g \), we have the result. ☐

**Lemma 4.1.** Let \( g_n : \Omega \times \mathbb{R}^n \to [0, +\infty) \) satisfy hypotheses \( H1 \) and \( H2 \), for all \( n \). Then, there exists a subsequence of \( \{g_n\} \) and a \( g : (a, b) \times \mathbb{R} \to [0, +\infty) \) such that \( g_n(\cdot, \xi) \) weak* converges to \( g(\cdot, \xi) \) for all \( \xi \in \mathbb{R} \).

**Theorem 4.1.** Let \( \{F_n\} \) be a sequence in \( F \) with integrand \( f_n \) and \( F \in F \) with integrand \( f \). Then the following statements are equivalent:

1. \( F_n(\cdot, I) \) \( \Gamma \)-converges to \( F(\cdot, I) \) in \( W^{1,p}(I) \), for all open intervals \( I \) of \( (a, b) \).
2. $f^\ast_n(\cdot, \xi^\ast)$ weak* converges to $f^\ast(\cdot, \xi^\ast)$, for all $\xi^\ast \in \mathbb{R}$.

The proof of above lemma and theorem are being skipped and can be found in [1].

Example 4.1. Let $0 < \alpha \leq a_\varepsilon(x) \leq \beta < +\infty$ and $g \in L^2(a, b)$. Let $F_\varepsilon: H^1_0(a, b) \to \mathbb{R}$ be defined as

$$F_\varepsilon(u) = \int_a^b \left\{ \frac{1}{2} a_\varepsilon(x)|u'|^2 - gu \right\} dx.$$  

The Euler-Lagrange equations yields that the minimizers $u_\varepsilon$,

$$\begin{cases} -\frac{d}{dx}(a_\varepsilon(x) \frac{du_\varepsilon}{dx}) = g \text{ in } (a, b) \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases}$$

Now, set $f_\varepsilon(x, \xi) := a_\varepsilon(x)|\xi|^2$. Then, $f_\varepsilon^\ast(x, \xi^\ast) = \frac{\xi^2}{4a_\varepsilon(x)}$. But, for each $\xi^\ast \in \mathbb{R}^n$, $f_\varepsilon^\ast(\cdot, \xi^\ast)$ converges weak* in $L^\infty(a, b)$ to $f^\ast(\cdot, \xi^\ast)$, where $f^\ast(x, \xi^\ast) = \frac{\xi^2}{4b(x)}$ and

$$\frac{1}{a_\varepsilon(x)} \rightharpoonup \frac{1}{b(x)}.$$

□

References


