PARTIAL DIFFERENTIAL EQUATIONS: A FIRST COURSE

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Notations

Symbols

$\mathbb{N}$ denotes the set of natural numbers

∇ $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$

Ω denotes an open subset of $\mathbb{R}^n$, not necessarily bounded

$\partial\Omega$ denotes the boundary of Ω

∂^α $\frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$

$\mathbb{R}$ denotes the real number line

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space

Δ $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$

A^t denotes the transpose of any matrix A

A^{-1} denotes the inverse of any matrix A

$e_i := (0, 0, \ldots, 1, 0, \ldots, 0)$ with 1 in the i-th place denotes the standard basis vectors of $\mathbb{R}^n$

Function Spaces

$C(X)$ is the class of all continuous functions on X

$C^k(\overline{X})$ is the class of C^k functions which admit a continuous extension to the boundary of X

$C^k(X)$ is the class of all k-times ($k \geq 1$) continuously differentiable functions on X

$C^\infty(X)$ is the class of all infinitely differentiable functions on X

$C^{j,k}(X \times Y)$ is the class of all j-times ($j \geq 0$) continuously differentiable functions on X and k-times ($k \geq 0$) continuously differentiable functions on Y

$C_c^\infty(X)$ is the class of all infinitely differentiable functions on X with compact support

General Conventions

∇_x, Δ_x or D_x^2 When a PDE involves both the space variable x and time variable t, the quantities like ∇, Δ, D^2, etc. are always taken with respect to the space variable x only. This is a standard convention. Sometimes the suffix, like ∇_x or Δ_x, is used to indicate that the operation is taken with respect to x.

ω_n denotes the surface area of a n-dimensional sphere of radius 1.

$B_r(x)$ denotes the open disk with centre at x and radius r

$S_r(x)$	denotes the circle or sphere with centre at x and radius r
BVP	Boundary value problem
IVP	Initial value problem

Chapter 1
Introduction

A partial differential equation (PDE) is an equation relating an unknown function of two or more variables and some or all of its partial derivatives to some known quantities/functions. The partial differential equation is a mathematical tool to express the phenomenons of nature. The process of understanding natural system can be divided in to three stages:

(i) Modelling the problem: deriving the mathematical equation describing the natural system. The derivation process could be facilitated by physical laws such as Newton's law, momentum, conservation laws, balancing forces etc.
(ii) Solving the equation: What constitutes as a solution to a equation?
(iii) Studying the properties of a solution: Most often the solution(s) of a differential equation may not have a nice formula or representation. How much information about the solution can one extract without any representation of a solution?

If I is an open interval of $\mathbb{R}$ then recall that the *derivative* of a function $u : I \to \mathbb{R}$, at $x \in \Omega$, is defined as

$$u'(x) := \lim_{h\to 0} \frac{u(x+h) - u(x)}{h},$$

provided the limit exists. Now, let $\Omega \subseteq \mathbb{R}^n$ be an open and connected (domain) subset, a convenient generalization of open interval I in higher dimension. The *directional derivative* of $u : \Omega \to \mathbb{R}$, at $x \in \Omega$ and in a prescribed direction $\xi \in \mathbb{R}^n$, is defined as

$$\partial_\xi u(x) := \lim_{h\to 0} \frac{u(x+h\xi) - u(x)}{h},$$

provided the limit exists. Let $e_i := (0, 0, \dots, 1, 0, \dots, 0)$, where 1 is in the i-th place, denote the standard basis vectors of $\mathbb{R}^n$. The *partial derivative* of u at x, in the i-th direction, is the directional derivative of u, at $x \in \Omega$ and along the direction e_i, and is denoted as

$$u_{x_i}(x) = \frac{\partial u}{\partial x_i}(x) := \lim_{h\to 0} \frac{u(x+he_i) - u(x)}{h}$$

and $\nabla u(x) := (u_{x_1}(x), u_{x_2}(x), \ldots, u_{x_n}(x))$ is the *gradient vector*. The directional derivative along any (unit) vector $\xi \in \mathbb{R}^n$ and the partial derivatives are related via the identity $\partial_\xi u(x) = \nabla u(x) \cdot \xi$. The *divergence* of a vector function $\mathbf{u} = (u_1, \ldots, u_n)$, denoted as $\operatorname{div}(\mathbf{u})$, is defined as $\operatorname{div}(\mathbf{u}) := \nabla \cdot \mathbf{u}$. The *curl* of a 3-dimensional vector function $\mathbf{u} = (u_1, u_2, u_3)$, denoted as $\operatorname{curl}(\mathbf{u})$, is defined as $\operatorname{curl}(\mathbf{u}) := \nabla \times \mathbf{u}$. The *curl* of a n-dimensional vector function $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is a $n \times n$ matrix function where each element is given as $(\operatorname{curl}(\mathbf{u}))_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$.

1.1 Multi-Index Notations

The multi-index notation is very convenient to denote polynomials in higher dimensions. This is also convenient to denote higher order partial derivatives in higher dimensions.

A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a n-tuple where $\alpha_i \in \mathbb{N} \cup \{0\}$, for each $1 \le i \le n$. Let $|\alpha| := \alpha_1 + \ldots + \alpha_n$. If α and β are two multi-indices, then $\alpha \le \beta$ means $\alpha_i \le \beta_i$, for all $1 \le i \le n$, and $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n)$. Also, $\alpha! = \alpha_1! \ldots \alpha_n!$ and, for any $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. With these notations, a k-degree polynomial in n-variables can be concisely written as $\sum_{|\alpha| \le k} a_\alpha x^\alpha$. The partial differential operator of *order* α is denoted as

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}.$$

One adopts the convention that the index α_j corresponds to the variable x_j. Hence, the order in which differentiation is performed is irrelevant as long as index j are matched. This is not a restrictive convention because the independence of order of differentiation is valid for smooth[1] functions. For instance, if $\alpha = (1,1,2)$ then one adopts the convention that

$$\frac{\partial^4}{\partial x_1 \partial x_2 \partial {x_3}^2} = \frac{\partial^4}{\partial x_2 \partial x_1 \partial {x_3}^2}.$$

If $|\alpha| = 0$, then $\partial^\alpha f = f$. For each $k \in \mathbb{N}$, $D^k u(x) := \{\partial^\alpha u(x) \mid |\alpha| = k\}$. The case $k = 1$ is the *gradient* vector,

$$\begin{aligned}\nabla u(x) := D^1 u(x) &= \left(\partial^{(1,0,\ldots,0)} u(x), \partial^{(0,1,0,\ldots,0)} u(x), \ldots, \partial^{(0,0,\ldots,0,1)} u(x)\right) \\ &= \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_n}\right).\end{aligned}$$

The case $k = 2$ is the *Hessian* matrix

[1] smooth, usually, refers to as much differentiability as required.

$$D^2u(x) = \begin{pmatrix} \frac{\partial^2 u(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 u(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u(x)}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 u(x)}{\partial x_2 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u(x)}{\partial x_n^2} \end{pmatrix}_{n\times n}.$$

Observe that the Hessian matrix is symmetric for continuously differentiable functions since the order in which partial derivatives are taken has no significance. The *Laplace* operator, denoted as Δ, is defined as the trace of the Hessian operator, i.e., $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Note that $\Delta = \nabla \cdot \nabla$.

Further, for a k-times differentiable function u, the n^k-*tensor* $D^k u(x) := \{\partial^\alpha u(x) \mid |\alpha| = k\}$ may be viewed as a map from $\mathbb{R}^n$ to $\mathbb{R}^{n^k}$. The magnitude of $D^k u(x)$ is

$$|D^k u(x)| := \left(\sum_{|\alpha|=k} |D^\alpha u(x)|^2 \right)^{\frac{1}{2}}.$$

In particular, $|\nabla u(x)| = (\sum_{i=1}^{n} u_{x_i}^2(x))^{\frac{1}{2}}$ or $|\nabla u(x)|^2 = \nabla u(x) \cdot \nabla u(x)$ and $|D^2 u(x)| = (\sum_{i,j=1}^{n} u_{x_i x_j}^2(x))^{\frac{1}{2}}$.

Example 1.1. Let $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be defined as $u(x,y) = ax^2 + by^2$. Then

$$\nabla u = (u_x, u_y) = (2ax, 2by)$$

and

$$D^2 u = \begin{pmatrix} u_{xx} & u_{yx} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

Observe that $\nabla u : \mathbb{R}^2 \to \mathbb{R}^2$ and $D^2 u : \mathbb{R}^2 \to \mathbb{R}^4 = \mathbb{R}^{2^2}$.

1.2 Definition and Classification of PDE

Definition 1.1. Let Ω be an open and connected subset of $\mathbb{R}^n$. A k-th *order* partial differential equation of an unknown function $u : \Omega \to \mathbb{R}$ is of the form

$$F\left(D^k u(x), D^{k-1} u(x), \ldots Du(x), u(x), x\right) = 0, \tag{1.1}$$

for each $x \in \Omega$, where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given map such that F depends, at least, on one k-th partial derivative u and is independent of $(k+j)$-th partial derivatives of u for all $j \in \mathbb{N}$.

The case $n = 1$ corresponds to a k-th order ordinary differential equation (ODE) $F(u^{(k)}(x), \ldots u(x), x) = 0$. For $n > 1$, the case $k = 1$ corresponds to a first order

PDE $F(Du(x), u(x), x) = 0$ and the case $k = 2$ corresponds to a second order PDE $F(D^2u(x), Du(x), u(x), x) = 0$. For instance, when $n = 2$ and $k = 1$, the PDE is first order with two unknown variable (x, y), represented as $F(u_x, u_y, u, x, y) = 0$ with F depending, at least, on one of u_x and u_y. Similarly, when $n = 3$ and $k = 1$, the PDE is a first order in three variables, represented as $F(u_x, u_y, u_z, u, x, y, z) = 0$ with F depending, at least, on one of u_x, u_y and u_z. We warn here about the abuse in the usage of x: in the n-variable case, $x \in \mathbb{R}^n$ is a n-tuple vector while in the two and three dimension case, x is the first component of the vector. The usage should be clear from the context.

The level of difficulty in solving a PDE may depend on its order k and linearity of F. Let us begin by classifying PDEs in a scale of linearity.

Definition 1.2. (i) A k-th order PDE is *linear* if F in (1.1) has the form

$$Fu := Lu - f(x)$$

where $Lu(x) := \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u(x)$ for given functions f and a_α's. It is called linear because L is linear in u for all derivatives , i.e., $L(\lambda u_1 + \mu u_2) = \lambda L(u_1) + \mu L(u_2)$ for $\lambda, \mu \in \mathbb{R}$. In addition, if $f \equiv 0$ then the PDE is linear and *homogeneous*.

(ii) A k-th order PDE is *semilinear* if F is linear only in the highest (k-th) order, i.e., F has the form

$$\sum_{|\alpha|=k} a_\alpha(x) \partial^\alpha u(x) + f(D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0.$$

(iii) A k-th order PDE is *quasilinear* if F has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x), \ldots, u(x), x) \partial^\alpha u + f(D^{k-1}u(x), \ldots, u(x), x) = 0,$$

i.e., the coefficient of its highest (k-th) order derivative depends on u and its derivative only upto the previous $(k-1)$-th orders.

(iv) A k-th order PDE is *fully nonlinear* if it depends nonlinearly on the highest (k-th) order derivatives.

Observe that the semilinear case excludes the possibility that f is linear in u and its derivatives because otherwise the PDE can be written in the linear form. Similarly, the quasilinear case excludes the possibility of either a_α (with $|\alpha| = k$) or f being linear.

Example 1.2. (i) $xu_y - yu_x = u$ is linear and homogeneous.
(ii) $xu_x + yu_y = x^2 + y^2$ is linear.
(iii) $u_{tt} - c^2 u_{xx} = f(x, t)$ is linear.
(iv) $y^2 u_{xx} + xu_{yy} = 0$ is linear and homogeneous.
(v) $u_x + u_y - u^2 = 0$ is semilinear.
(vi) $u_t + uu_x + u_{xxx} = 0$ is semilinear.
(vii) $u_{tt}^2 + u_{xxxx} = 0$ is semilinear.

(viii) $u_x + uu_y - u^2 = 0$ is quasilinear.
(ix) $uu_x + u_y = 2$ is quasilinear.
(x) $u_x u_y - u = 0$ is nonlinear.
(xi) $u_x^2 + u_y^2 = 1$ is nonlinear.

Example 1.3 (Some important linear PDE). For $x \in \mathbb{R}^n$ and $t \in (0,\infty)$:

(i) (Transport Equation) $u_t(x,t) + b \cdot \nabla_x u(x,t) = 0$, for a given $b \in \mathbb{R}^n$. The transport equation may be thought of as describing the transport of a non-diffusing and non-decaying pollutant along a flow channel with speed b. The unknown u is the concentration of the pollutant.
(ii) (Laplace Equation) $\Delta u(x) = 0$. The Laplace operator $\Delta := \sum_{i=1}^n \partial_{x_i}^2$ is a second order partial differential operator. Harmonic functions are solutions to the PDE $\Delta u = 0$. The Laplace equation describes the steady state system of both heat and wave equation.
(iii) (Poisson Equation) $\Delta u(x) = f(x)$, the inhomogeneous version of Laplace equation, where f usually represents a source. The Poisson equation appears in the study of Newtonian potential and electrostatics. It also finds its application in the study of geometry and topology of Riemannian manifolds.
(iv) (Helmholtz Equation) $\Delta u(x) + k^2 u(x) = 0$, for a given constant k.
(v) (Heat Equation) The heat operator $\partial_t - c\Delta_x$ is a second order operator in $\mathbb{R}^{n+1}$. It describes transfer and diffusion phenomena. Fourier studied this operator in the context of heat transfer. $u_t(x,t) - c\Delta_x u(x,t) = 0$, for non-zero $c \in \mathbb{R}$, describes the conduction of heat in a homogeneous and isotropic medium where c captures the conductivity of the material. $u(x,t)$ represents the temperature at x and time t.
(vi) (Kolmogorov's Equation) $u_t(x,t) - A \cdot D_x^2 u(x,t) + b \cdot \nabla_x u(x,t) = 0$, for given $n \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^n$. The first product is the matrix scalar product and the second is the vector scalar product.
(vii) (Wave Equation) The wave operator $\partial_t^2 - \Delta_x$ is a second order operator in $\mathbb{R}^{n+1}$. It describes oscillatory phenomena and wave propogation. Note that the operator Δ_x involves only the space variable x. $u_{tt}(x,t) - c^2\Delta_x u(x,t) = 0$, for a non-zero $c \in \mathbb{R}$, describes the propogation of transversal waves (or vibration) of elastic medium such as strings (one dimension), membrane (two dimension). In three dimensions it describes the propogation of electromagnetic waves in vacuum or sound waves. c represents the propogation speed and u represents the displacement or wave amplitude.
(viii) (General Wave Equation) $u_{tt}(x,t) - A \cdot D_x^2 u(x,t) + b \cdot \nabla_x u(x,t) = 0$, for given $n \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^n$. The first product is the matrix scalar product and the second is the vector scalar product.
(ix) (Vibrating Plate) $u_{tt}(x,t) - \Delta^2 u(x,t) = 0$, for $x \in \mathbb{R}^2$, describes the vibrations of a homogeneous isotropic two dimensional plate. The *biharmonic operator*, Δ^2, is the square of the Laplacian operator which is a fourth order operator given as
$$\Delta^2 := \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}.$$

(x) (Schrödinger Equation) The Schrödinger operator $-\imath\partial_t - \Delta_x$ was introduced to describe the behaviour of the electron and elementary particles. It almost looks like the heat operator except the $-\imath$ and that makes all the difference.
(xi) (Airy's Equation) $u_t(x,t) + u_{xxx}(x,t) = 0$.
(xii) (Beam Equation) $u_t(x,t) + u_{xxxx}(x,t) = 0$.
(xiii) The Cauchy-Riemann (CR) operator is the first order differential operator $\partial_x + \imath\partial_y$. If $f : \mathbb{R}^2 \to \mathbb{C}$ is a holomorphic function then the CR equation is given by $f_x + \imath f_y = 0$. If $z = x + \imath y$ then $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2\imath}$. By chain rule, $2\partial_{\bar{z}} = (\partial_x + \imath\partial_y)$ is the CR operator. In other words, any solution to the CR equation is independent of the $\bar{z}$ variable. The above discussion also suggests that $2\partial_z = \partial_x - \imath\partial_y$, thus motivating the notion of "anti" CR operator $\partial_x - \imath\partial_y$. Also, we obtain the relation that $4\partial^2_{z\bar{z}} = \Delta$ suggesting a possible connection between Laplacian and CR equation.

Example 1.4 (Some nonlinear PDE).

(i) (Poisson Equation) For any given f nonlinear in u, $\Delta u(x) = f(u)$.
(ii) (Inviscid Burgers' Equation) $u_t(x,t) + u(x,t)u_x(x,t) = 0$, for $x \in \mathbb{R}$, describes the one dimensional flux of a non-viscous fluid. It also models traffic dynamics.
(iii) (Eikonal Equation) $|\nabla u(x)| = f(x)$ is a first order nonlinear equation. The level surfaces $\{x \mid u(x) = t\}$ describe the position of a light wave front at time t. This equation also arises in optimal control and computer vision etc. In fact, the name "eikon" is the transliteration of the Greek word for image.
(iv) (Hamilton-Jacobi Equation) $u_t(x,t) + H(\nabla_x u(x,t), x) = 0$.
(v) (Minimal Surface Equation) The equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = f(x)$$

arises in geometry. The graph of the solution u defined on the domain Ω (say convex domain, for simplicity) has the given mean curvature $f : \Omega \to \mathbb{R}$. When $f \equiv 0$ the equation is called *minimal surface equation*.
(vi) (Monge-Ampére Equation) $\det(D^2 u(x)) = f(x, u, \nabla u)$ is a fully nonlinear PDE encountered in optimal transport problems. The gradient of the solution, ∇u, maps optimal transportation path.
(vii) (Schrödinger Equation) $\imath u_t(x,t) + \Delta_x u(x,t) - V(u)u(x,t) = 0$. The wavefunction $u : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{C}$ is associated with the evolution of a particle driven by a potential $V(u)$.
(viii) (Korteweg de Vries (KdV) Equation)
$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0$.

Exercise 1.1. Classify the PDE's in terms of their linearity:

(i) $(y-u)u_x + xu_y = xy + u^2$.
(ii) $uu_x^2 - xu_y = \frac{2}{x}u^3$.
(iii) $x^2 u_x + (y-x)u_y = y \sin u$.

(iv) $(\sin y)u_x - e^x u_y = e^y u$.
(v) $u_x + \sin(u_y) = u$.
(vi) $uu_x + x^2 u_{yyy} + \sin x = 0$.
(vii) $u_x + e^{x^2} u_y = 0$.
(viii) $u_{tt} + (\sin y)u_{yy} - e^t \cos y = 0$.
(ix) $x^2 u_{xx} + e^x u = x u_{xyy}$.
(x) $e^y u_{xxx} + e^x u = -\sin y + 10 x u_y$.
(xi) $y^2 u_{xx} + e^x u u_x = 2 x u_y + u$.
(xii) $u_x u_{xxy} + e^x u u_y = 5 x^2 u_x$.
(xiii) $u_t = k^2(u_{xx} + u_{yy}) + f(x,y,t)$.
(xiv) $x^2 u_{xxy} + y^2 u_{yy} - \log(1+y^2)u = 0$.
(xv) $u_x + u^3 = 1$.
(xvi) $u_{xxyy} + e^x u_x = y$.
(xvii) $uu_{xx} + u_{yy} - u = 0$.
(xviii) $u_{xx} + u_t = 3u$.
(xix) $u_{tt} - u_{xx} + u^3 = 0$.
(xx) $u_{tt} + u_{xxxx} = 0$.
(xxi) $(\cos xy^2)u_x - y^2 u_y = \tan(x^2 + y^2)$.

1.3 Solution of PDE

Definition 1.3. A function $u : \Omega \to \mathbb{R}$ is said to be a (classical) *solution* to the PDE (1.1), if $\partial^\alpha u$ exists for all α explicitly present in (1.1) and u satisfies the equation (1.1) pointwise for all $x \in \Omega$.

Example 1.5. Consider the first order equation $u_x(x,y) = 0$ in $\mathbb{R}^2$. Freezing the y-variable, the PDE can be viewed as an ODE in x-variable. On integrating both sides with respect to x, we get $u(x,y) = f(y)$ for any arbitrary function $f : \mathbb{R} \to \mathbb{R}$. Therefore, for every choice of $f : \mathbb{R} \to \mathbb{R}$, there is a solution u of the PDE. Note that the solution u is not necessarily in $C^1(\mathbb{R}^2)$, in contrast to the situation in ODE. By choosing a discontinuous function f, one obtains a solution which is discontinuous in the y-direction. Further, in contrast to the ODE situation where the parameter was a constant depending on the order of the equation here the parameter is an arbitrary function.

Example 1.6. Consider the second order equation $u_{xx}(x,y) = 0$ in $\mathbb{R}^2$. Freezing the y-variable, the PDE can be viewed as a second order ODE in x-variable. On integrating both sides with respect to x, we get $u_x(x,y) = f(y)$ for any arbitrary function $f : \mathbb{R} \to \mathbb{R}$. On integrating again both sides with respect to x, we get $u(x,y) = xf(y) + g(y)$ for any arbitrary function $g : \mathbb{R} \to \mathbb{R}$. Therefore, for every choice of $f, g : \mathbb{R} \to \mathbb{R}$, there is a solution u of the PDE.

Example 1.7. Consider the second order PDE $u_{xy}(x,y) = 0$ in $\mathbb{R}^2$. In contrast to the previous two examples, the PDE involves derivatives in both variables. Integrate both sides with respect to y to obtain $u_x(x,y) = F(x)$, for any arbitrary integrable function $F : \mathbb{R} \to \mathbb{R}$. Now, integrating both sides with respect to x, we get $u(x,y) = f(x) + g(y)$ for an arbitrary $g : \mathbb{R} \to \mathbb{R}$ and a $f \in C^1(\mathbb{R})$ where $f' = F$. But the u obtained above is not a solution to $u_{yx}(x,y) = 0$ if g is not differentiable. If mixed derivatives have to be same then assume that $f,g \in C^1(\mathbb{R})$ for the above solution to exist.

Example 1.8. Consider the first order equation $u_t(x,t) = u(x,t)$ in $\mathbb{R} \times (0,\infty)$ such that $u(x,t) \neq 0$, for all (x,t). Freezing the x-variable, the PDE can be viewed as an ODE in t-variable. Integrate both sides with respect to t to obtain $u(x,t) = f(x)e^t$, for some arbitrary (not necessarily continuous) positive-valued function $f : \mathbb{R} \to (0,\infty)$.

Example 1.9. Consider the second order equation $u_{tt}(x,t) = -u(x,t)$ in $\mathbb{R} \times (0,\infty)$. Freezing the x-variable, the PDE can be viewed as a second order ODE with constant coefficients in t-variable. Thus, $u(x,t) = f(x)\cos t + g(x)\sin t$, for some arbitrary (not necessarily continuous) function $f,g : \mathbb{R} \to \mathbb{R}$.

Example 1.10. Consider the first order equation $u_x(x,y) = u_y(x,y)$ in $\mathbb{R}^2$. On first glance, the PDE does not seem simple to solve. But, by change of coordinates, the PDE can be rewritten in a simpler form. Choose the coordinates $w = x+y$ and $z = x-y$ and, by chain rule, $u_x = u_w + u_z$ and $u_y = u_w - u_z$. In the new coordinates, the PDE becomes $u_z(w,z) = 0$ which is in the form considered in Example 1.5. Therefore, its solution is $u(w,z) = f(w)$ for any arbitrary $f : \mathbb{R} \to \mathbb{R}$ and, hence, $u(x,y) = f(x+y)$. However, now f can no longer be arbitrary.

Exercise 1.2. Rewrite the following PDE in the coordinates w and z.

(i) $u_x + u_y = 1$ for $w = x+y$ and $z = x-y$.
(ii) $au_t + bu_x = u$ for $w = ax - bt$ and $z = t/a$ where $a,b \neq 0$.
(iii) $au_x + bu_y = 0$ for $w = ax+by$ and $z = bx - ay$, where $a^2 + b^2 > 0$.
(iv) $u_{tt} = c^2 u_{xx}$ for $w = x+ct$ and $z = x-ct$.
(v) $u_{xx} + 2u_{xy} + u_{yy} = 0$ for $w = x$ and $z = x-y$.
(vi) $u_{xx} - 2u_{xy} + 5u_{yy} = 0$ for $w = x+y$ and $z = 2x$.
(vii) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$ for $w = y-2x$ and $z = x$. (should get $u_{zz} = 0$).
(viii) $u_{xx} + 2u_{xy} - 3u_{yy} = 0$ for $w = y-3x$ and $z = x+y$.

The family of solutions, obtained in the above examples, may not be the only family that solves the given PDE. Following example illustrates a situation where three different family of solutions exist (more may exist too) for the same PDE.

Example 1.11. Consider the second order PDE $u_t(x,t) = u_{xx}(x,t)$.

(i) Note that $u(x,t) = c$ is a solution of the PDE, for any constant $c \in \mathbb{R}$. This is a family of solutions indexed by $c \in \mathbb{R}$.
(ii) The function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as $u(x,t) = \frac{x^2}{2} + t + c$, for any constant $c \in \mathbb{R}$, is also a family of solutions of the PDE. Because $u_t = 1$, $u_x = x$ and $u_{xx} = 1$. This family is not covered in the first case.

(iii) The function $u(x,t) = e^{c(x+ct)}$ is also a family of solutions to the PDE, for each $c \in \mathbb{R}$. Because $u_t = c^2 u$, $u_x = cu$ and $u_{xx} = c^2 u$. This family is not covered in the previous two cases.

Exercise 1.3. Determine a and b so that $u(x,y) = e^{ax+by}$ is a solution to

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$$

Exercise 1.4. Determine the relation between a and b for $u(x,y) = f(ax+by)$ to be a solution to $3u_x - 7u_y = 0$, for any differentiable function f such that $f'(z) \neq 0$ for all $z \in \mathbb{R}$.

Recall that the family of solutions of an ODE is indexed by constants. In contrast to ODE, observe that the family of solutions of a PDE is indexed by either functions or constants. A family of surfaces indexed by constants or an arbitrary function, upon differentiation, may lead to a PDE that has the surfaces as its solution.

Example 1.12. Consider the two parameter family of surfaces

$$u(x,y;a,b) = (x-a)^2 + (y-b)^2$$

for parameters $(a,b) \in \mathbb{R}^2$. Differentiating with respect to each variable, one obtains $u_x = 2(x-a)$ and $u_y = 2(y-b)$. Eliminating a and b, one notes that the family of surfaces is a solution to the PDE

$$u_x^2 + u_y^2 - 4u = 0.$$

Example 1.13. Let $u(x,y) = xy + f(x^2+y^2)$ be a surface for an arbitrary differentiable function f. Differentiating the given equation with respect to x and y, one obtains

$$u_x = y + 2xf'(x^2+y^2), \quad u_y = x + 2yf'(x^2+y^2),$$

respectively. Eliminate f' after multiplying y and x, respectively, to obtain the PDE

$$yu_x - xu_y = y^2 - x^2.$$

Example 1.14. Let $u(x,y) = f(x/y)$ be a surface given by an arbitrary differentiable function f. Differentiating wth respect to x and y, one obtains

$$u_x = \frac{1}{y} f', \quad u_y = \frac{-x}{y^2} f',$$

respectively. Eliminate f' after multiplying x and y, respectively, to obtain the PDE

$$xu_x + yu_y = 0.$$

Example 1.15. Let $u(x,y) = (x+a)(y+b)$ be a two parameter family of surfaces. Differentiating the given equation with respect to x and y, one obtains

$$u_x = y + b, \quad u_y = x + a,$$

respectively. Eliminate a and b to obtain the PDE

$$u_x u_y = u.$$

Example 1.16. Let $u(x,y) = ax + by$ be a two parameter family of surfaces. Differentiating with respect to x and y, one obtains

$$u_x = a, \quad u_y = b$$

respectively. Eliminate a and b to obtain

$$xu_x + yu_y = u.$$

Exercise 1.5. Find the first order PDE, by eliminating the arbitrary function f, satisfied by $u(x,y) = e^x f(2x - y)$. (Answer: $u_x + 2u_y - u = 0$).

Exercise 1.6. Find the first order PDE, by eliminating the arbitrary function f, satisfied by $u(x,y) = e^{-4x} f(2x - 3y)$. (Answer: $3u_x + 2u_y + 12u = 0$).

1.4 Cauchy Problem

In the theory of ordinary differential equations (ODEs), one may have come across the initial value problems, i.e.

$$\begin{cases} F\left(y^{(k)}, \ldots, y', y, x\right) = 0 & \text{in } I \\ y(x_0) = y_0 & \text{for some } x_0 \in I \\ y^{(i)}(x_0) = y_0^{(i)} & \forall i = \{1, 2, \ldots, k-1\} \text{ for some } x_0 \in I \end{cases} \tag{1.2}$$

where I is a subinterval of $\mathbb{R}$, and $x_0 \in \bar{I}$. Our experience with ODE hints that the prescription of initial values at x_0 are useful in computing the arbitrary constants appearing in the general solution of the ODE. Thus, to solve for the unknown in the one-dimensional interval I an 'initial' value was prescribed on a set $\{x_0\}$ which is one dimension less than I. Motivated from this, one expects that for a PDE given on a n-dimensional domain, the unknown should be prescribed on a $n-1$-dimensional subset (or surface) called the *hypersurface*.

Definition 1.4. A set $\Gamma \subset \mathbb{R}^n$ is said to be C^k-*hypersurface* if for every $p \in \Gamma$ there exists a neighbourhood U_p of p in $\mathbb{R}^n$ and a C^k map $\phi_p : U_p \to \mathbb{R}$ with $\nabla \phi_p \neq 0$ on U_p such that $\Gamma \cap U_p = \{x \in U_p \mid \phi_p(x) = 0\}$.

The above definiton says that a hypersurface is locally graph of some function with domain in $\mathbb{R}^{n-1}$. Under the non-zero gradient condition, the implicit function theorem states that the zero set, i.e. $\{\phi_p(x) = 0\}$ of a smooth function is, for some j, locally of the form $x_j = \psi(\hat{x}_j)$, i.e. locally graph of ψ where $\hat{x}_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ is the $(n-1)$ tuple. Henceforth, without loss of generality (by suitably rearranging the coordinate system), assume $j = n$, i.e. the local graph is given as $x_n = \psi(x')$ where $x' = (x_1, \ldots, x_{n-1})$ the first $(n-1)$ tuple. The dimension is $n-1$ because the domain of ψ is a subset of $\mathbb{R}^{n-1}$. Locally, the function $(x', x_n) \mapsto (x', x_n - \psi(x'))$ "flattens" the hyperspace to the hyperplane $\{y_n = 0\}$ where $y_i = x_i$ for all $i \neq n$ and $y_n = x_n - \psi(x')$. Further, a hypersurface is given by an equation $\{\phi = 0\}$ where $\phi : \mathbb{R}^n \to \mathbb{R}$. If $\phi \in C^k(\mathbb{R}^n)$ then the hypersurface is C^k-hypersurface.

Example 1.17. Any surface in $\mathbb{R}^3$ is a hypersurface in $\mathbb{R}^3$. Any curve in $\mathbb{R}^2$ is a hypersurface in $\mathbb{R}^2$. For instance, the unit circle $\phi(x,y) := x^2 + y^2 - 1 = 0$ is a curve and not graph of any function. But, locally, it is graph of the functions $\psi_{\pm}(x) = \pm\sqrt{1-x^2}$. Consider the coordinate transformation $(x,y) \mapsto (w,z)$ where $w = x$ and $z = \phi(x,y)$. Then observe that all horizontal lines $\{y = \pm c\}$ in xy-plane gets mapped to the parabolae with intercepts $c^2 - 1$ in wz-plane and all circles of radius r in xy-plane gets mapped to horizontal line $\{z = r^2 - 1\}$ in the wz-plane. In particular, the circle of radius of one is mapped to the axis $\{z = 0\}$. See Figure 1.1.

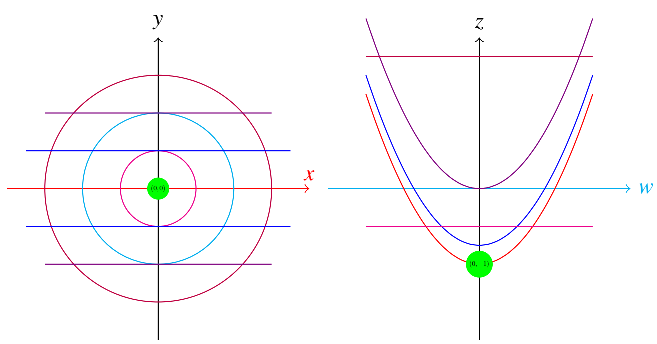

Fig. 1.1 Flattening a Circle

Any domain $\Omega \subset \mathbb{R}^n$ is C^k if its boundary $\partial\Omega$ is a C^k-hypersurface. Let $\Gamma \subset \Omega$ be a smooth hypersurface with a unit normal vector field ν. Recall that the *normal* derivative is the directional derivative along the normal vector $\nu(x)$ for each $x \in \Gamma$. The k-th normal derivative of u at $x \in \Gamma$ is defined by

$$\partial_\nu^k u(x) = \sum_{|\alpha|=k} \nu^\alpha(x)\partial^\alpha u(x).$$

The *Cauchy Problem* is a generalization of the above question to PDE: given the knowledge of the unknown u on a smooth hypersurface $\Gamma \subset \Omega$, can one find the unknown u satisfying the PDE? The prescription of u on Γ is said to be the *Cauchy data*. It is desirable to know the minimum required Cauchy data in order to solve the Cauchy problem. The IVP (1.2) motivates us to define the Cauchy problem as

$$\begin{cases} F\left(D^k u(x),\ldots,Du(x),u(x),x\right) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{on } \Gamma \\ \partial_\nu^i u(x) = u_i(x) & \text{on } \Gamma \ \forall i = \{1,2,\ldots,k-1\} \end{cases} \tag{1.3}$$

where Ω is an open and connected subset (domain) of $\mathbb{R}^n$ and Γ is a hypersurface contained in Ω. A natural question at this juncture is whether the knowledge of u and all its normal derivative upto order $(k-1)$ on Γ is sufficient to compute all order derivatives of u on Γ. Posing the above question is natural because if the Cauchy problem admits an analytic solution (a natural starting point before 20th century) then one should be able to compute all the terms of the power series.

1.5 Well-posedness of PDE

It has been illustrated via examples that a PDE has a family of solutions. The choice of one solution from the family of solutions is made by imposing boundary conditions (boundary value problem) or initial conditions (initial value problem). If too many initial/boundary conditions are specified, then the PDE may have *no solution*. If too few initial/boundary conditions are specified, then the PDE may have *many solutions*. Even with right amount of initial/boundary conditions, but at wrong places, the solution may fail to be stable, i.e., may not depend continuously on the initial or boundary data. It is, usually, desirable to solve a *well-posed* problem, in the sense of Hadamard. A PDE, along with the boundary condition or initial condition, is said to be well-posedness if the PDE

(a) admits a solution (existence);
(b) the solution is unique (uniqueness);
(c) and the solution depends continuously on the data given (stability).

Any PDE not meeting the above criteria is said to be *ill-posed*. If the PDE (with boundary/initial conditions) is viewed as a map then the well-posedness of the PDE is expressed in terms of the surjectivity, injectivity and continuity of the "inverse" map. The existence and uniqueness condition depends on the notion of solution in consideration. There are three notions of solution, viz., classical solutions, weak solutions and strong solutions. This textbook, for the most part, is in the classical situation. Further, the stability condition means that a small "change" in the data reflects a small "change" in the solution. The *change* is measured using a *metric* or "distance" in the function space of data and solution, respectively. Though this text deals with only well-posed problems, there are ill-posed problems which are also of interest.

The following example illustrates the idea of continuous dependence of solution on data in the *uniform metric* on the space of continuous functions.

Example 1.18. The initial value problem (IVP)

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = u_t(x,0) & = 0 \end{cases}$$

has the trivial solution $u(x,t) = 0$. Consider the IVP with a small change in data,

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 0 \\ u_t(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right) \end{cases}$$

which has the unique[2] solution $u_\varepsilon(x,t) = \varepsilon^2 \sin(x/\varepsilon)\sin(t/\varepsilon)$. The change in solution of the IVP is measured using the uniform metric as

$$\sup_{(x,t)}\{|u_\varepsilon(x,t) - u(x,t)|\} = \varepsilon^2 \sup_{(x,t)}\{|\sin(x/\varepsilon)\sin(t/\varepsilon)|\} = \varepsilon^2.$$

Thus, a small change in data induces a small enough change in solution under the uniform metric[3].

Example 1.19 (Hadamard's Counterexample). The initial value problem (IVP)

$$\begin{cases} u_{xx}(x,t) = -u_{tt}(x,t) & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0,\infty) \\ u(\pm\pi/2, t) = 0 & \{t > 0\} \\ u(x,0) = 0 & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ u_t(x,0) = e^{-\sqrt{n}} \cos nx & \frac{-\pi}{2} < x < \frac{\pi}{2} \end{cases}$$

admits a family of solution

$$u_n(x,t) = \frac{1}{n} e^{-\sqrt{n}} \cos nx \sinh ny \text{ for odd integer } n.$$

Observe that $\|u_n(\cdot,0)\|_\infty \to 0$ as $n \to \infty$ and, for all $t > 0$ and $p = 2,\infty$

$$\|u_n(\cdot,y)\|_p \to \infty, \text{ as } n \to \infty.$$

Exercise 1.7. Consider the IVP

$$\begin{cases} u_t(x,t) = -u_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 1. \end{cases}$$

(a) Show that $u(x,t) \equiv 1$ solves the IVP.
(b) Show that $u_n(x,t) = 1 + \frac{e^{n^2 t}}{n} \sin(nx)$ solves the IVP

$$\begin{cases} u_t(x,t) = -u_{xx}(x,t) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 1 + \frac{\sin(nx)}{n}. \end{cases}$$

(c) Find $\sup_x\{|u_n(x,0) - 1|\}$.
(d) Find $\sup_{(x,t)}\{|u_n(x,t) - 1|\}$.
(e) Conclude that the IVP is not stable in the chosen metric and, hence, ill-posed.

[2] This claim will be proved in later chapters.

[3] The space $\mathbb{R} \times (0,\infty)$ is not compact and the metric is not complete. The example is only to explain the notion of stability at an elementarty level.

1.6 Equation of Conservation of Mass

We end this chapter with a derivation of the equation of conservation of mass. Let us consider an ideal compressible fluid (say, gas) occupying a bounded region $\Omega \subset \mathbb{R}^n$ (in practice, $n = 3$ but the derivation is true for any dimension). Let $\rho(x,t)$ denote the density of the fluid for $x \in \Omega$ at time $t \in I \subset \mathbb{R}$, for some open interval I. For mathematical rigor, let us assume Ω to be a bounded open subset of $\mathbb{R}^n$ and $\rho \in C^1(\Omega \times I)$. Cut a region $\Omega_t \subset \Omega$ and follow the deformation of Ω_t at time t, as t varies in I. For mathematical rigor, let Ω_t have C^1 boundaries (cf. Definition B.1). The motion of Ω_t gives rise to a trajectory $x(t)$ for each point $x(t) \in \Omega_t$. Let the velocity of deformation be $\mathbf{b}(x,t)$. We also assume that the deformation of Ω_t is smooth, i.e., $\mathbf{b}(x,t)$ is continuous in a neighbourhood of $\Omega \times I$.

Recall that mass is the product of density and volume and, hence, the mass of Ω_t is given as

$$\int_{\Omega_t} \rho(x,t)\,dx.$$

The law of *conservation of mass* states that during motion the mass is conserved. Hence, the mass of Ω_t is constant for all t. Thus, its derivative with respect to t should vanish, i.e.,

$$\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx = 0.$$

But

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx &= \lim_{h\to 0}\frac{1}{h}\left(\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t)\,dx\right) \\
&= \lim_{h\to 0}\int_{\Omega_t} \frac{\rho(x,t+h)-\rho(x,t)}{h}\,dx \\
&\quad + \lim_{h\to 0}\frac{1}{h}\left(\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx\right).
\end{aligned}$$

The first limit is

$$\lim_{h\to 0}\int_{\Omega_t} \frac{\rho(x,t+h)-\rho(x,t)}{h}\,dx = \int_{\Omega_t} \frac{\partial\rho}{\partial t}(x,t)\,dx$$

and the second integral reduces as,

$$\begin{aligned}
\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx &= \int_{\Omega} \rho(x,t+h)\left(\chi_{\Omega_{t+h}} - \chi_{\Omega_t}\right) \\
&= \int_{\Omega_{t+h}\setminus\Omega_t} \rho(x,t+h)\,dx \\
&\quad - \int_{\Omega_t\setminus\Omega_{t+h}} \rho(x,t+h)\,dx,
\end{aligned}$$

where χ_E is the *characteristic* or *indicator* function

$$\chi_E(y) = \begin{cases} 1 & y \in E \\ 0 & y \notin E. \end{cases}$$

To evaluate the Riemann integral on the RHS, partition the set $(\Omega_{t+h} \setminus \Omega_t) \cup (\Omega_t \setminus \Omega_{t+h})$ with cylinders and evaluate the integral by making the cylinders as small as possible. For each fixed t, choose $0 < s \ll 1$ and a polygon that covers $\partial\Omega_t$ from outside such that the area of each face of the polygon is less than s and the faces are tangent to some point $x_i \in \partial\Omega_t$. Let the polygon have m faces $F_1, F_2, \ldots, F_m$ that are tangential at the points $\{x_1, x_2, \ldots x_m\} \in \partial\Omega_t$. Since Ω_{t+h} is the position of Ω_t, after time h, any point $x(t) \in \Omega_t$ moves to $x(t+h) = \mathbf{b}(x,t)h$. Hence, the cylinders with base F_i and height $\mathbf{b}(x_i,t)h$ is expected to cover the region (in the integral) depending on whether the deformation is inward or outward. Thus, $\mathbf{b}(x_i,t) \cdot \nu(x_i)$ is positive or negative depending on whether Ω_{t+h} deformes outward or inward, where $\nu(x_i)$ is the unit outward normal at $x_i \in \partial\Omega_t$. Thus,

$$\begin{aligned} &\int_{\Omega_{t+h}\setminus\Omega_t} \rho(x,t+h)\,dx \\ &- \int_{\Omega_t\setminus\Omega_{t+h}} \rho(x,t+h)\,dx = \lim_{s\to 0} \sum_{i=1}^{m} \rho(x_i,t)\mathbf{b}(x_i,t)\cdot\nu(x_i)hs \\ &\qquad\qquad\qquad\qquad\qquad\quad = h\int_{\partial\Omega_t} \rho(x,t)\mathbf{b}(x,t)\cdot\nu(x)\,d\sigma \end{aligned}$$

and

$$\frac{1}{h}\left(\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx\right) \stackrel{h\to 0}{\to} \int_{\partial\Omega_t} \rho(x,t)\mathbf{b}(x,t)\cdot\nu(x)\,d\sigma.$$

But, by Gauss divergence theorem (cf. (B.1)),

$$\int_{\partial\Omega_t} \rho(x,t)\mathbf{b}(x,t)\cdot\nu(x)\,d\sigma = \int_{\Omega_t} \operatorname{div}(\rho(x,t)\mathbf{b}(x,t))\,dx$$

and, hence,

$$\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx = \int_{\Omega_t} \left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{b})\right)dx.$$

Invoking the law of conservation of mass, one obtains

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{b}) = 0 \text{ in } \Omega\times\mathbb{R}. \tag{1.4}$$

(1.4) is called the *equation of continuity*. Any quantity that is conserved as it moves in an open set Ω satisfies (1.4), the equation of continuity. To summarise, the law of *conservation of mass* states that the rate of change of mass in Ω is equal to the sum of the rate at which mass flows in and out of Ω and the rate at which mass is produced and destroyed due to sources/sinks in Ω. Thus,

$$\rho_t(x,t) + \mathrm{div}_x[\mathbf{q}(x,t)] = f(x,t) \tag{1.5}$$

where f represents the sources in Ω and $\mathbf{q}(x) = (q_1, \ldots, q_n)$ denotes the *flux* vector such that $\mathbf{q} \cdot \nu$ gives the mass per unit area per unit time crossing the boundary $\partial\Omega$ of Ω with unit exterior normal $\nu = (\nu_1, \ldots, \nu_n)$.

Remark 1.1. By *Fick's law of diffusion*, the flux $\mathbf{q}$ is related to ρ as

$$\mathbf{q}(x,t) = -c\nabla_x \rho(x,t) + \rho(x,t)\mathbf{b}(x) \tag{1.6}$$

where $\mathbf{b}(x) = (b_1, \ldots, b_n)$ represents the velocity of the fluid which is known *a priori*. The quantity $c > 0$ is the diffusion coefficient which may depend on ρ but we shall assume that it is constant. Substituting (1.6) in (1.5) yields the second order PDE

$$\rho_t(x,t) = c\Delta_x \rho(x,t) - \mathrm{div}_x[\rho(x,t)\mathbf{b}(x)] + f(x,t) \tag{1.7}$$

which describes the diffusion-dispersion phenomena.

Remark 1.2. If there is no diffusion and no source then $c = f = 0$ and (1.7) is the first order PDE

$$\rho_t(x,t) + \mathbf{b}(x) \cdot \nabla_x \rho(x,t) + \rho(x,t)\mathrm{div}_x \mathbf{b} = 0. \tag{1.8}$$

Remark 1.3. If the substance decays with no source then $f(x,t) = -d\rho(x,t)$, where d indicates decay, and the phenomenon is reflected by the equation

$$\rho_t(x,t) + \mathbf{b}(x) \cdot \nabla_x \rho(x,t) + \rho(x,t)\mathrm{div}_x \mathbf{b} + d\rho(x,t) = c\Delta_x \rho(x,t). \tag{1.9}$$

The case $d > 0$ represents decaying and $d < 0$ represents increasing.

Chapter 2
First Order PDE

In this chapter one studies the Cauchy problem associated with the first order PDE of the form

$$\begin{cases} F(\nabla u(x), u(x), x) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{on } \Gamma \end{cases}$$

We have already introduced the notion of non-characteristic hypersurface in § 1.4 for linear partial differential operator. The discussion in § 1.4 can be generalised to first order quasilinear operator.

2.1 Characteristic Hypersurfaces for Quasilinear

Definition 2.1. For any given vector field $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$, let $Lu := \mathbf{a}(x,u) \cdot \nabla u - f(x,u)$ be the first order quasilinear partial differential operator defined in a neighbourhood of $x_0 \in \mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. Then Γ is *non-characteristic* at x_0 if

$$\mathbf{a}(x_0, u_0(x_0)) \cdot \nu(x_0) \neq 0$$

where $\nu(x_0)$ is the normal to Γ at x_0. Otherwise, we say Γ is *characteristic* at x_0 with respect to L. If Γ is *(non)characteristic* at each of its point then we say Γ is (non)characteristic.

If $\gamma(r) := (\gamma_1(r), \ldots, \gamma_n(r))$, for $r \in \mathbb{R}^{n-1}$, be the parametrization of the initial data curve Γ then the non-characteristic condition means that the vectors $\mathbf{a}(\gamma(r), u_0(r))$ and $\nabla_r \gamma_j$, for all j, are linearly independent for all r, i.e. the determinant is non-zero for the matrix

$$\begin{vmatrix} \mathbf{a}(\gamma(r), u_0(r)) \\ \partial_{r_1} \gamma \\ \vdots \\ \partial_{r_{n-1}} \gamma \end{vmatrix}_{n\times n} \neq 0 \quad \forall r. \tag{2.1}$$

The motivation to name the property "non-characteristic" will become obvious when we encounter the method of characteristics for the first order PDE. Geometrically, a hypersurface is non-characteristic for first order if the coefficient vector field **a** is not in its tangential direction, i.e. the coefficient vector **a** is not a tangent vector to Γ. The non-characteristic condition depends on the initial hypersurface and the coefficients of first order derivatives in the linear case. In the quasilinear case, it also depends on the initial data.

Theorem 2.1. *For any $f \in C(\mathbb{R}^{n+1})$ and* **a** *such that $|\mathbf{a}| \neq 0$, let u be a smooth solution to the first order quasilinear Cauchy problem*

$$\begin{cases} \mathbf{a}(x,u(x)) \cdot \nabla u(x) = f(x,u) \text{ in } \mathbb{R}^n \\ \qquad\qquad\qquad u(x) = u_0(x) \quad \text{on } \Gamma. \end{cases} \tag{2.2}$$

If Γ is a non-characteristic hypersurface then it is possible to compute all order partial derivatives of u on Γ in terms the initial data viz. the hypersurface Γ, the initial condition $\{u_0\}$ and the coefficients **a**.

Proof. Let us first consider the case when Γ is the hyperplane $\{x_n = 0\}$. We seek to know whether all first order derivatives of u on $\{x_n = 0\}$ can be computed. Without loss of generality, let us compute at $x = 0$, i.e $u(0)$ and $\nabla u(0) := (\nabla_{x'} u(0), \partial_{x_n} u(0))$ where $x = (x', x_n)$ where x' is the $(n-1)$-tuple. If the initial condition u_0 is a smooth function then the x' derivative of u is computed to be the x'-derivatve of u_0, i.e. $\nabla_{x'} u(0) = \nabla_{x'} u_0(0)$. It only remains to compute $\partial_{x_n} u(0)$. Using the PDE, whenever $a_n(0, u_0(0)) \neq 0$, we have

$$\partial_{x_n} u(0) = \frac{-1}{a_n(0, u_0(0))} \left(\mathbf{a}'(0, u_0(0)) \cdot \nabla_{x'} u(0) - f(0, u_0(0)) \right).$$

Now, for a general hyperspace Γ given by the equation $\{\phi = 0\}$ for a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in a neighbourhood of the origin with $\nabla \phi \neq 0$. Recall that $\nabla \phi$ is normal to Γ. Without loss of generality, we assume $\phi_{x_n}(x_0) \neq 0$. Consider the change of coordinate $(x', x_n) \mapsto y := (x', \phi(x))$, then its Jacobian matrix is given by

$$\begin{pmatrix} I_{(n-1)\times(n-1)} & \mathbf{0}_{n-1} \\ \nabla_{x'} \phi & \phi_{x_n} \end{pmatrix}_{n \times n}$$

and its determinant at x_0 is non-zero because $\phi_{x_n}(x_0) \neq 0$. The change of coordinates has mapped the hypersurface to the hyperplane $\{y_n = 0\}$. Rewriting the given PDE in the new variable y, we get

$$Lu = \mathbf{a}(x, u(x)) \cdot \nabla \phi \, \partial_{y_n} u + \text{ terms not involving } \partial_{y_n} u$$

and the initial conditions are given on the hyperplane $\{y_n = 0\}$. Thus, the necessary condition is $\mathbf{a}(x, u(x)) \cdot \nabla \phi \neq 0$ for all $x \in \Gamma$. Recall that $\nabla \phi$ is the normal to the hypersurface Γ.

Example 2.1. In the two dimension case, the curve $\Gamma = \{\gamma_1(r), \gamma_2(r)\} \subset \Omega \subset \mathbb{R}^2$ is *non-characteristic* for the quasilinear Cauchy problem

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma \end{cases} \tag{2.3}$$

if Γ is nowhere tangent to $(a(\gamma_1, \gamma_2, u_0), b(\gamma_1, \gamma_2, u_0))$, i.e.

$$(a(\gamma_1, \gamma_2, u_0), b(\gamma_1, \gamma_2, u_0)) \cdot (-\gamma_2', \gamma_1') \neq 0 \quad \text{for all } r.$$

Example 2.2. Let Γ be a straight line $y = mx + c$ in $\mathbb{R}^2$. The equation of Γ is $\phi(x,y) = y - mx - c$. Then, $\nabla\phi = (-m, 1)$. If using parametrization then the line is $\Gamma(r) := (r, mr + c)$ for $r \in \mathbb{R}$. Let us verify the characteristic property of these straight lines with respect to various first order PDEs.

(a) Consider the equation $2u_x(x,y) + 3u_y(x,y) = 1$ in $\mathbb{R}^2$. Therefore,

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2,3) \cdot (-m, 1) = 3 - 2m.$$

Thus, the line is not a non-characteristic for $m = 3/2$, i.e., all lines with slope $3/2$ is not a non-characteristic.

(b) Consider the equation $u_x(x,y) + u_y(x,y) = 1$ in $\mathbb{R}^2$. Therefore,

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (1,1) \cdot (-m, 1) = 1 - m.$$

Thus, the line is not a non-characteristic for $m = 1$, i.e., all lines with slope 1 is not a non-characteristic.

2.2 Normal Vector of a Surface

Let $\phi(x,y,z) = 0$ be the equation of a surface S in $\mathbb{R}^3$. Fix $p_0 = (x_0, y_0, z_0) \in S$. What is the normal vector at p_0? Fix an arbitrary curve C lying on S and passing through p_0. Let $r(t) = (x(t), y(t), z(t))$ be the parametric form of C with $r(t_0) = p_0$. Since C lies on S, $\phi(r(t)) = \phi(x(t), y(t), z(t)) = 0$, for all t. Differentiating w.r.t t (using chain rule),

$$\begin{aligned} \frac{\partial\phi}{\partial x}\frac{dx(t)}{dt} + \frac{\partial\phi}{\partial y}\frac{dy(t)}{dt} + \frac{\partial\phi}{\partial z}\frac{dz(t)}{dt} &= 0 \\ (\phi_x, \phi_y, \phi_z) \cdot (x'(t), y'(t), z'(t)) &= 0 \\ \nabla\phi(r(t)) \cdot r'(t) &= 0. \end{aligned}$$

In particular, $\nabla\phi(p_0) \cdot r'(t_0) = 0$. Since $r'(t_0)$ is the slope of the tangent, at t_0, to the curve C, the vector $\nabla\phi(p_0)$ is perpendicular to the tangent vector at p_0. Since the argument is valid for any curve in S that passes through p_0, $\nabla\phi(p_0)$ is normal vector to the tangent plane at p_0. If, in particular, the equation of the surface is given as $\phi(x,y,z) = u(x,y) - z$, for some $u : \mathbb{R}^2 \to \mathbb{R}$, then

$$\begin{aligned}\nabla\phi(p_0) &= (\phi_x(p_0), \phi_y(p_0), \phi_z(p_0))\\ &= (u_x(x_0,y_0), u_y(x_0,y_0), -1) = (\nabla u(x_0,y_0), -1).\end{aligned}$$

2.3 Method of Characteristics for Quasilinear

Let us consider the quasilinear PDE $\mathbf{a}(x,u)\cdot\nabla u - f(x,u) = 0$ in a domain $\Omega \subset \mathbb{R}^n$. Solving for the unknown $u : \Omega \to \mathbb{R}$ is equivalent to determining the surface S in $\mathbb{R}^{n+1}$ given by

$$S = \{(x,z) \in \Omega \times \mathbb{R} \mid u(x) - z = 0\}$$

The equation of the surface S is given by $\{\phi(x,z) := u(x) - z = 0\}$. The smoothness of surface S depends on the smoothness of u. The normal vector to S is given by $\nabla_{(x,z)}\phi = (\nabla u(x), -1)$. But using the PDE satisfied by u, we know that

$$(\mathbf{a}(x,u(x)), f(x,u(x)))\cdot(\nabla u(x), -1) = 0.$$

Thus, the data vector field $V(x,z) := (\mathbf{a}(x,z), f(x,z)) \in \mathbb{R}^{n+1}$ is perpendicular to the normal of S at every point of S. Thus, V must lie on the tangent plane of S, at each of its point. Hence, we seek a surface S such that the given data field V is tangential to S, at every point of S. This motivates the definition of integral surface.

Definition 2.2. A smooth curve in $\mathbb{R}^n$ is said to be an *integral curve* w.r.t a given vector field, if the vector field is tangential to the curve at each of its point. Further, a smooth surface in $\mathbb{R}^n$ is said to be an *integral surface* w.r.t a given vector field, if the vector field is tangential to the surface at each of its point.

In the spirit of above definition and arguments, finding a solution to the first order quasilinear PDE is equivalent to determining an integral surface S corresponding to the coefficient vector field $V = (\mathbf{a}, f)$. An integral surface w.r.t V is an union[1] of integral curves w.r.t V. The integral curves are also called *characteristics* and recall that the non-characteristic condition implied the coefficient vector $\mathbf{a}$ is not tangential to the hypersurface. Hence, the name non-characteristic.

Thus, finding the unknown u is equivalent to finding the integral surface corresponding to the data vector field $V(x,z)) = (\mathbf{a}(x,z), f(x,z))$. Thus, if $\Gamma = \{x(s), z(s)\}$ is an integral curve corresponding to V lying on the solution surface S, i.e. V is tangential to Γ at all its points, then the curve is described by the $(n+1)$ system of ODEs,

$$\frac{dx}{ds} = \mathbf{a}(x(s), z(s)) \text{ and } \frac{dz}{ds} = f(x(s), z(s)). \tag{2.4}$$

[1] the union is in the sense that every point in the integral surface belongs to exactly one characteristic

The $n+1$ ODEs obtained are called *characteristic equations*. The method of characteristics converts a first order PDE to a system of ODE whose solution describe the integral curves.

Remark 2.1. If the coefficients $\mathbf{a}$ are independent of u, i.e. the PDE is linear then the characteristic curves will not intersect. Because, if the curves intersect then, at the point of intersection, they have the same tangent, which is not possible! If the coefficients $\mathbf{a}$ are constants (independent of both x and u) then the characteristic curves are straight lines.

Theorem 2.2. *Let* $\mathbf{a}$ *and* f *are smooth*[2] *functions in their arguments. A n-dimensional surface S is an integral surface for* (2.2) *iff it is the union of characteristics.*

Proof. The surface S, locally, has the equation $z = u(x)$ for some C^1 map u subject to appropriate relabelling of the coordinate variables and components of $\mathbf{a}$. For any (x_0, z_0), let $s \mapsto (x(s), z(s))$ denote the characteristic curve passing through (x_0, z_0). Set $v(s) := z(s) - u(x(s))$. Note that $x(0) = x_0$ and $z(0) = z_0$. Since $(x_0, z_0) \in S$, we have $v(0) = 0$. Then we have the IVP in v

$$\frac{dv}{ds} = \frac{dz}{ds} - \nabla u \cdot \frac{dx}{ds} = f(x,z) - \mathbf{a}(x,z) \cdot \nabla u(x) = f(x, v + u(x)) - \mathbf{a}(x, v + u(x)) \cdot \nabla u(x).$$

The above IVP has unique solution, locally. If a integral surface S is given by $z = u(x)$ then $v = 0$ is a solution of the above IVP and, hence, is its only solution. Thus, $z(s) = u(x(s), s)$ and S is the union of characteristics. Conversely, if S is union of characteristics then $v = 0$ and the above IVP implies that S is an intergal surface.

Theorem 2.3 (Existence and Uniqueness). *Let the coefficients* $\mathbf{a}$ *and* f *of* (2.2) *admit continuous partial derivatives with respect to the variables* x, u. *Let* $\{\gamma(r)\}$ *be parametrization of the initial data curve* Γ *and* u_0 *are continuously differentiable. If the data curve is non-characteristic, i.e.,* (2.1) *is satisfied then there exists a unique solution of* (2.2) *in some neighbourhood of* Γ .

Proof. For $r \in \mathbb{R}^{n-1}$, let $\gamma(r) := (\gamma_1(r), \ldots, \gamma_n(r))$ be the parametrization of Γ. The characteristic curves are solution to the ODE's

$$\frac{dx}{ds}(r,s) = \mathbf{a}(x,y,u) \text{ and } \frac{du}{ds}(r,s) = f(x,y,u)$$

such that on Γ, $x(r,s) = \gamma(r)$ and $u(r,s) = u_0(r)$. For each fixed $r \in \mathbb{R}^{n-1}$, any solution of the system of ODEs above is local in the s-variable. Note that the map $(r,s) \mapsto x(r,s)$ is invertible in a neighbourhood of Γ because Γ is non-characteristic and the Jacobian of the map is non-zero, i.e.

$$\begin{vmatrix} \partial_s x \\ \partial_{r_1} \gamma \\ \vdots \\ \partial_{r_{n-1}} \gamma \end{vmatrix}_{n \times n} = \begin{vmatrix} \mathbf{a}(\gamma(r), u_0(r)) \\ \partial_{r_1} \gamma \\ \vdots \\ \partial_{r_{n-1}} \gamma \end{vmatrix}_{n \times n} \neq 0.$$

[2] smooth refers to as much continuous differentiability as required

By inverse function theorem, one can solve for r and s in terms of x, locally, in the neighbourhood of Γ and $u(x) := u(r(x), s(x))$ is a solution of (2.2) in that neighbourhood of Γ.

$$\begin{aligned}
\mathbf{a} \cdot \nabla u(x) &= \sum_{i=1}^{n-1} u_{r_i}(\mathbf{a} \cdot \nabla r_i) + u_s(\mathbf{a} \cdot \nabla s) \\
&= \sum_{i=1}^{n-1} u_{r_i}(x'(s) \cdot \nabla r_i) + u_s(x'(s) \cdot \nabla s) \\
&= u_r \cdot \frac{dr}{ds} + u_s \frac{ds}{ds} = \frac{du}{ds} = f(x,u).
\end{aligned}$$

Also, on Γ, $u(x) = u(\gamma(r)) = u_0(\gamma(r))$.

Example 2.3 (Linear Transport Equation).

The transport of a substance in a fluid flowing (one dimenisonal flow) with constant speed a, with neither source or sink of substance, is given by

$$u_t(x,t) + au_x(x,t) + du(x,t) = cu_{xx}(x,t) \quad (x,t) \in \mathbb{R} \times (0, \infty)$$

where c is the diffusive coefficient of the substance and d is the rate of decay of the substance. Note that the case of no diffusion $(c = 0)$ is a linear first order equation which will be studied in this section. Consider the homogeneous linear transport equation in two variable,

$$u_t + au_x = 0, \quad x \in \mathbb{R} \text{ and } t \in (0, \infty),$$

where the constant $a \in \mathbb{R}$ is given. Thus, the given vector field $V(x,t) = (a, 1, 0)$. The characteristic equations are

$$\frac{dx}{ds} = a, \quad \frac{dt}{ds} = 1, \text{ and } \frac{dz}{ds} = 0.$$

Solving the 3 ODE's, we get

$$x(s) = as + c_1, \quad t(s) = s + c_2, \text{ and } z(s) = c_3.$$

Eliminating the parameter s, we get the curves (lines) $x - at =$ a constant in the xt-plane (see figure 2.1). The ODE for z determines the value of u along the lines $x - at = x_0$ and, hence, $z =$ a constant on the characteristic lines. Therefore, $u(x,t) = g(x - at)$ is the general solution, for an arbitrary function g.

Example 2.4. We shall now compute the solution of the Cauchy problem

$$\begin{cases} u_t + au_x = 0 & x \in \mathbb{R} \text{ and } t \in (0, \infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{2.5}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a given smooth function. We now check for non-characteristic property of Γ. Note that $\Gamma \equiv \{(x,0)\}$, the x-axis of xt-plane, is the (boundary) curve

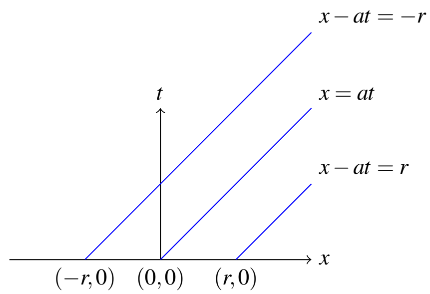

Fig. 2.1 Characteristics Curves

on which the value of u is given. Thus, $(\Gamma, u_0) = \{(x, 0, u_0(x))\}$ is the known curve on the solution surface of u. The curve Γ is given by the equation $\{t = 0\}$ and, hence, its normal is $(0,1)$. Γ is non-characteristic, because $(a,1)\cdot(0,1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dx(r,s)}{ds} = a, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions,

$$x(r,0) = r, \quad t(r,0) = 0, \text{ and } z(r,0) = u_0(r).$$

Solving the ODE's, we get

$$x(r,s) = as + c_1(r), \quad t(r,s) = s + c_2(r)$$

and $z(r,s) = c_3(r)$ with initial conditions

$$x(r,0) = c_1(r) = r$$

$$t(r,0) = c_2(r) = 0, \text{ and } z(r,0) = c_3(r) = u_0(r).$$

Therefore,

$$x(r,s) = as + r, \quad t(r,s) = s, \text{ and } z(r,s) = u_0(r).$$

We solve for r,s in terms of x,t and set $u(x,t) = z(r(x,t), s(x,t))$.

$$r(x,t) = x - at \text{ and } s(x,t) = t.$$

Therefore, $u(x,t) = z(r,s) = u_0(r) = u_0(x-at)$. See figure 2.2.

Remark 2.2. The *one space* dimension transport equation describes the transport of an insoluble[3] substance P in a fluid flowing with constant speed a. If we consider two observers, one a fixed observer A and another observer B, moving with speed a

[3] assuming no diffusion and no decay of substance.

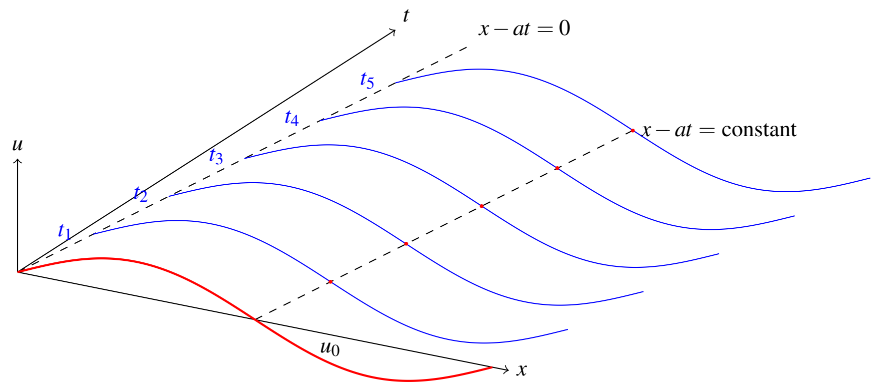

Fig. 2.2 Transportation of Initial Data u_0

and in the same direction as the substance P. For B, the substance P would appear stationary while for A, the fixed observer, the substance P would appear to travel with speed a. While the given PDE is the transport equation with respect to the observer A, the PDE corresponding to $z(r,s)$ in the characteristic equation describes the transport equation of the "stationary" substance P from the viewpoint of the moving observer B. The reference frame of B is given by r and s. The relation of r and s in terms x and t is the coordinate transformation of the reference frame of A to the reference frame of B. Fix a point x at time $t=0$. After time t, the point x remains as x for the fixed observer A, while for the moving observer B, the point x is now $x-at$. Therefore, the coordinate system for B is (r,s) where $r=x-at$ and $s=t$. Let $z(r,s)$ describe the motion of P from B's perspective. Since B sees P as stationary, the PDE describing the motion of P is $z_s(r,s)=0$. Therefore, $z(r,s)=g(r)$, for some arbitrary function g (sufficiently differentiable), is the solution from B's perspective. To solve the problem from A's perspective, note that

$$u_t = z_r r_t + z_s s_t = -az_r + z_s \text{ and}$$

$$u_x = z_r r_x + z_s s_x = z_r.$$

Therefore, $u_t + au_x = -az_r + z_s + az_r = z_s$ and, hence, $u(x,t) = z(r,s) = g(r) = g(x-at)$. The choice of g is based on our restriction to be in a classical solution set-up. Note that, for any choice of g, we have $g(x) = u(x,0)$. The line $x-at=r$, for some constant r, in the xt-plane tracks the flow of the substance placed at r at time $t=0$ (cf. Fig 2.1).

2.3.1 General Solution by Lagrange's Method

Consider the first order system of n ODEs

$$x'(t) = f(x(t)) \tag{2.6}$$

where the unknown is $x : I \to \Omega$, $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ are open interval and subset, respectively. The solution of the (2.6) is a curve or trajectory in $\mathbb{R}^n$ indexed by $t \in I$.

Definition 2.1 *A point $x_0 \in \mathbb{R}^n$ is said to be a* critical *or* stationary *point if $f(x_0) = 0$ otherwise it is called* regular *or* non-stationary *point.*

Definition 2.3. A non-constant C^1 function $\phi : \Omega_0 \subset \Omega \to \mathbb{R}$ is a *prime or first integral* of (2.6) if ϕ is constant for any solution $x(t)$ of (2.6) that lies within Ω_0.

Theorem 2.1 *A C^1-map $\phi : \Omega_0 \to \mathbb{R}$ is a prime integral of* (2.6) *iff $\nabla\phi(x) \cdot f(x) = 0$ for all $x \in \Omega_0$.*

Proof. Let ϕ be a prime integral of the first order system, i.e. ϕ is constant for any solution $x(t)$ of the system that lies within Ω_0. Thus,

$$0 = \frac{d}{dt}\phi(x(t)) = \nabla\phi \cdot x'(t) = \nabla\phi \cdot f(x(t)).$$

Since for any point in $x_0 \in \Omega_0$ there is a solution of the system through x_0, we have the orthogonality condition valid for all $x \in \Omega_0$. The above argument is reversible because the equality is valid in the reverse direction too. Thus the condition is both necessary and sufficient.

Definition 2.4. A collection of $k \le n$ C^1-map $\{\phi_1, \phi_2, \ldots, \phi_k\}$ are *functionally independent* in a neighbourhood $x_0 \in \mathbb{R}^n$ if the rank of the Jacobian matrix is k, i.e.

$$\text{rank} \begin{pmatrix} \nabla\phi_1(x_0) \\ \vdots \\ \nabla\phi_k(x_0) \end{pmatrix}_{k\times n} = k.$$

In other words, above definition means that $\{\nabla\phi_i\}_{1\le i\le k}$ are linearly independent. The following result says that among all possible prime integrals of (2.6) one can choose functionally independent prime integrals.

Theorem 2.2 *There exists exactly $(n-1)$ functionally independent prime integrals of* (2.6) *in a neighbourhood of a regular (non-stationary) point $x_0 \in \mathbb{R}^n$.*

The functionally independent $(n-1)$ prime integrals plays the same role as the linearly independent fundamentla system solutions of a linear differential equations.

Theorem 2.3 *Let $\{\phi_i\}_{1\le i\le n-1}$ be functionally independent prime integrals of* (2.6) *in a neighbourhood V_a of $x_0 \in \mathbb{R}^n$. If ψ is a prime integral of* (2.6) *in some neighbourhood of x_0 then there exists an open neighbourhood U of $\{\phi_i(x_0)\}_{1\le i\le n-1} \in \mathbb{R}^{n-1}$, an open neighbourhood $W_a \subset V_a$ and a C^1-map $G : U \to \mathbb{R}$ such that $\psi(x) = G(\phi_1(x), \ldots, \phi_{n-1}(x))$ for all $x \in W_a$.*

Theorem 2.4. *Let $\{\phi_i(x,u)\}_{1\le i\le n}$ be n functionally independent prime integrals of the $n+1$ system of characteristic ODEs (2.4) and $G = G(h)$ be a C^1 function on $\mathbb{R}^n$. If $\sum_{i=1}^n G_{h_i}\partial_u\phi_i \neq 0$, then $G(\phi_1(x,u),\dots,\phi_n(x,u)) = 0$ is the general solution of (2.2) in the implicit form.*

Proof. Define $\Phi(x,u) := G(\phi_1(x,u),\dots,\phi_n(x,u))$. We first observe that Φ is a prime integral of the system of $n+1$ equations (2.4). Note that $\nabla\Phi = \sum_{i=1}^n G_{h_i}\nabla\phi_i$ and, hence, $\nabla\Phi\cdot(\mathbf{a},f) = \sum_{i=1}^n G_{h_i}\nabla\phi_i\cdot(\mathbf{a},f) = 0$. By Theorem 2.1, Φ is a prime integral of the $(n+1)$ system of characteristic equations (2.4) because each ϕ_i are prime integrals. Also Φ satisfies the equation

$$\mathbf{a}(x,u)\cdot\nabla_x\Phi - f(x,u)\Phi_u = 0. \tag{2.7}$$

Note that $\Phi_u \neq 0$ because $\Phi_u = \sum_{i=1}^n G_{h_i}\partial_u\phi_i \neq 0$. Since Φ is a prime integral and $\Phi_u \neq 0$, then the equation $\Phi(x,u) = 0$ defines implicitly an integral surface $u = u(x,y)$ of (2.2). By the Implicit Function Theorem, the u is such that $u_x = -\frac{\Phi_x}{\Phi_u}$ and $u_y = -\frac{\Phi_y}{\Phi_u}$ and from (2.7) we get easily (2.2).

Example 2.5. Let us compute the general solution of the first order PDE $xu_x(x,y) + yu_y(x,y) = u(x,y)$. The characteristic equations are

$$\frac{dx}{ds} = x(s) \quad \frac{dy}{ds} = y(s) \text{ and } \frac{dz}{ds} = z(s).$$

Thus, $x(s) = c_1e^s$, $y(s) = c_2e^s$ and $z(s) = c_3e^s$. Eliminating the parameter s, we get $y/x = c_4$ and $z/x = c_5$. Alternately, we can solve the ODEs

$$\frac{dy}{dx} = \frac{y}{x} \text{ and } \frac{dz}{dx} = \frac{z}{x},$$

we get $y/x = c_4$ and $z/x = c_5$. Set $\phi(x,y,z) := y/x$ and $\psi(x,y,z) := z/x$. For $x \neq 0$, the functions ϕ and ψ are *functionally independent*, i.e., their Jacobian has maximal rank:

$$\operatorname{rank}\begin{pmatrix}\phi_x & \phi_y & \phi_z\\ \psi_x & \psi_y & \psi_z\end{pmatrix} = \operatorname{rank}\begin{pmatrix}-\frac{y}{x^2} & \frac{1}{x} & 0\\ -\frac{z}{x^2} & 0 & \frac{1}{x}\end{pmatrix} = 2.$$

Thus, the general solution is $G(y/x, z/x) = 0$ for an arbitrary function G. Compare this answer with Example 1.16.

Example 2.6. Let us compute the general solution of the first order PDE $yu_x - xu_y = 2xyu$. The characteristic equations are

$$\frac{dx}{ds} = y(s) \quad \frac{dy}{ds} = -x(s) \text{ and } \frac{dz}{ds} = 2xyz.$$

To avoid cumbersome ODE, let us begin by assuming $y \neq 0$, then dividing the entire equation by y, we get

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = -x(s)/y(s) \text{ and } \frac{dz}{ds} = 2xz.$$

Solving which we get $x(s) = s + c_1$, $y(s) = -s^2 - 2c_1 s + 2c_2$ and $|z(s)| = c_3 e^{s^2+2c_1 s}$. Eliminating s between x and y, we get the characteristic curves to be $y^2 + x^2 =$ a constant and $z = c_4 e^{x^2}$. Alternately, we can solve

$$\frac{dy}{dx} = -x(s)/y(s) \text{ and } \frac{dz}{dx} = 2xz.$$

we get the characteristic curves to be $y^2 + x^2 =$ a constant and $z = c_4 e^{x^2}$. Thus, the general solution is $G(y^2 + x^2, e^{-x^2} z) = 0$. Explicitly,

$$u(x,y) = g(y^2 + x^2) e^{x^2}$$

for some arbitrary smooth functions g.

Example 2.7. Let us compute the general solution of the first order PDE $u_x + 2xu_y = u^2$. The characteristic equations are

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = 2x(s) \text{ and } \frac{dz}{ds} = z^2(s).$$

Solving which we get $x(s) = s + c_1$, $y(s) = s^2 + 2c_1 s + c_2$ and $z(s) = -1/(s + c_3)$. Eliminating s between x and y, we get the characteristic curves to be $y - x^2 =$ a constant and $x + 1/z =$ a constant. Alternately, we can solve

$$\frac{dy}{dx} = 2x(s) \text{ and } \frac{dz}{dx} = z^2.$$

we get the characteristic curves to be $y - x^2 =$ a constant and $x + 1/z$ = a constant. Thus, the general solution is $G(y - x^2, x + 1/z) = 0$. Explicitly,

$$u(x,y) = \frac{-1}{x + g(y - x^2)}$$

for some arbitrary smooth functions g.

Example 2.8. Let us compute the general solution of the first order PDE $u_{x_1} + e^{x_1} u_{x_2} + e^{x_3} u_{x_3} = (2x_1 + e^{x_1}) e^u$. The characteristic equations are

$$\frac{dx_1}{ds} = 1 \quad \frac{dx_2}{ds} = e^{x_1} \quad \frac{dx_3}{ds} = e^{x_3} \text{ and } \frac{dz}{ds} = (2x_1 + e^{x_1}) e^{z(s)}.$$

Solving which we get $x_1(s) = s + c_1$, $x_2(s) = e^{c_1} e^s + c_2$, $e^{-x_3(s)} = -s + c_3$ and

$$e^{-z(s)} = -s^2 - 2c_1 s - e^{s+c_1} + c_4.$$

Eliminating s between x_1 and x_2, we get $x_2 - e^{x_1} =$ a constant, $e^{-x_3} + x_1 =$ a constant. Altenately, we can solve

$$\frac{dx_2}{dx_1} = e^{x_1} \quad \frac{dx_3}{dx_1} = e^{x_3} \text{ and } \frac{dz}{dx_1} = (2x_1 + e^{x_1})e^{z(s)}$$

we get $x_2 - e^{x_1} =$ a constant, $e^{-x_3} + x_1 =$ a constant and $x_1^2 + e^{x_1} + e^{-z} =$ a constant. Then the general solution is $G(x_2 - e^{x_1}, e^{-x_3} + x_1, e^{-u} + x_1^2 + e^{x_1}) = 0$. Explcitly,

$$e^{-u} = g(x_2 - e^{x_1}, e^{-x_3} + x_1) - x_1^2 - e^{x_1}$$

for some arbitrary smooth functions g.

Example 2.9. Let us compute the general solution of the first order PDE $yuu_x + xuu_y = xy$. The characteristic equations are

$$\frac{dx}{ds} = yz, \quad \frac{dy}{ds} = xz \text{ and } \frac{dz}{ds} = xy.$$

Hence,

$$\begin{aligned} 0 &= x\frac{dx}{ds} - y\frac{dy}{ds} \\ &= \frac{d(x^2)}{ds} - \frac{d(y^2)}{ds} \\ &= \frac{d(x^2 - y^2)}{ds}. \end{aligned}$$

Thus, $x^2 - y^2 = c_1$ and, similarly, $x^2 - z^2 = c_2$. Alternately, we can solve

$$\frac{dy}{dx} = \frac{x}{y} \text{ and } \frac{dz}{dx} = \frac{x}{z}.$$

Hence, the general solution is $G(x^2 - y^2, x^2 - z^2) = 0$ for some arbitrary function G. Explicitly, for some g or h,

$$u^2(x,y) = x^2 + g(x^2 - y^2) \text{ or } u^2(x,y) = y^2 + h(x^2 - y^2).$$

Example 2.10. Let us compute the general solution of the first order PDE $2yu_x + uu_y = 2yu^2$. The characteristic equations are

$$\frac{dx}{ds} = 2y(s) \quad \frac{dy}{ds} = z(s) \text{ and } \frac{dz}{ds} = 2y(s)z^2(s).$$

Solving in the parametric form is quite cumbersome, because we will have a second order nonlinear ODE of y, $y''(s) = 2y(y')^2$. However, for $u \neq 0$, we get $\frac{dz}{dy} = 2y(s)z(s)$ solving which we get $\ln|z| = y^2 + c_1$ or $z = c_2 e^{y^2}$. Similarly, $\frac{dy}{dx} = \frac{z}{2y} = \frac{c_2 e^{y^2}}{2y}$ solving which we get $c_2 x + e^{-y^2} = c_3$ or, equivalently, $xze^{-y^2} + e^{-y^2} = c_3$. Thus, the general solution is

$$G(e^{-y^2}(1 + xz), e^{-y^2} z) = 0.$$

or

$$u(x,y) = g\left(e^{-y^2}(1+xu)\right)e^{y^2}$$

for some arbitrary smooth functions g. Note that $u \equiv 0$ is a solution if we choose $g = 0$. The characteristic curves are $(1+xu)e^{-y^2}$ = a constant and along these curves ue^{-y^2} is constant.

Example 2.11. The *higher dimension homogeneous* transport problem is given by

$$u_t(x,t) + \mathbf{a} \cdot \nabla u(x,t) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{2.8}$$

where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown and $\mathbf{a} \in \mathbb{R}^n$. By setting $y := (x,t)$ in (2.8),

$$(\mathbf{a},1) \cdot \nabla_y u(y) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Thus, the directional derivative of u, for all $y \in \mathbb{R}^n \times (0,\infty)$, along the direction $(\mathbf{a},1)$ is zero. Hence, u must be constant along all lines in the direction of $(\mathbf{a},1)$. The parametric equation of a line passing through (x,t) and parallel to $(\mathbf{a},1)$ is $(x,t)+s(\mathbf{a},1) = (x+s\mathbf{a},t+s)$. For the lines to lie in the half-space $\mathbb{R}^n \times [0,\infty)$, it is enough to consider $s \geq -t$. Loosely speaking, since u is constant on the line $(x+s\mathbf{a},t+s)$, the value of u at $s = 0$ and $s = -t$ must coincide. Hence, $u(x,t) = u(x-t\mathbf{a},0)$. To derive this conclusion in a precise way, we set $v(s) := u(x+s\mathbf{a},t+s)$, for all $s \in (-t,\infty)$, and for a fixed $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Thus,

$$\begin{aligned}
\frac{dv(s)}{ds} &= \nabla u(x+s\mathbf{a},t+s) \cdot \frac{d(x+s\mathbf{a})}{ds} + u_t(x+s\mathbf{a},t+s)\frac{d(t+s)}{ds} \\
&= \mathbf{a} \cdot \nabla u(x+s\mathbf{a},t+s) + u_t(x+s\mathbf{a},t+s) = 0.
\end{aligned}$$

The last equality is due to (2.8) and, hence, v is a constant function. Therefore, $v(0) = v(-t)$ which implies that $u(x,t) = u(x-t\mathbf{a},0)$. For instance, if the value of u is known at time $t = 0$, say $u(x,0) = g(x)$ on $\mathbb{R}^n \times \{t = 0\}$ for a function $g : \mathbb{R}^n \to \mathbb{R}$, then

$$u(x,t) = u(x-t\mathbf{a},0) = g(x-t\mathbf{a}).$$

Since (x,t) was arbitrary in $\mathbb{R}^n \times (0,\infty)$, we have $u(x,t) = g(x-t\mathbf{a})$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Thus, $g(x-t\mathbf{a})$ is a classical solution to (2.8) whenever $g \in C^1(\mathbb{R}^n)$. We remark that if $g \notin C^1(\mathbb{R}^n)$ then $g(x-t\mathbf{a})$ can be interpreted as a *weak* solution of (2.8).

Remark 2.3. Observe that in the Cauchy problem of Transport equation (2.5), suppose we choose a discontinuous u_0, viz.

$$\phi(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

then $u(x,t)$ inherits this jump continuity. The characteristic curve passing through the point of disconitnuity will propagate the jump in the soluton surface. Thus, u is no longer a smooth solution. In applications it is often necessary to consider such solutions which, by our definition, is not even a differentiable, hence, not a solution. Such situations give rise to the need of relaxing the notion of solution.

Example 2.12. The *inhomogeneous* transport problem is

$$u_t(x,t) + \mathbf{a} \cdot \nabla u(x,t) = f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{2.9}$$

where $\mathbf{a} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ are given. The f represents the intensity of an external source. Motivated from the homogeneous case, set $v(s) := u(x + s\mathbf{a}, t + s)$, for all $s \in \mathbb{R}$, for a fixed $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Thus,

$$\frac{dv(s)}{ds} = \mathbf{a} \cdot \nabla u(x + s\mathbf{a}, t + s) + u_t(x + s\mathbf{a}, t + s) = f(x + s\mathbf{a}, t + s).$$

In the homogeneous case, we concluded that $u(x,t) - u(x - t\mathbf{a}, 0) = v(0) - v(-t) = 0$. Therefore, consider

$$\begin{aligned} u(x,t) - u(x - t\mathbf{a}, 0) &= v(0) - v(-t) \\ &= \int_{-t}^{0} \frac{dv}{ds}\, ds \\ &= \int_{-t}^{0} f(x + s\mathbf{a}, t + s)\, ds \\ &= \int_{0}^{t} f(x + (s - t)\mathbf{a}, s)\, ds. \end{aligned}$$

The last equality is obtained by a change of variable $s := t + s$. Hence,

$$u(x,t) = u(x - t\mathbf{a}, 0) + \int_0^t f(x + (s - t)\mathbf{a}, s)\, ds$$

solves (2.9).

Example 2.13. The *one space* dimension transport equation, with no diffusion but decay, is

$$u_t(x,t) + a u_x(x,t) + D u(x,t) = 0 \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

with $a, D \in \mathbb{R}$. The characteristic equation

$$\frac{dx}{ds} = a, \quad \frac{dt}{ds} = 1, \text{ and } \frac{dz}{ds} = -Dz.$$

Solving the 3 ODE's, we get

$$x(s) = as + c_1, \quad t(s) = s + c_2, \text{ and } z(s) = c_3 e^{-Ds}.$$

Eliminating the parameter s, we get the curves (lines) $x - at =$ a constant in the xt-plane (see figure 2.1). The ODE for z determines the value of u along the lines

$x - at = x_0$. Therefore, $u(x,t) = g(x-at)e^{-Dt}$ is the general solution, for an arbitrary function g.

2.3.2 *Data on Characteristic Curves, Non-Global and Blow-up of Solutions*

Exercise 2.1 *(i) Find the general solution of the first order PDE* $2u_x(x,y) + 3u_y(x,y) + 8u(x,y) = 0$.

(ii) For the PDE given above, check for the characteristic property of the following curves

(a) $y = x$ *in the xy-plane*
(b) $y = \frac{3x-1}{2}$.

(iii) Discuss the particular solutions of the above PDE, corresponding to

(a) $u(x,x) = x^4$ *on* $y = x$
(b) $u(x,(3x-1)/2) = x^2$ *on* $y = (3x-1)/2$
(c) $u(x,(3x-1)/2) = e^{-4x}$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

Proof. (i) The characteristic equations are

$$\frac{dx}{ds} = 2, \quad \frac{dy}{ds} = 3 \text{ and } \frac{dz}{ds} = -8z.$$

Hence,

$$x(s) = 2s + c_1 \quad y(s) = 3s + c_2 \text{ and } z(s) = c_3 e^{-8s}.$$

Thus, $3x - 2y = c_4$ and $z = c_4 e^{-4x}$ or $z = c_5 e^{-8y/3}$. Hence, the general solution is $G(3x-2y, e^{4x}z) = 0$. Explicitly, for some h or g,

$$u(x,y) = h(3x-2y)e^{-4x} \text{ or } u(x,y) = g(3x-2y)e^{-8y/3}.$$

(ii) (a) Parametrise the curve $y = x$ as $\Gamma(r) : r \mapsto (r,r)$. Thus $\gamma_1(r) = \gamma_2(r) = r$. Since the coefficients of the PDE are $a(r) = 2$ and $b(r) = 3$, we have

$$(a,b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2,3) \cdot (-1,1) = -2 + 3 = 1 \neq 0.$$

Hence Γ is non-characteristic.

(b) Parametrise the curve $y = (3x-1)/2$ as $\Gamma(r) : r \mapsto (r, (3r-1)/2)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = (3r-1)/2$ and

$$(a,b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2,3) \cdot (-3/2, 1) = -3 + 3 = 0.$$

Hence Γ is a characteristic curve.

(iii) Recall that the general solution is $G(3x-2y, e^{4x}z) = 0$ or

$$u(x,y) = h(3x-2y)e^{-4x} \text{ or } u(x,y) = g(3x-2y)e^{-8y/3}.$$

(a) Now, $u(x,x) = x^4$ implies $G(x, e^{4x}x^4) = 0$. Thus,

$$e^{4x}z = e^{12x-8y}(3x-2y)^4$$

and

$$u(x,y) = (3x-2y)^4 e^{8(x-y)}.$$

Thus, we have a unique solution u.

(b) Using the given condition, we have $G(1, x^2e^{4x}) = 0$. Either $h(1) = x^2e^{4x}$ or $h(x^2e^{4x}) = 1$. The first case is not valid (multi-valued function). The second case corresponds to $z = e^{-4x}$ which will not satisfy the Cauchy data. Hence there is no solution u solving the given PDE with the given data.

(c) Once again using the given condition, we have $G(1, x^2e^{4x}) = 0$. Either $h(1) = x^2e^{4x}$ or $h(x^2e^{4x}) = 1$. The first case is not valid (multi-valued function). The second case corresponds to $z = e^{-4x}$ which will satisfy the Cauchy data. Since there many choices of h that satisfies $h(x^2e^{4x}) = 1$, we have infinite number of solutions (or choices for) u that solves the PDE.

Exercise 2.2 *(i) Find the general solution (in terms of arbitrary functions) of the first order PDE $u_x(x,y) + u_y(x,y) = 1$.*

(ii) For the PDE given above, check for the characteristic property of the following curves

(a) the x-axis, $\{(x,0)\}$, in the xy-plane

(b) $y = x$.

(iii) Discuss the particular solutions of the above PDE, corresponding to

(a) $u(x,0) = \phi(x)$ on x-axis.

(b) $u(x,x) = x$ on $y = x$.

(c) $u(x,x) = 1$ on $y = x$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

Proof. (i) The characteristic equations are

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 1 \text{ and } \frac{dz}{ds} = 1.$$

Hence,

$$x(s) = s + c_1 \quad y(s) = s + c_2 \text{ and } z(s) = s + c_3.$$

Thus, $y - x = c_4$ and $z - x = c_5$ or $x - y = c_4$ and $z - y = c_5$. Hence, for some h and g,

$$u(x,y) = x + h(y-x) \text{ or } u(x,y) = y + g(x-y).$$

(ii) (a) Parametrise the curve x-axis as $\Gamma(r) : r \mapsto (r,0)$. Thus $\gamma_1(r) = r$ and $\gamma_2(r) = 0$. Since the coefficients of the PDE are $a(r) = 1$ and $b(r) = 1$, we have

$$(a,b)\cdot(-\gamma_2'(r), \gamma_1'(r)) = (1,1)\cdot(0,1) = 1 \neq 0.$$

Hence Γ is non-characteristic.

(b) Parametrise the curve $y = x$ as $\Gamma(r) : r \mapsto (r,r)$. Hence $\gamma_1(r) = r = \gamma_2(r)$ and

$$(a,b)\cdot(-\gamma_2'(r), \gamma_1'(r)) = (1,1)\cdot(-1,1) = -1+1 = 0.$$

Hence Γ is a characteristic curve.

(iii) Recall that the general solution is

$$u(x,y) = x + h(y-x) \text{ or } u(x,y) = y + g(x-y).$$

(a) Now, $u(x,0) = \phi(x)$ implies $h(x) = \phi(-x) + x$ or $g(x) = \phi(x)$, and

$$u(x,y) = y + \phi(x-y).$$

Thus, we have a unique solution u.

(b) Using the given condition, we have $h(0) = 0$ or $g(0) = 0$. One has many choices of function satisying these conditions. Thus, we have infinite number of solutions (or choices for) u that solves the PDE.

(c) Once again using the given condition, we have $h(0) = 1 - x$ or $g(0) = 1 - x$ for all $x \in \mathbb{R}$. This implies h and g are not well defined. We have no function h and g, hence there is no solution u solving the given PDE with the given data.

Exercise 2.3 *(i) Find the general solution (in terms of arbitrary functions) of the first order PDE $xu_x(x,y) + 2xuu_y(x,y) = u(x,y)$.*

(ii) For the PDE given above, check if the following curves in xy-plane are non-characteristic and discuss the particular solutions of the PDE

(a) $y = 2x^2 + 1$ and $u(x, 2x^2+1) = 2x$.
(b) $y = 3x^3$ and $u(x, 3x^3) = 2x^2$.
(c) $y = x^3 - 1$ and $u(x, x^3-1) = x^2$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

Proof. (i) The characteristic equations are

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = 2xz \text{ and } \frac{dz}{ds} = z.$$

Hence,

$$x(s) = c_1 e^s \quad z(s) = c_2 e^s \text{ and } y(s) = c_1 c_2 e^{2s} + c_3.$$

Thus, $y = c_2/c_1x^2 + c_3$ and $z = c_2/c_1x$. Therefore, $y - zx = c_3$ and, for some g,

$$u(x,y) = xg(y - xu).$$

The characteristic curves are $y - xu =$ a constant which depends on u.

(ii) (a) Parametrise the curve $y = 2x^2 + 1$ as $\Gamma(r) : r \mapsto (r, 2r^2 + 1)$. Thus $\gamma_1(r) = r$ and $\gamma_2(r) = 2r^2 + 1$. Since the coefficients of the PDE are $a(r) = r$ and $b(r,u) = 4r^2$, we have

$$(a,b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 4r^2) \cdot (-4r, 1) = -4r^2 + 4r^2 = 0.$$

Hence Γ is not non-characteristic. But on the characteristic curves $y - 2x^2 = 1$ the function $u = 2x$ solves the PDE. Elsewhere the solution is non-unique and there are many choices because $u(x, 2x^2 + 1) = 2x$ implies $g(1) = 2$. Thus, we have infinite number of solutions (or choices for) u that solves the PDE on other characterisitc curves.

(b) Parametrise the curve $y = 3x^3$ as $\Gamma(r) : r \mapsto (r, 3r^3)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = 3r^3$ and

$$(a,b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 4r^3) \cdot (-9r^2, 1) = -9r^3 + 4r^3 = -5r^3 \neq 0$$

for $r \neq 0$. Hence Γ is a non-characteristic curve. Using $(u(x, 3x^3) = 2x^2$ we get $2x^2 = xg(3x^3 - 2x^3)$ which implies $2x = g(x^3)$. Thus, $g(x) = 2x^{1/3}$ and $u(x,y) = 2x(y - xu)^{1/3}$ or $u^3(x,y) + 8x^4u = 8x^3y$. Thus, we have a unique solution. The characteristic curves are $y - 2x^3 =$ a constant and the data $u = 2x^2$ is given on $y - 3x^3 = 0$.

(c) Parametrise the curve $y = x^3 - 1$ as $\Gamma(r) : r \mapsto (r, r^3 - 1)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = r^3 - 1$ and

$$(a,b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 2r^3) \cdot (-3r^2, 1) = -3r^3 + 2r^3 = -r^3 \neq 0$$

for $r \neq 0$. Hence Γ is a non-characteristic curve. Using $(u(x, x^3 - 1) = x^2$ we get $x = g(-1)$. Thus g is not well defined and, hence there is no solution. The characteristic curves are $y - x^3 =$ a constant and $u = x^2$ given on $y - x^3 = -1$ is not a solution.

Example 2.14. Consider the Burgers' equation $u_y(x,y) + u(x,y)u_x(x,y) = 1$ in $\mathbb{R}^2$.

(a) Consider the Cauchy data $u(x,x) = 2$ on $\{y = x\}$. The normal to the data curve is $(-1,1)$. Therefore, $(u,1) \cdot (-1,1) = -u + 1 = -2 + 1 = -1 \neq 0$. Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = z(r,s), \frac{dy(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = 1$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = r, \text{ and } z(r,0) = 2.$$

Therefore, $y(r,s) = s + c_1(r)$, $z(r,s) = s + c_2(r)$ and $x(r,s) = s^2/2 + c_2(r)s + c_3(r)$. Using the initial conditions, we get $c_1(r) = r = c_3(r)$ and $c_2(r) = 2$. Solving for r and s, in terms of x, y and z, we get $s = y - r = z - 2$ and $2x = s^2 + 4s + 2r$. Solving the system for x and y, by eliminating s, we get the characteristic curves as the parabolae $C(x,y,r) := 2(x+2-r) - (y+2-r)^2 = 0$. See the blue curves in 2.3. For each $r \in \mathbb{R}$, the parabola has the vertex at $(r-2, r-2)$. Using the equation of s and r in x, we get $z^2 - 2z - 2(x-y) = 0$ and, hence, $z = 1 \pm \sqrt{1+2(x-y)}$. The choice of minus sign before radical is ruled out because it does not satisfy the initial data. Hence, $u(x,y) = 1 + \sqrt{1+2(x-y)}$. The solution u is constant along the lines $y - x =$ a constant and $u \equiv 1$ on the line $x - y = -1/2$. The solution does not exist in the region $x - y < -1/2$ because the term inside square root is negative here. The region to the left of the cyan coloured line in 2.3. The line $x - y = -1/2$ is the envelope of the family of characteristic curves $C(x,y,r) = 0$. To see this differentiate C with respect to r and eliminate r between $C_r = 0$ and $C = 0$. Note that $C_r(x,y,r) = y + 1 - r = 0$. Using this r in C, we get the line $x - y = -1/2$.

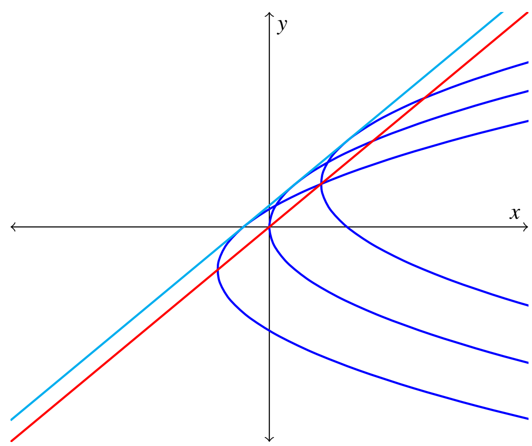

Fig. 2.3 Non-Global Solutions

(b) Consider the Cauchy data $u(x^2, 2x) = x$. The normal to the data curve is $(-2, 2x)$. Therefore, $(u,1) \cdot (-2, 2x) = -2u + 2x = 0$. Hence, the given initial curve is characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = z(r,s), \frac{dy(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = 1$$

with initial conditions

$$x(r,0) = r^2, \quad y(r,0) = 2r, \text{ and } z(r,0) = r.$$

Therefore, $y(r,s) = s + c_1(r)$, $z(r,s) = s + c_2(r)$ and $x(r,s) = s^2/2 + c_2(r)s + c_3(r)$. Using the initial conditions, we get $c_1(r) = 2r$, $c_2(r) = r$ and $c_3(r) = r^2$. Solving for r and s, in terms of x, y and z, we get $s = y - 2r = z - r$ and $2x = s^2 + 2rs + 2r^2$. Solving the system for x and y, by eliminating s, we get the characteristic curves as the parabolae $C(x,y,r) := 2x - r^2 - (y-r)^2 = 0$. Using the equation of s and r in x, we get $2z^2 + 2yz + y^2 - 2x = 0$ and, hence, $u(x,y) == y/2 \pm \sqrt{x - y^2/4}$. Both the choice of sign before the radical is valid because it satisfies the initial data. The solution is two-valued in the region $\{x > y^2/4\}$ and no solution exists in the region $\{x < y^2/4\}$ because the term inside square root is negative here. The parabola $x - y^2/4 = 0$ is the envelope of the family of characteristic curves $C(x,y,r) = 0$. To see this differentiate C with respect to r and eliminate r between $C_r = 0$ and $C = 0$. Note that $C_r(x,y,r) = 2y - 4r = 0$. Using this r in C, we get the parabola $x - y^2/4 = 0$.

(c) Consider the Cauchy data $u(x^2/2, x) = x$. The normal to the data curve is $(-1, x)$. Therefore, $(u,1)\cdot(-1,x) = -u + x = 0$. Hence, the given initial curve is characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = z(r,s), \frac{dy(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = 1$$

with initial conditions

$$x(r,0) = r^2/2, \quad y(r,0) = r, \text{ and } z(r,0) = r.$$

Therefore, $y(r,s) = s + c_1(r)$, $z(r,s) = s + c_2(r)$ and $x(r,s) = s^2/2 + c_2(r)s + c_3(r)$. Using the initial conditions, we get $c_1(r) = r = c_2(r)$ and $c_3(r) = r^2/2$. Solving for r and s, in terms of x, y and z, we get $s = y - r = z - r$ and $2x = s^2 + 2rs + r^2 = (s+r)^2$. Solving the system for x and y, by eliminating s, we get the characteristic curves as the parabola $C(x,y,r) := x - y^2/2 = 0$ for all r. Using the equation of s and r in x, we get $z^2 = 2x + g(z-y)$ for any arbitrary function g such that $g(0) = 0$.

Example 2.15. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Let $\Gamma := \{(x,0) \mid x \in \mathbb{R}\}$. Consider the semi-linear PDE

$$\begin{cases} u_x(x,y) + u_y(x,y) = u^2(x,y) & \text{in } \Omega \\ \qquad\qquad u(x,0) = \phi(x) & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,0)$ for all $r \in \mathbb{R}$. Therefore,

$$(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (1,1)\cdot(0,1) = 1 \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = 1, \frac{dy(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = z^2(r,s)$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = 0, \text{ and } z(r,0) = \phi(r).$$

Therefore, $x(r,s) = s + c_1(r)$, $y(r,s) = s + c_2(r)$ and $z(r,s) = \frac{-1}{s+c_3(r)}$. Using the initial conditions, we get $c_1(r) = r$, $c_2(r) = 0$ and $c_3(r) = -(\phi(r))^{-1}$. Note that this makes sense only if $\phi(r) \neq 0$ for all r. To overcome this situation, we write $z(r,s) = \frac{\phi(r)}{1-\phi(r)s}$. Also, we have $(x(r,s), y(r,s)) = (s+r, s)$, where $s \geq 0$ and $r \in \mathbb{R}$. Moreover, $r = x - y$ and $s = y$. Therefore, $u(x,y) = z(r,s) = \frac{\phi(r)}{1-\phi(r)s} = \frac{\phi(x-y)}{1-\phi(x-y)y}$. Note that the non-linearity in the z-variable, even though the equation is linear, may cause a possible blow-up (or singularity) in the solution. For instance, even if we assume ϕ is bounded, a very large value of y may induce a singularity.

Example 2.16. Consider the PDE $u_x + u_y = u^2$. Let us find the solution of the PDE that passes through the curve $u = x$ on $y = -x$. The characteristic equations (ODE's) are

$$\frac{dx}{ds} = 1 = \frac{dy}{ds} \text{ and } \frac{dz}{ds} = z^2(s).$$

The characteristic curves are $x - y =$ a constant and $z^{-1} + x =$ a constant. Note that we have used the fact that $u \not\equiv 0$ because zero solution does not satisfy the Cauchy data. Thus, the general solution is $u(x,y) = \frac{1}{g(x-y)-x}$, for some arbitrary smooth functions g. Using the Cauchy data we obtain, $g(x) = \frac{4+x^2}{2x}$ and hence,

$$u(x,y) = \frac{2(x-y)}{4+y^2-x^2}.$$

Note that along the hyperbola $x^2 - y^2 = 4$ the solution is infinite.

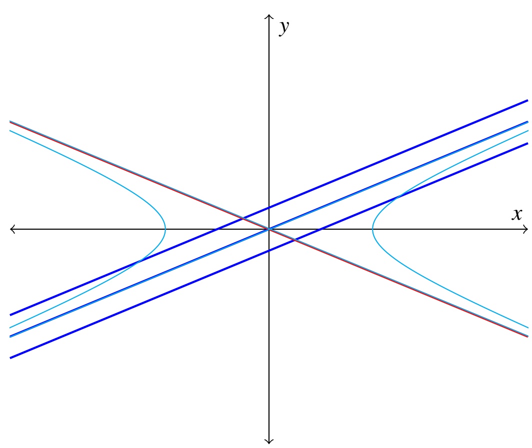

Fig. 2.4 Blow-up of Solutions

2.3.3 Inflow Characteristics

For a well-posed Cauchy problem, the Cauchy data is prescribed only on the *inflow boundary* part of the boundary of the domain. The inflow boundary part $\Gamma_i \subset \Gamma$ is defined as

$$\Gamma_i := \{x \in \Gamma \mid \mathbf{a} \cdot \nu(x) < 0\}$$

where $\mathbf{a}$ is the coefficient vector of the first order semilinear PDE $\mathbf{a} \cdot \nabla u(x) = f(x,u)$ and $\nu(x)$ is the unit outward normal at x.

Example 2.17. Let us consider the linear transport equation in two variable in the domain $\Omega := (0,\infty) \times (0,\infty)$.

$$u_t + a u_x = 0, \quad x \in (0,\infty) \text{ and } t \in (0,\infty),$$

where the constant $a \in \mathbb{R}$ is given. The boundary of Ω is $\Gamma := \{(0,t) \mid t > 0\} \cup \{(x,0) \mid x > 0\}$. Let us check the non-characteristic condition of the data curve Γ. Its parametrization is $(r,0)$ when $r \geq 0$ and $(0,-r)$ when $r < 0$. Then

$$(a,1) \cdot (\gamma_2'(r), -\gamma_1'(r)) = \begin{cases} -1 & r \geq 0 \\ -a & r < 0. \end{cases}$$

Thus, the γ is non-characteristic. Though, it seems natural to define the initial value u_0 on the whole of Γ, the initial value prescription depends on the boundary with the characteristics flowing in to the boundary. This is called *inflow boundary*. In other words, we include the first point of intersection of the projected characteristics with the boundary Γ. For instance, if $a > 0$ then the projected characteristics curves inflow in to the entire boundary Γ and, hence, u_0 should be prescribed on all of Γ. However, if $a < 0$ then it is enough to prescribe u_0 on the subset $\{(x,0) \mid x > 0\}$ of Γ. See fig 2.5.

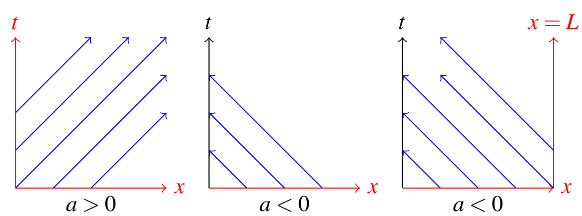

Fig. 2.5 Inflow Characteristics

Example 2.18. Let us consider the linear transport equation in two variable in the domain $\Omega := (0,L) \times (0,\infty)$.

$$u_t + a u_x = 0, \quad x \in (0,L) \text{ and } t \in (0,\infty),$$

where the constant $a \in \mathbb{R}$ is given. The boundary of Ω is $\Gamma := \{(0,t) \mid t > 0\} \cup \{(0,L) \times \{0\}\} \cup \{(L,t) \mid t > 0\}$. Let us check the non-characteristic condition of the data curve Γ. Its parametrization is

$$(\gamma_1(r), \gamma_2(r)) = \begin{cases} (r,0) & 0 \le r \le L \\ (L, r-L) & L < r < \infty \\ (0,-r) & r < 0. \end{cases}$$

Then

$$(a,1) \cdot (\gamma_2'(r), -\gamma_1'(r)) = \begin{cases} -1 & 0 \le r \le L \\ a & L < r < \infty \\ -a & r < 0. \end{cases}$$

Thus, the γ is non-characteristic. If $a > 0$ then the projected characteristics curves inflow in to a subset of Γ and, hence, u_0 should be prescribed on the subset $\{(0,t) \mid t > 0\} \cup \{(0,L) \times \{0\}\}$ of Γ. However, if $a < 0$ then it is enough to prescribe u_0 on the the subset $\{(0,L) \times \{0\}\} \cup \{(L,t) \mid t > 0\}$ of Γ. See fig 2.5.

2.3.4 Intersection of Characteristic Curves: Shocks

Example 2.19 (Solution in Implicit Form). Consider the PDE of the form $u_t(x,t) + a(u)u_x(x,t) = 0$ in $\mathbb{R} \times (0,\infty)$. The characteristic equations are:

$$\frac{dx}{ds} = a(z), \quad \frac{dt}{ds} = 1, \text{ and } \frac{dz}{ds} = 0.$$

On eliminating the parameter s, we get $\frac{dx}{dt} = a(z)$ and $\frac{dz}{dt} = 0$. Solving the system of ODE, we get the general solution $u(x,t) = g[x - ta(u)]$ in the implicit form. In particular, the *Burgers' equation* $u_t(x,t) + u(x,t)u_x(x,t) = 0$ in $\mathbb{R} \times (0,\infty)$ is the situation corresponding to $a(u) \equiv u$. The general solution is $u(x,t) = g(x - tu)$ in the implicit form.

Example 2.20. We now consider the Cauchy problem of *Burgers' equation*

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We first check for non-characteristic property of Γ. Note that $(\Gamma, u_0) = \{(x, 0, u_0(x))\}$ is the known curve on the solution surface of u. We parametrize the curve Γ with r-variable, i.e., $\Gamma = \{(r,0)\}$. Γ is non-characteristic, because $(u,1) \cdot (0,1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dx(r,s)}{ds} = z, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions $x(r,0) = r, \quad t(r,0) = 0,$ and $z(r,0) = u_0(r)$. Solving the ODE corresponding to z, we get $z(r,s) = c_3(r)$ with initial conditions $z(r,0) = c_3(r) = u_0(r)$. Thus, $z(r,s) = u_0(r)$. Using this in the ODE of x, we get

$$\frac{dx(r,s)}{ds} = u_0(r).$$

Solving the remaining ODEs, we get $x(r,s) = u_0(r)s + c_1(r), \quad t(r,s) = s + c_2(r)$ with initial conditions $x(r,0) = c_1(r) = r$ and $t(r,0) = c_2(r) = 0$. Therefore, $x(r,s) = u_0(r)s + r$ and $t(r,s) = s$. Eliminating the s variable between x and t we obtain the projected characteristic curves are given by $x = u_0(r)t + r$ passing through $(r,0) \in \Gamma$ and u takes the constant value $u_0(r)$ along these lines. Solving r and s, in terms of x,t and z, we get $s = t$ and $r = x - zt$. Therefore, $u(x,t) = u_0(x - tu)$ is the solution in the implicit form.

Example 2.21. If the initial data $u_0(x) = c$, for some constant c, then $u(x,t) = c$ and the characteristic curves are $t = x/c + \text{constant}$.

Example 2.22. If we choose the initial data to be $u_0(x) = x$, then the characteristic curves $C(x,t;r) = 0$ are $x = r(1+t)$ and $u(x,t) = \frac{x}{1+t}$ in $\mathbb{R} \times (0,\infty)$. u takes the constant value r along the line $t = x/r - 1$ and all these curves intersect at $(0,-1)$. Let us find the point of intersection of these curves. To do so we differentiate C with respect to r, i.e. $C_r = 0$ which gives $t = -1$, for all r. Using the expression of r in $C(x,t;r) = 0$ equation, we get the intersection point $(0,-1)$. Thus, u is multiply defined or undefined at $(0,-1)$ but this point is not in the domain of interest. This example also illustrates that the non-existence of a solution in a region has no relevance to the discontinuity or non-smoothness of the data. See fig 2.6.

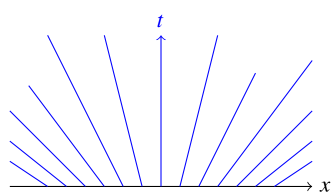

Fig. 2.6 Characteristics Curves for the initial data $u_0(x) = x$

Example 2.23. In the Burgers' equation, suppose we choose u_0 to be the function

$$u_0(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ 1 & 1 < x. \end{cases}$$

Then the characteristic curves $C(x,t;r) = 0$ are

$$x = \begin{cases} -t+r & r < -1 \\ r(t+1) & -1 \le r \le 1 \\ t+r & 1 < r. \end{cases}$$

Therefore,

$$u(x,t) = \begin{cases} -1 & x+t < -1 \\ \frac{x}{t+1} & -(t+1) \le x \le (t+1) \\ 1 & 1 < x-t. \end{cases}$$

Let us find the point of intersection of these curves. To do so we differentiate C with respect to r, i.e. $C_r = 0$ which gives $t = -1$, for all r. Using the expression of r in $C(x,t;r) = 0$ equation, we get the intersection point $(0,-1)$. Thus, u is multiply defined or undefined at $(0,-1)$ but this point is not in the domain of interest. See fig. 2.7.

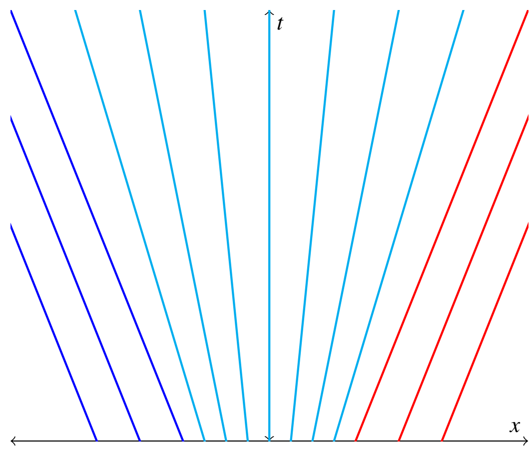

Fig. 2.7 Characteristics Curves for the initial data increasing data

Example 2.24. If we choose the initial data to be $u_0(x) = -x$, then the characteristic curves are $C(x,t;r) := r(1-t) - x = 0$. Let us find the point of intersection of these curves. To do so we differentiate C with respect to r, i.e. $C_r = 0$ which gives $1-t = 0$ or $t = 1$. Using this value of t in $C = 0$ gives $x = 0$. Thus, observe that $(0,1)$ lies on all the characteristic curves, i.e. the curves intersect at $(0,1)$. Therefore, $u(x,t) = \frac{x}{t-1}$ in $\mathbb{R} \times (0,1)$. See fig. 2.8. The solution blows-up on the $\{t = 1\}$.

Remark 2.4. In contrast to the transport equation, the slope $\frac{1}{u_0(r)}$ of the projected characteristic curves of the Burgers equation is not fixed and depends on the initial condition. If r_1 and r_2 are such that $r_1 < r_2$ but $u_0(r_1) > u_0(r_2)$ then the characteristic curves will necessarily intersect at

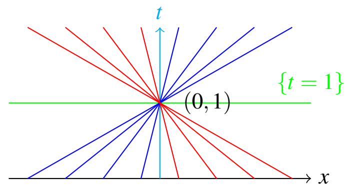

Fig. 2.8 Characteristics Curves for the initial data $u_0(x) = -x$

$$(x_0,t_0) := \left(\frac{r_2u_0(r_1) - r_1u_0(r_2)}{u_0(r_1) - u_0(r_2)}, \frac{r_2 - r_1}{u_0(r_1) - u_0(r_2)}\right)$$

because the slope of the line passing through $(r_2,0)$ is bigger than the one passing through $(r_1,0)$. This situation leads to a multi-valued solution because

$$u(x_0,t_0) = u(r_2,0) = u_0(r_2) < u_0(r_1) = u(r_1,0) = u(x_0,t_0).$$

Physically, the intersection of characteristic curves describes the phenomenon of wave breaking, i.e, faster moving waves overtaking slower waves. This situation is called the *shock*. Thus, even if one begins with a smooth decreasing initial data one may not be able to find a solution for all time t.

Example 2.25. If we choose the initial data to be

$$u_0(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & x \geq 1. \end{cases}$$

Then the characteristic curves $C(x,t;r) = 0$ are

$$x = \begin{cases} t+r & r \leq 0 \\ (1-r)t+r & 0 < r < 1 \\ r & r \geq 1. \end{cases}$$

Therefore,

$$u(x,t) = \begin{cases} 1 & x \leq t \text{ and } x < 1 \\ \frac{1-x}{1-t} & t < x < 1 \\ 0 & x \geq 1. \end{cases}$$

Let us find the point of intersection of these curves. To do so we differentiate C with respect to r, i.e. $C_r = 0$ which gives $1-t = 0$ or $t = 1$ for all $0 < r < 1$ and using this in $C = 0$ equation we get the envelope with vertex at $(1,1)$ bounded by the lines $\{x = t\}$ and $\{x = 1\}$. Note that the solution behaves well outside the wedge shaped region with vertex at $(1,1)$ and bounded by the lines $\{x = 1\}$ and $\{t = x\}$, i.e., in the closed region $\{(x,t) \mid 1 \le x \le t\}$, the characteristics start *crossing* each other and $u(x,t)$ takes three values. See figure 2.9.

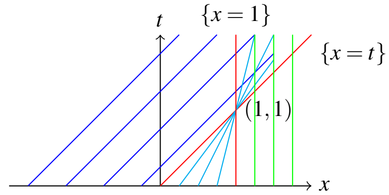

Fig. 2.9 Characteristics Curves for the initial data linearly decreasing data

Example 2.26. In the Burgers' equation, suppose we choose u_0 to be the function

$$u_0(x) = \begin{cases} \cos 2\pi x & 0 \le x \le 1 \\ 1 & x \le 0 \text{ and } x \ge 1. \end{cases}$$

Then the characteristic curves $C(x,t;r) = 0$ are

$$x = \begin{cases} t\cos(2\pi r) + r & 0 \le r \le 1 \\ t + r & r \le 0 \text{ and } r \ge 1. \end{cases}$$

To find the envelope, we find $C_r = 0$ which gives $2\pi t \sin(2\pi r) - 1 = 0$ for $0 \le r \le 1$. Thus, $r = \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{2t\pi}\right)$. Since $r \ge 0$, the domain of $\sin^{-1}$ on which its range is positive is $[0,1]$. Thus, $0 \le 1/2t\pi \le 1$ implies that $t \ge 1/2\pi$. The envelope first forms at $t = 1/2\pi$, i.e. at the vertex $(1/4, 1/2\pi)$. Using the expression of r in $C(x,t;r) = 0$ equation, we get two branches of the envelope, for all $t \ge 2\pi$,

$$x = \sqrt{t^2 - \frac{1}{4\pi^2}} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{2\pi t}\right) \quad \forall x \in (1/4, \infty)$$
$$x = -\sqrt{t^2 - \frac{1}{4\pi^2}} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{2\pi t}\right) \quad \forall x \in (-\infty, 1/4).$$

We have used the identity that $\cos(\sin^{-1}(s)) = \pm\sqrt{1-s^2}$. Note that the solution behaves well outside the wedge shaped region with vertex at $(1/4, 1/2\pi)$ and bounded by the lines passing through $(1/10, 0)$ and $(2/5, 0)$, see the magenta lines in 2.10. The characteristics start *crossing* each other and $u(x,t)$ takes three values inside the wedge (cf. figure 2.10).

Example 2.27. In the Burgers' equation, suppose we choose u_0 to be the function

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}$$

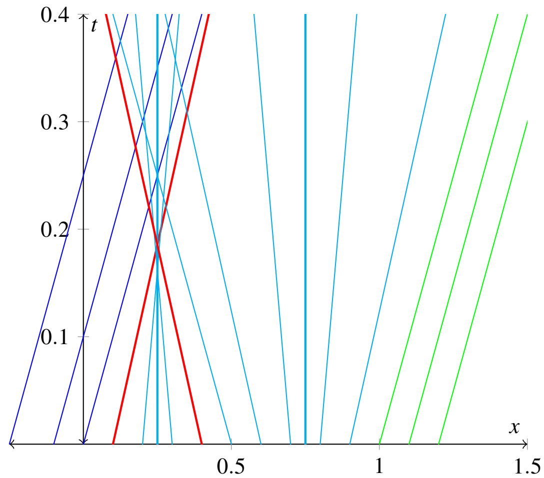

Fig. 2.10 Characteristics Curves for the non-monotone initial data

The initial data is decreasing and non-smooth. The solution is multi-valued for any $x > 0$.

Example 2.28. If the initial data u_0 is a non-decreasing function, i.e. $u_0' \geq 0$ then the characteristic curves do not intersect. However, in such a situation, it is possible that we do not have enough information to compute the solution in the entire region. Such situations also give rise to the need of relaxing the notion of solution. In the Burgers' equation, suppose we choose u_0 to be the function

$$u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

The initial data is non-decreasing and the soluton is

$$u(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > t, \end{cases}$$

but there is no information of u on the wedge $\{0 < x < t\}$.

Example 2.29. Consider the Burgers' equation

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = x & \text{in } \mathbb{R} \times (0,\infty) \\ \qquad\qquad\qquad\quad u(x,0) = 1 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

The normal to the data curve is $(0,1)$. Therefore, $(u,1)\cdot(0,1) = 1 \neq 0$. Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = z(r,s), \frac{dt(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = x(r,s)$$

with initial conditions

$$x(r,0) = r, \quad t(r,0) = 0, \text{ and } z(r,0) = 1.$$

Therefore, $t(r,s) = s + c_1(r)$, $x(r,s) = c_2(r)e^s + c_3(r)e^{-s}$ and $z(r,s) = c_4(r)e^s + c_5(r)e^{-s}$. Using the initial conditions, we get $c_1(r) = 0$, $c_2(r) = (r+1)/2 = c_4(r)$, $c_3(r) = (r-1)/2$ and $c_5(r) = (1-r)/2$. Solving for r and s, in terms of x,t and z, we get $s = t$ and

$$r = \frac{2x - e^t + e^{-t}}{e^t + e^{-t}}.$$

Therefore, $u(x,t) = x\frac{e^{2t}-1}{e^{2t}+1} + \frac{2e^t}{e^{2t+1}}$.

Example 2.30. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. Let $\Gamma := \{(x,0) \mid x > 0\}$. Consider the linear PDE

$$\begin{cases} xu_y(x,y) - yu_x(x,y) = u(x,y) & \text{in } \Omega \\ \qquad\qquad\quad u(x,0) = \phi(x) & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,0)$ for $r > 0$. Therefore,

$$(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (0,r) \cdot (0,1) = r \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = -y; \frac{dy(r,s)}{ds} = x \text{ and } \frac{dz(r,s)}{ds} = z(s)$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = 0, \text{ and } z(r,0) = \phi(r).$$

Note that

$$\frac{d^2x(r,s)}{ds} = -x(r,s) \text{ and } \frac{d^2y(r,s)}{ds} = -y(r,s).$$

Then, $x(r,s) = c_1(r)\cos s + c_2(r)\sin s$ and $y(r,s) = c_3(r)\cos s + c_4(r)\sin s$. Using the initial condition, we get $c_1(r) = r$ and $c_3(r) = 0$. Also,

$$0 = -y(r,0) = \frac{dx(r,s)}{ds}\mid_{s=0} = -c_1(r)\sin 0 + c_2(r)\cos 0 = c_2(r).$$

and, similarly, $c_4(r) = r$. Also, $z(r,s) = c_5(r)e^s$ where $c_5(r) = \phi(r)$. Thus, we have $(x(r,s), y(r,s)) = (r\cos s, r\sin s)$, where $r > 0$ and $0 \leq s \leq \pi/2$. Hence, $r = (x^2 + y^2)^{1/2}$ and $s = \arctan(y/x)$. Therefore, for any given (x,y), we have

$$u(x,y) = z(r,s) = \phi(r)e^s = \phi(\sqrt{x^2+y^2})e^{\arctan(y/x)}.$$

Exercise 2.1. Find the general solution (in terms of arbitrary functions) of the given first order PDE

(i) $xu_x + yu_y = xe^{-u}$ with $x > 0$. (Answer: $u(x,y) = f(y/x)$ for some arbitrary f).
(ii) $u_x + u_y = y + u$. (Answer: $u(x,y) = -(1+y) + f(y-x)e^x$).
(iii) $x^2u_x + y^2u_y = (x+y)u$. (Answer: $u(x,y) = f((1/x)-(1/y))(x-y)$).
(iv) $x(y^2-u^2)u_x - y(u^2+x^2)u_y = (x^2+y^2)u$. (Answer: $u(x,y) = \frac{x}{y}f(x^2+y^2+u^2)$).
(v) $(\ln(y+u))u_x + u_y = -1$.
(vi) $x(y-u)u_x + y(u-x)u_y = (x-y)u$.
(vii) $u(u^2+xy)(xu_x - yu_y) = x^4$.
(viii) $(y+xu)u_x - (x+yu)u_y = x^2 - y^2$.
(ix) $(y^2+u^2)u_x - xyu_y + xu = 0$.
(x) $(y-u)u_x + (u-x)u_y = x - y$.
(xi) $x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$.
(xii) $\sqrt{1-x^2}u_x + u_y = 0$.
(xiii) $(x+y)uu_x + (x-y)uu_y = x^2 + y^2$.

Exercise 2.2. Find the general solution of the following PDE. Check if the given data curve is non-characteristic or not. Also find the solution(s) (if it exists) given the value of u on the prescribed curves.

(i) $2u_t + 3u_x = 0$ with $u(x,0) = \sin x$.
(ii) $u_x - u_y = 1$ with $u(x,0) = x^2$.
(iii) $u_x + u_y = u$ with $u(x,0) = \cos x$.
(iv) $u_x - u_y = u$ with $u(x,-x) = \sin x$.
(v) $4u_x + u_y = u^2$ with $u(x,0) = \frac{1}{1+x^2}$.
(vi) $au_x + u_y = u^2$ with $u(x,0) = \cos x$.
(vii) $u_x + 4u_y = x(u+1)$ with $u(x,5x) = 1$.
(viii) $(1-xu)u_x + y(2x^2+u)u_y = 2x(1-xu)$. Also, when $u(0,y) = e^y$ on $x = 0$.
(ix) $e^{2y}u_x + xu_y = xu^2$. Also, when $u(x,0) = e^{x^2}$ on $y = 0$.
(x) $u_x - 2xuu_y = 0$. Also, when $u(x,2x) = x^{-1}$ on $y = 2x$ and when $u(x,x^3) = x$ on $y = x^3$.
(xi) $-3u_x + u_y = 0$ with $u(x,0) = e^{-x^2}$. (Answer: $u(x,y) = e^{-(x+3y)^2}$).
(xii) $yu_x + xu_y = x^2 + y^2$ with $u(x,0) = 1 + x^2$ and $u(0,y) = 1 + y^2$. (Answer: $u(x,y) = xy + |x^2 - y^2|$).
(xiii) $yu_x + xu_y = 4xy^3$ with $u(x,0) = -x^4$ and $u(0,y) = 0$.
(xiv) $yu_x + xu_y = u$ with $u(x,0) = x^3$.
(xv) $u_x + yu_y = y^2$ with $u(0,y) = \sin y$.
(xvi) $u_x + yu_y = u^2$ with $u(0,y) = \sin y$.
(xvii) $u_x + yu_y = u$ with $u(x,3e^x) = 2$.
(xviii) $u_x + yu_y = u$ with $u(x,e^x) = e^x$.
(xix) $u_x + xu_y = u$ with $u(1,y) = \phi(y)$.
(xx) $xu_x + u_y = 3x - u$ with $u(x,0) = \arctan x$.
(xxi) $xu_x + u_y = 0$ with $u(x,0) = \phi(x)$.
(xxii) $xu_x + yu_y = u$ with $u(x,1) = 2 + e^{-|x|}$.
(xxiii) $xu_x + yu_y = xe^{-u}$ with $u(x,x^2) = 0$.

(xxiv) $xu_x - yu_y = 0$ with $u(x,x) = x^4$.
(xxv) $e^{2y}u_x + xu_y = xu^2$ with $u(x,0) = e^{x^2}$.
(xxvi) $uu_x + u_y = 1$ with $u(2r^2, 2r) = 0$ for $r > 0$. (Answer: No solution for $y^2 > 4x$).
(xxvii) $(y-u)u_x + (u-x)u_y = x - y$ with $u(x, 1/x) = 0$.
(xxviii) $x(y^2+u)u_x - y(x^2+u)u_y = (x^2 - y^2)u$ with $u(x, -x) = 1$.
(xxix) $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$.

Exercise 2.3. Solve the equation $xu_x + 2yu_y = 0$ with $u(1,y) = e^y$. Does a solutions exist with data on $u(0,y) = g(y)$ or $u(x,0) = h(x)$? What happens to characteristic curves at $(0,0)$?

Exercise 2.4. Solve the equation $yu_x + xu_y = 0$ with $u(0,y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?

Exercise 2.5. Solve the equation $u_x + yu_y = 0$ with $u(x,0) = 1$. Also, solve the equation with $u(x,0) = x$. If there is no solution, give reasons for non-existence.

2.4 Characteristic Hypersurfaces for Fully Nonlinear

For any given smooth map $F : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$, a fully nonlinear first order PDE

$$F(\nabla u(x), u(x), x) = 0 \quad \text{in } \Omega, \tag{2.10}$$

where $\Omega \subset \mathbb{R}^n$ is an open subset and u is the unknown. We set $p := \nabla u$, $z := u$ for convenience. Since F is smooth in its arguments, its derivatives F_{p_i}, F_z and F_{x_i} exists. Let us denote $\nabla_p F := (F_{p_1}, \ldots, F_{p_n})$ and $\nabla_x F := (F_{x_1}, \ldots, F_{x_n})$.

In the quasilinear situation, $F(p,z,x) = \mathbf{a} \cdot p - f(x,z)$. Thus, the tangent vector was known a priori as $(\mathbf{a}, f)$. This information is lacking in fully nonlinear. For instance, consider the fully nonlinear eikonal equation $|\nabla u|^2 = 1$. This PDE can be rewritten as $(p,1) \cdot (p,-1) = 0$. Thus, in contrast to the quasilinear case, the tangent vector now depends on ∇u which is not known, a priori. Thus, we need to determine characteristic curve $x(s)$, the value of u along $x(s)$ and the tangent plane at each s of the curve which is called the *characteristic strip*.

For any fixed $(x,z) \in \Omega \times \mathbb{R}$, set $W(x_0, z_0) := \{p \in \mathbb{R}^n \mid F(p, z_0, x_0) = 0\}$. Let S denote the unknown graph (surface) of u in $\mathbb{R}^{n+1}$ given by the equation $\{\phi(x,z) := u(x) - z = 0\}$. Thus, solving for u in (2.10) is equivalent to finding a $u \in C^1(\Omega)$ and a $p \in W(x, u(x))$ such that $p = \nabla u(x)$, for all $x \in \Omega$. For each $(x_0, z_0) \in S$, consider the family of planes

$$(z - z_0) = p \cdot (x - x_0) \quad \forall p \in W(x_0, z_0).$$

Since S is the solution surface among these planes there is one plane which is tangential to S at (x_0, z_0) because for that choice of p, $(p, -1)$ is normal at (x_0, z_0) on S. The *envelope*[4] of the family of planes form a cone $C(x_0,z_0)$, with vertex at (x_0,z_0). To see this note that, for each $(x,z)\in C(x_0,z_0)$, there exists a $p=p(x)$ such that the corresponding plane and the cone $C(x_0,z_0)$ have the same normal. Thus, the equation of the cone $C(x_0,z_0)$ is $z-z_0=p(x)\cdot(x-x_0)$. The normal of the envelope cone and the tangent plane are same. Thus, for each $1\le j\le n$, $p_j(x)=p_j(x)+\partial_j p\cdot(x-x_0)$ or $\partial_j p\cdot(x-x_0)=0$. But differentiating $F(x_0,z_0,p(x))=0$ with respect x_j gives $\nabla_p F(x_0,z_0,p(x))\cdot\partial_j p=0$. Thus, in a neighbourhood of x_0, $(x-x_0)$ is parallel to $\nabla_p F(x_0,z_0,p(x))$, i.e. there exists a $\lambda(x)$ such that $\nabla_p F(x_0,z_0,p(x))=\lambda(x)(x-x_0)$. Solving for $(p(x),\lambda(x))$ using above n equations and $F(x_0,z_0,p(x))=0$ yields a $p(x)$ appearing in the equation of the cone $C(x_0,z_0)$. Thus, the equation of the cone $C(x_0,z_0)$ is determined by the $p=p(x)$ which can be found by solving the $n+1$ system of equations above. The cone $C(x_0,z_0)$ is tangential to S and is called the *Monge cone*.

Definition 2.5. A surface S in $\mathbb{R}^{n+1}$ is said to be an *integral surface* associated to (2.10) if at each point $(x_0,z_0)\in S\subset\mathbb{R}^{n+1}$ it is tangential to the Monge cone with vertex at (x_0,z_0).

We make a note that the coefficients of the first order derivatives in the quasilinear case, i.e. $F(p,z,x)=\mathbf{a}\cdot p-f(x,z)$ satisfies the relation $\mathbf{a}=\nabla_p F$.

Definition 2.6. Let $F(\nabla u,u,x)=0$ be a fully nonlinear PDE defined in a neighbourhood of $x_0\in\mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. Then Γ is non-characteristic at x_0 with respect to F, if there exists a function v such that $v=u$ on Γ, $F(\nabla v(x_0),v(x_0),x_0)=0$ and

$$\nabla_p F(\nabla v(x_0),v(x_0),x_0)\cdot\nu(x_0)\neq 0$$

where $\nu(x_0)$ is the normal to Γ at x_0. Otherwise, we say Γ is *characteristic* at x_0 with respect to F. If Γ is *(non)characteristic* at each of its point then we say Γ is (non)characteristic.

The arguments leading to the above definition of non-characteristic hypersurface is similar to the arguments given in Theorem 2.1 for the first order quasilinear PDE case, except that now the coefficients of first order derivatives are written in terms of p-derivatives of F. In particular, in the two dimension case, the initial data curve $\Gamma=\{\gamma_1(r),\gamma_2(r)\}$ in $\Omega\subset\mathbb{R}^2$ is *non-characteristic* for (2.10) with Cauchy data $u=u_0$ on Γ, if there exists function $\alpha(r)$ and $\beta(r)$ such that $F(\alpha(r),\beta(r),u_0(r),\gamma_1(r),\gamma_2(r))=0$ and $u_0'(r)=\alpha(r)\gamma_1'(r)+\beta(r)\gamma_2'(r)$ and Γ satisfies

$$(F_q(\gamma_1,\gamma_2,u_0,\alpha,\beta),F_p(\gamma_1,\gamma_2,u_0,\alpha,\beta))\cdot(-\gamma_2',\gamma_1')\neq 0.$$

In the case of two dimension we shall switch to the notation (p,q) instead of (p_1,p_2) for convenience sake.

[4] Envelope of a family of planes is that surface which is tangent, at each of its point, to some plane from the family.

Example 2.31. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ and $\Gamma := \{(0,y) \mid y \in \mathbb{R}\}$. Consider the fully nonlinear PDE

$$\begin{cases} u_x u_y = u(x,y) & \text{in } \Omega \\ u(0,y) = y^2 & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (0,r)$ for all $r \in \mathbb{R}$ and $u_0(r) = r^2$. We seek functions α and β such that

$$\alpha(r)\beta(r) = r^2 \text{ and } 2r = \beta(r).$$

Solving them, we get $\beta(r) = 2r$ and $\alpha(r) = r/2$. Since $F(p,q,z,x,y) = pq - z$, we have

$$F_q \gamma_2'(r) = p = \alpha(r) = r/2 \neq 0, \quad \text{for } r \neq 0.$$

Hence, the given initial curve is non-characteristic except at $(0,0)$.

Example 2.32. Consider the fully nonlinear *eikonal* equation $|\nabla u|^2 = 1$ in $\mathbb{R}^{n-1} \times (0,\infty)$ and the hypersurface $\Gamma := \{x_n = 0\}$. Let us check whether Γ is non-characteristic for the given PDE. Note that $F(p,z,x) := |p|^2 - 1$. Let us define $v(x',x_n) := u(x',0) + cx_n$ for a c that will be chosen appropriately later. Then $|\nabla v(x_0)|^2 = |\nabla_{x'} u(x_0)|^2 + c^2$. Let us choose

$$c = \pm\sqrt{1 - |\nabla_{x'} u(x_0)|^2}.$$

We have the corresponding v satisfying $F(\nabla v(x_0), v(x_0), x_0) = 0$. Further,

$$\nabla_p F(\nabla v(x_0), v(x_0), x_0) \cdot \nu(x_0) = F_{p_n}(x_0) = 2v_{x_n}(x_0) = 2c.$$

If $|\nabla_{x'} u(x_0)| < 1$ then $c \neq 0$ and $\nabla_p F(\nabla v(x_0), v(x_0), x_0) \cdot \nu(x_0) \neq 0$. Thus, Γ is non-characteristic at x_0 when the initial data u_0 is given such that $|\nabla_{x'} u_0(x_0)| < 1$ for the given fully nonlinear PDE. Observe that for a nonlinear PDE the concept of characteristics also depend on initial values. The case when $|\nabla_{x'} u(x_0)| = 1$ the data curve is characteristic but the Cauchy problem may admit a solution. For instance, for the Cauchy data $u_0(x') = x_i$, for fixed $1 \leq i \leq n-1$, we have the solution $u(x) = x_i$. However, for any choice of u_0 such that $|\nabla_{x'} u_0(x')|^2 > 1$, there is no solution to the Cauchy problem.

2.5 Method of Characteristics for Fully Nonlinear

We shall now derive the characteristics equation for a fully nonlinear PDE. In contrast to the quasilinear case, in the fully nonlinear case the characetristic curves that we seek carry with them a tangent plane that also needs to be found. Thus, we need to determine a *characteristic strip*. Equivalently, the system of n ODEs that was a complete system by itself in the quasilinear case is no longer a complete system in the fully nonlinear. For instance, for the eikonal equation $|\nabla u|^2 = 1$ we have $(\nabla u, 1)\cdot(\nabla u, -1) = 0$. Thus, the characteristic curves $(x(s), z(s))$ satisfy the ODE

$$\frac{d}{ds}x(s) = p(s) \text{ and } \frac{d}{ds}z(s) = 1.$$

The above system of ODEs is not a complete system with $2n+1$ unknowns, including p, but with only $(n+1)$ equations.

To make the system complete, one needs to introduce ODEs corresponding to the unknown varibale p along the curve. The fact that the PDE is first order imposes the condition that $|\nabla_p F|^2 \neq 0$. Differentiating F with respect to the x_i-variable in (2.10), one obtains

$$\sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z p_i + F_{x_i} = 0.$$

The idea is to choose a curve $x(s)$ in Ω such that one can compute u and ∇u along the curve. Thus, one seeks to find $x(s)$ such that

$$\sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s)) u_{x_j x_i}(x(s)) + p_i(s) F_z(p(s), z(s), x(s)) \\ + F_{x_i}(p(s), z(s), x(s)) = 0.$$

The first term involves a second derivative of u. Since u should solve a first order PDE, one wishes to eliminate the second order derivatives. Differentiate $p_i(s)$ w.r.t s to obtain

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \frac{dx_j(s)}{ds}.$$

By setting

$$\frac{dx_j(s)}{ds} = F_{p_j}(p(s), z(s), x(s))$$

one obtains

$$\begin{aligned} \frac{dp_i(s)}{ds} &= \sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s)) u_{x_j x_i}(x(s)) \\ &= -p_i(s) F_z(p(s), z(s), x(s)) - F_{x_i}(p(s), z(s), x(s)). \end{aligned}$$

Thus, we have a system of n ODEs for computing p along the curve $x(s)$, i.e.

$$\frac{dp(s)}{ds} = -p(s) F_z(p(s), z(s), x(s)) - \nabla_x F(p(s), z(s), x(s)). \tag{2.11}$$

In obtaining above ODEs we had set the curve $x(s)$ to satisfy another n system of ODEs, i.e.

$$\frac{dx(s)}{ds} = \nabla_p F(p(s), z(s), x(s)). \tag{2.12}$$

Now to compute u along the curve $x(s)$, we differentiate $z(s)$ with respect to the s-variable to obtain

$$\frac{dz(s)}{ds} = \sum_{j=i}^{n} u_{x_j}(x(s))\frac{dx_j(s)}{ds} = \sum_{j=i}^{n} u_{x_j}(x(s))F_{p_j}(p(s),z(s),x(s)).$$

Thus, we have one nore ODE corresponding to the value of z along the curve $x(s)$, i.e.

$$\frac{dz(s)}{ds} = p(s)\cdot\nabla_p F(p(s),z(s),x(s)). \tag{2.13}$$

The $2n+1$ system of first order ODE are called the *characteristic equations* of (2.10). Thus, along the characteristic curves the PDE degenerates to a system of ODEs. A fully nonlinear PDE can be solved if all the $2n+1$ ODEs are solvable. The $n+1$ system of ODEs (2.12) and (2.13) can be used compute the characteristic curve $(x(s),z(s))$ which lies on the integral surface, while the system of ODEs (2.11) give the normal vector at each point of the curve and, hence, determines a tangent plane. Thus, the solution of the $2n+1$ system of ODEs is called a *characteristic strip*. Note that for linear, semilinear and quasilinear PDE the coefficients are independent of ∇u and, hence, there is no need to compute ∇u, a priori. Thus, in these cases, it is enough to solve the $n+1$ ODEs (2.12) and (2.13) because they form a determined system.

Remark 2.5. There are some cases of fully nonlinear PDE which are as easy to solve as quasilinear PDE. This arises when the right hand side (RHS) of (2.11) is a simple situation. For instance, the RHS is zero for the possible two situations. The first one being when the fully nonlinear PDE is of the form $F(p) = 0$, i.e. F is independent of x and z variables. The other possible case is the nonlinear PDE of the form $F(p,z,x) := p\cdot x - z + f(p) = 0$ where $f:\mathbb{R}^n\to\mathbb{R}$ is given. This form of PDE is called the *Clairaut's* equation. Another simple case is when the PDE is of the form $F(p) = z$, i.e. F is independent of x variable. In this case, RHS is p itself.

2.6 Complete, Singular and General Integrals

Consider the first order PDE (2.10), i.e. $F(\nabla u(x),u(x),x) = 0$ in Ω. Let $A\subset\mathbb{R}^n$ be an open set which is the parameter set. Let us introduce the $n\times(n+1)$ matrix

$$(D_a u, D^2_{xa}u) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \dots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \dots & u_{x_n a_n} \end{pmatrix}.$$

Definition 2.7. A C^2 function $u = u(x;a)$ is said to be a *complete integral* in $\Omega\times A$ if $u(x;a)$ solves (2.10) for each $a\in A$ and the rank of the matrix $(D_a u, D^2_{xa}u)$ is n.

The condition on the rank of the matrix means that $u(x;a)$ strictly depends on all the n components of a. The complete intergal has as many constant parameters as the independent variables.

Definition 2.8. Let $\Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be open subsets and let $u = u(x;a)$ be a C^1 function of both x and a. Suppose the equation $D_a u(x;a) = 0$ is solvable for a, as a C^1 function of x, say $a = \phi(x)$, i.e., $D_a u(x;\phi(x)) = 0$, then $v(x) := u(x;\phi(x))$ is the *envelope* of the functions $\{u(\cdot;a)\}_{a\in A}$.

The idea is that for each $x \in \Omega$, the graph of v is tangent to the graph of $u(\cdot;a)$ for $a = \phi(x)$. The envelope v is called the *singular integral* when v solves the PDE (2.10). The singular integral denotes the equation of envelope of the surface represented by the complete integral of the given PDE. The singular integral is obtained by eliminating the constant a between the n equations $D_a u = 0$ and the complete integral.

Theorem 2.5. *Suppose for each $a \in A$, $u(\cdot;a)$ is a solution to (2.10) and the envelope v of u, given as $v(x) = u(x,\phi(x))$, exists then v also solves (2.10).*

Proof. Since $v(x) = u(x;\phi(x))$, for each $i = 1,2,\ldots,n$,

$$v_{x_i}(x) = u_{x_i} + \sum_{j=1}^{n} u_{a_j}\phi^j_{x_i}(x) = u_{x_i}$$

because $D_a u(x;\phi(x)) = 0$. Therefore,

$$F(\nabla v(x), v(x), x) = F(\nabla u(x;\phi(x)), u(x;\phi(x)), x) = 0.$$

Definition 2.9. The general integral is the C^1 envelope v (provided it exists) of the functions $u(x;a',h(a'))$ where $a' := (a_1,a_2,\ldots,a_{n-1})$ and $h : A' \subset \mathbb{R}^{n-1} \to \mathbb{R}$.

The general intergal is obtained by eliminating the constant a' between the $n-1$ equations $D_{a'}u = 0$ and $\phi(x,z;a',h(a'))$.

Remark 2.6. Let us illustrate in the two dimension set-up the procedure to obtain a general solution from a complete integral. Let $G(x,y,z,a,b)$ be a complete integral of $F(p,q,z,x,y) = 0$. Impose $b = h(a)$ in the complete integral G to obtain $G(x,y,z,a,h(a))$ and impose the envelope condition $D_a G = 0$. Thus, $0 = G_a + h'(a)G_h$. This gives $a = \phi(x,y,z)$. So, the general solution is $G(x,y,z,\phi,h\circ\phi) = 0$.

Remark 2.7. Note that if the fully nonlinear PDE is of the form $F(p) = 0$, i.e. F is independent of x and z variables. Thus, the RHS of (2.11) is zero which implies p are constants in the s variable. Using this information and (2.12) in (2.13), we get $z(x;a,b) = \bar{a}\cdot\bar{x} + bx_n + a_n$ is a solution to PDE provided $F(\bar{a},b) = 0$, i.e. $b = \psi(\bar{a})$ for some ψ. Thus, the *complete integral* is of the form $z(x;a) = (\bar{a},\psi(\bar{a}))\cdot x + a_n$. For singular integral, we eliminate a between $z = \bar{a}.\bar{x} + \psi(\bar{a})x_n + a_n$, $z_{\bar{a}} = 0$ and $z_{a_n} = 0$. But observe that $z_{a_n} = 1$ and is never zero. Thus, *singular integral* does not exist for PDE of type $F(p) = 0$. The *general integral* is found by choosing h

such that $h(\bar{a}) = a_n$ and eliminate $\bar{a}$ between $z = (\bar{a}, \psi(\bar{a})) \cdot x + h(\bar{a})$ and $z_{\bar{a}} = 0$, i.e. $0 = \bar{x} + x_n \psi'(\bar{a}) + h'(\bar{a})$.

Example 2.33. Consider the PDE $u_x u_y = c$ for some constant $c \in \mathbb{R}$. Thus, $z = ax + dy + b$ is a complete integral if $ad = c$ or $d = c/a$. Thus, $u(x,y;a,b) = ax + \frac{c}{a}y + b$ is a complete integral.

Example 2.34. Consider the PDE $u_x^2 + u_y^2 = u_x u_y$. Thus, $z = ax + cy + b$ is a complete integral if $a^2 + c^2 = ac$ or $c = \frac{a \pm \sqrt{a^2 - 4a^2}}{2} = \frac{a \pm \iota a \sqrt{3}}{2}$. Thus, $u(x,y;a,b) = ax + \frac{a \pm \iota a \sqrt{3}}{2} y + b$ is a complete integral.

Example 2.35. Consider the PDE $u_x^2 + u_y = 0$. Thus, $z = ax + cy + b$ is a complete integral if $c = -a^2$. Thus, $u(x,y;a,b) = ax - a^2 y + b$. We will show that the complete integral $\phi(x,y,u,a,b) := u(x,y) - ax + ya^2 - b = 0$ gives a solution to the given PDE. To eliminate the parameters we differentiate ϕ with respect to both x and y variable to obtain $u_x = a$ and $u_y = -a^2$ and eliminating a and b among them is precisely the PDE.

Example 2.36. Consider the *eikonal* equation $|\nabla u| = 1$. Thus, $z = a \cdot x + b$ is a complete integral if $|a| = 1$. The complete integral $\phi(x,u,a,b) := u(x;a,b) - a \cdot x - b$, for all $a \in S_1(0)$ where $S_1(0)$ is the sphere of radius one centred at origin and $b \in \mathbb{R}$, property is checked because on diferentiating ϕ with respect to x, we get the relation $\nabla u = a$ and $|a| = 1$.

Example 2.37. A complete integral of the eikonal equation $|\nabla u(x,y)| = 1$ in two dimension, i.e., $\sqrt{u_x^2 + u_y^2} = 1$ is $u(x,y;a,b) = (x,y) \cdot (\cos a, \sin a) + b$. Consider $h \equiv 0$, i.e. set $b = 0$, then $u(x,y;a,h(a)) = (x,y) \cdot (\cos a, \sin a)$. Thus, solving for a in $D_a u = -x \sin a + y \cos a = 0$, we get $a = \arctan(y/x)$. Since

$$\cos(\arctan(z)) = \frac{1}{\sqrt{1+z^2}} \text{ and } \sin(\arctan(z)) = \frac{z}{\sqrt{1+z^2}},$$

we have the envelope $v(x,y) = \pm\sqrt{x^2+y^2}$ for non-zero vectors is the general integral.

Remark 2.8. For any given $f : \mathbb{R}^n \to \mathbb{R}$, the *Clairaut's* equation is given as $x \cdot \nabla u + f(\nabla u) = u$. The motivation for the above form of nonlinear PDE $F(p,z,x) := p \cdot x - z + f(p) = 0$ is motivated as one possible case where the RHS of (2.11) is zero which implies p are constants in the s variable. Using the form of the PDE and (2.12) in (2.13), we get $z(x;a,b) = a \cdot x + b$ is a solution to PDE provided $f(a) = b$, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Thus, the complete integral is of the form $z(x;a) = a \cdot x + f(a)$. Set $\phi(x,u,a) := u(x;a) - a \cdot x - f(a) = 0$. To eliminate the parameters we differentiate ϕ with respect to both each x_i variable to obtain $\nabla u = a$. Eliminating a in the complete integral we get the required PDE.

Example 2.38. The Hamilton-Jacobi is a special case of the nonlinear equation where $F(x,z,p) = p_n + H(x, p_1, \ldots, p_{n-1})$ where H is independent of z and p_n.

For any given $H : \mathbb{R}^n \to \mathbb{R}$, the complete integral of the *Hamilton-Jacobi* equation $u_t + H(\nabla u) = 0$ is $\phi(x,t,u,a,b) := u(x,t;a,b) - a \cdot x + tH(a) - b$ for all $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. On differentiating with respect to the variable (x,t), we get $\nabla u = a$ and $u_t = -H(a)$.

Example 2.39. A complete integral of the Hamilton-Jacobi equation $u_t + |\nabla u|^2 = 0$ is $u(x,t;a,b) = x \cdot a - t|a|^2 + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Consider $h \equiv 0$, then $u(x,t;a,h(a)) = x \cdot a - t|a|^2$. Thus, solving for a in $D_a u = x - 2ta = 0$, we get $a = \phi(x,t) := \frac{x}{2t}$. We get the envelope $v(x,t) = \frac{|x|^2}{4t}$.

Example 2.40. The complete integral of the nonlinear PDE $u^2(1 + |\nabla u|^2) = 1$ is $u(x;a) = \pm(1 - |x-a|^2)^{1/2}$ with $|x-a| < 1$. Now, solving $D_a u = \pm\frac{x-a}{(1-|x-a|^2)^{1/2}} = 0$ for a, we get $a = \phi(x) := x$. Thus, the envelope is $v(x) = \pm 1$.

Example 2.41. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$. Let $\Gamma := \{(0,y) \mid y \in \mathbb{R}\}$. Consider the fully nonlinear Cauchy problem

$$\begin{cases} u_x u_y = u(x,y) & \text{in } \Omega \\ u(0,y) = y^2 & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (0,r)$ for all $r \in \mathbb{R}$. We have already verified that the data curve is non-characteristic except at $(0,0)$ in Example 2.31. The characteristic equations are

$$\begin{cases} \left(\frac{dx(r,s)}{ds}, \frac{dy(r,s)}{ds}\right) = (q(r,s), p(r,s)), \ (using (2.12)) \\ \left(\frac{dp(r,s)}{ds}, \frac{dq(r,s)}{ds}\right) = (p(r,s), q(r,s)), \ (using (2.11)) \\ \frac{dz(r,s)}{ds} = (p(r,s), q(r,s)) \cdot (q(r,s), p(r,s)) = 2p(r,s)q(r,s) \quad (using (2.13)) \end{cases}$$

with initial conditions $x(r,0) = 0, y(r,0) = r, z(r,0) = r^2, p(r,0) = \alpha(r) = \frac{r}{2}$ and $q(r,0) = \beta(r) = 2r$. Thus, on integrating, we get $p(r,s) = (r/2)e^s$ and $q(r,s) = 2re^s$, for all $s \in \mathbb{R}$. Using (p,q), we solve for x to get $x(r,s) = 2r(e^s - 1)$ and $y(r,s) = (r/2)(e^s + 1)$. Solving for z, we get $z(r,s) = r^2 e^{2s}$. Solving r and s in terms of x and y, we get

$$r = \frac{4y - x}{4} \text{ and } e^s = \frac{x + 4y}{4y - x}.$$

Hence $u(x,y) = z(r(x,y), s(x,y)) = \frac{(x+4y)^2}{16}$.

Example 2.42. Let us consider the same non-linear PDE as in above example but with a different datum curve and data, i.e.

$$\begin{cases} u_x u_y = u(x,y) & \text{in } \mathbb{R}^2 \\ u(x, 1+x) = x^2 & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r, 1+r)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that

$$\alpha(r)\beta(r) = r^2 \text{ and } 2r = \alpha(r) + \beta(r).$$

Solving them, we get $\beta(r) = r = \alpha(r)$. Since $F(p,q,z,x,y) = pq - z$, we have

$$F_p\gamma_1' - F_q\gamma_2'(r) = q - p = \beta(r) - \alpha(r) = r - r = 0, \quad \forall r.$$

Hence, the given initial curve is characteristic. The characteristic equations are

$$\begin{cases} \left(\frac{dx(r,s)}{ds}, \frac{dy(r,s)}{ds}\right) = (q(r,s), p(r,s)), \ (using(2.12)) \\ \left(\frac{dp(r,s)}{ds}, \frac{dq(r,s)}{ds}\right) = (p(r,s), q(r,s)), \ (using(2.11)) \\ \frac{dz(r,s)}{ds} = (p(r,s), q(r,s)) \cdot (q(r,s), p(r,s)) = 2p(r,s)q(r,s) \quad (using(2.13) \end{cases}$$

with initial conditions $x(r,0) = r$, $y(r,0) = 1 + r$, $z(r,0) = r^2$, $p(r,0) = \alpha(r) = r$ and $q(r,0) = \beta(r) = r$. Thus, on integrating, we get $p(r,s) = re^s$ and $q(r,s) = re^s$, for all $s \in \mathbb{R}$. Using (p,q), we solve for x to get $x(r,s) = re^s$ and $y(r,s) = re^s + 1$. Solving for z, we get $z(r,s) = r^2e^{2s}$. Note that there is no unique way of solving r and s in terms of x and y. This is because the Jacobian is zero. In fact, we have three possible representation of u, viz., $u = x^2$, $u = (y-1)^2$ and $u = x(y-1)$. Of these three possibilities, only $u = x(y-1)$ satisfies the PDE.

Example 2.43. For any given $\lambda \in \mathbb{R}$, consider the fully non-linear PDE

$$\begin{cases} u_x^2 + u_y^2 = 1 & \text{in } \mathbb{R}^2 \\ u(x,x) = \lambda x & \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,r)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that $\alpha^2(r) + \beta^2(r) = 1$ and $\lambda = \alpha(r) + \beta(r)$. We can view $\alpha(r) = \cos\theta$ and $\beta(r) = \sin\theta$ where θ is such that $\cos\theta + \sin\theta = \lambda$. Thus, $\lambda^2 - 1 = \sin(2\theta)$ or $\theta = \frac{1}{2}\sin^{-1}(\lambda^2 - 1)$. This imposes that $|\lambda| \leq \sqrt{2}$. Since $F(p,q,z,x,y) = p^2 + q^2 - 1$, we have

$$F_p\gamma_1' - F_q\gamma_2'(r) = 2(p - q) = 2(\cos\theta - \sin\theta) = 0, \quad \forall \theta = \pi/4 + k\pi,$$

where $k = 0, 1, 2, \ldots$. Hence, the given initial curve is non-characteristic for $\theta \neq \pi/4 + k\pi$ for all k and $\cos\theta + \sin\theta = \lambda$, i.e. for all $|\lambda| < \sqrt{2}$. The characteristic equations are

$$\begin{cases} \left(\frac{dx(r,s)}{ds}, \frac{dy(r,s)}{ds}\right) = (2p(r,s), 2q(r,s)), \ (using(2.12)) \\ \left(\frac{dp(r,s)}{ds}, \frac{dq(r,s)}{ds}\right) = (0,0), \qquad (using(2.11)) \\ \frac{dz(r,s)}{ds} = (p(r,s), q(r,s)) \cdot (2p(r,s), 2q(r,s)) = 2(p^2 + q^2) = 2 \quad (using(2.13)) \end{cases}$$

with initial conditions $x(r,0) = r$, $y(r,0) = r, z(r,0) = \lambda r, p(r,0) = \cos\theta$ and $q(r,0) = \sin\theta$. Thus, on integrating, we get $p(r,s) = \cos\theta$ and $q(r,s) = \sin\theta$. Using (p,q), we solve for x to get $x(r,s) = 2s\cos\theta + r$ and $y(r,s) = 2s\sin\theta + r$. Solving for z, we get $z(r,s) = 2s + \lambda r$. Solving r and s in terms of x and y, we get

$$s = \frac{x-y}{2(\cos\theta - \sin\theta)} \quad \text{and } r = \frac{x\sin\theta - y\cos\theta}{\sin\theta - \cos\theta}.$$

Therefore, the general solution is

$$\begin{aligned} u(x,y) &= \frac{x-y+\lambda(y\cos\theta - x\sin\theta)}{\cos\theta - \sin\theta} \\ &= \frac{x-y+(\cos\theta+\sin\theta)(y\cos\theta - x\sin\theta)}{\cos\theta - \sin\theta} \\ &= \frac{x(1-\sin\theta\cos\theta - \sin^2\theta)+y(\cos^2\theta+\sin\theta\cos\theta - 1)}{\cos\theta - \sin\theta} \\ &= x\cos\theta + y\sin\theta. \end{aligned}$$

Example 2.44. Consider the equation $y(u_x^2 - u_y^2) + uu_y = 0$ with $u(2y,y) = 3y$. Note that $F(p,q,z,x,y) = y(p^2-q^2)+zq = 0$. Thus the ODEs become

$$(dx,dy) = (2yp, z-2yq) \text{ and } (dp,dq) = -(pq,p^2).$$

Thus, $\frac{dq}{dp} = \frac{-p}{q}$. On integrating, we get $p^2 - q^2 = a$. Using the PDE, we get $ya + uq = 0$. Thus,

$$q = -\frac{ay}{z} \text{ and } p = \pm\sqrt{a + \frac{a^2y^2}{z^2}} = \pm\frac{1}{z}\left[a(z^2+ay^2)\right]^{\frac{1}{2}}$$

Now, $dz = (p,q)\cdot(dx,dy)$ or $zdz = (\pm\left[a(z^2+ay^2)\right]^{\frac{1}{2}}, -ay)\cdot(dx,dy)$. Hence,

$$\begin{aligned} \mp\sqrt{a}dx &= -\frac{1}{\sqrt{z^2+ay^2}}(zdz+aydy) = -\frac{1}{2\sqrt{z^2+ay^2}}d(z^2+ay^2) = -d(\sqrt{z^2+ay^2}) \\ \mp\sqrt{a}x \mp b &= -\sqrt{z^2+ay^2} \\ u^2(x,y) &= (b+x\sqrt{a})^2 - ay^2. \end{aligned}$$

Using the initial data, we get $9y^2 = (b+2y\sqrt{a})^2 - ay^2$ which is satisfied with $b=0$ and $a=3$. Thus, $u(x,y)^2 = 3(x^2-y^2)$. The solution is valid in the domain $\{|x| > |y|\}$.

Example 2.45. Find the characteristic strip of the equation $F(p,q,z,x,y) := x^2y^2pq - xp - yq - z + 1 = 0$ passing through the initial data curve $\Gamma(r) := (r,1,-r)$. Let us compute

$$F_p^2 + F_q^2 = x^4y^4(p^2+q^2) + (x^2+y^2) - 2x^2y^2(xq+yp).$$

Thus the ODEs are

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (x^2y^2q - x, x^2y^2p - y),$$

$$\left(\frac{dp(s)}{ds}, \frac{q(s)}{ds}\right) = (2p(1-xy^2q), 2q(1-x^2yp)).$$

Thus,

$$\left(\frac{dp}{p}, \frac{dq}{q}\right) = -2\left(\frac{dx}{x}, \frac{dy}{y}\right)$$

which gives $p = ax^{-2}$ and $q = by^{-2}$. Using this information in the ODE corresponding to z, we get

$$dz = ax^{-2}dx + by^{-2}dy.$$

On integration, we get $u(x,y) = -ax^{-1} - by^{-1} + c$. Using this information in $F = 0$, we get $c = ab + 1$. Thus, $u(x,y) = -ax^{-1} - by^{-1} + ab + 1$. INCOMPLETE!!! FIND CONSTANTS USING INITIAL DATA.

Example 2.46. Let us compute a complete integral of the first order PDE $u_x u_y = u(x,y)$. The equation is of the form $F(p,q,z,x) = pq - z$. The characteristic equations are (using (2.12))

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (q(s), p(s)),$$

(using (2.11))

$$\left(\frac{dp(s)}{ds}, \frac{dq(s)}{ds}\right) = (p(s), q(s)).$$

and (using (2.13))

$$\frac{dz(s)}{ds} = (p(s), q(s)) \cdot (q(s), p(s)) = 2p(s)q(s).$$

Thus, on integrating, we get $p(s) = c_1 e^s$ and $q(s) = c_2 e^s$. Solving for z, we get

$$z(s) = c_1 c_2 e^{2s} + c_3.$$

Using (p,q), we solve for x to get $x(s) = c_2 e^s + b$ and $y(s) = c_1 e^s + a$.Therefore,

$$u(x,y) = (y-a)(x-b) + c_3$$

is a complete integral for arbitrary constants a and b, if we choose $c_3 = 0$.

Example 2.47. A complete integral of the nonlinear equation $u_x u_y = u$, considered in Example 2.46, is $u(x,y;a,b) = xy + ab + (a,b) \cdot (x,y)$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(a) = a$, then $u(x,y;a,h(a)) = u(x,y;a) = xy + a^2 + a(x+y)$. Solving for a in $D_a u = 2a + x + y = 0$ yields $a = \phi(x,y) := \frac{-(x+y)}{2}$. Therefore, the envelope $v(x,y) = u(x,y;\phi(x,y)) = -(x-y)^2/4$.

Example 2.48. Let us find a different complete integral of the nonlinear PDE $u_x u_y = u$. Note that $F(p,q,z,x) = pq - z$. Then the ODE

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (q(s), p(s)),$$

$$\left(\frac{dp(s)}{ds}, \frac{dq(s)}{ds}\right) = (p(s), q(s)).$$

Thus, on integrating, we get $p(s) = c_1 e^s$ and $q(s) = c_2 e^s$. Therefore, $p/q = a$. Using this equation with $pq = z$, we get $p = \pm\sqrt{az}$ and $q = \pm\sqrt{z/a}$. Now,

$$\begin{aligned}
\frac{dz(s)}{ds} &= (p(s), q(s)) \cdot \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) \\
&= \pm\sqrt{az}\frac{dx(s)}{ds} \pm \sqrt{z/a}\frac{dy(s)}{ds} \\
\frac{1}{\sqrt{z}}\frac{dz(s)}{ds} &= \pm\left(\sqrt{a}\frac{dx(s)}{ds} + 1/\sqrt{a}\frac{dy(s)}{ds}\right) \\
2\sqrt{z} &= \pm\left(\sqrt{a}x + y/\sqrt{a}\right) + c_3.
\end{aligned}$$

Thus,

$$u(x,y) = \left[b + \frac{1}{2}\left(\sqrt{a}x + y/\sqrt{a}\right)\right]^2$$

is a complete integral, if we had chosen $a > 0$.

Note that previous two examples compute two different complete integral for same equation. However, in both examples, no choice of a and b will give the zero solution $u \equiv 0$. Thus, $u \equiv 0$ is called *singular solution*.

Example 2.49. Consider the equation $xu_xu_y + yu_y^2 = 1$ with $u(x,0) = x/2$. The parametrisation of the initial curve is $\Gamma(r) := (r,0)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that $r\alpha(r)\beta(r) = 1$ and $1/2 = \alpha(r)$. Solving them, we get $\beta(r) = 2/r$. Since $F(p,q,z,x,y) = xpq + yq^2 - 1$, we have

$$F_p\gamma_1' - F_q\gamma_2'(r) = 2xq = 2r\beta(r) = 4 \neq 0, \quad \forall r.$$

Hence, the given initial curve is non-characteristic. Thus the ODE

$$\begin{cases} \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (xq, xp + 2yq), \\ \left(\frac{dp(s)}{ds}, \frac{dq(s)}{ds}\right) = -(pq, q^2). \end{cases}$$

Thus, on integrating, we get $p/q = a$. Using the PDE, we get $(xa+y)q^2 = 1$. Thus,

$$q = \pm\frac{1}{\sqrt{xa+y}} \quad p = \pm\frac{a}{\sqrt{xa+y}}.$$

Now,

$$\begin{aligned}
\frac{dz(s)}{ds} &= (p(s),q(s))\cdot\left(\frac{dx(s)}{ds},\frac{dy(s)}{ds}\right)\\
&= \pm\frac{1}{\sqrt{xa+y}}\left(a\frac{dx(s)}{ds}+\frac{dy(s)}{ds}\right)\\
u(x,y) &= \pm 2\sqrt{ax+y}+b.
\end{aligned}$$

Thus, $(u(x,y)-b)^2 = 4(ax+y)$ is a complete integral. Using the initial data, we get $(r-2b)^2 = 16ar$. Differentiating w.r.t r, we get $r = 8a+2b$ and using in first equation we get $r = 4a$ and $b = h(a) := -2a$. Hence, $(u+2a)^2 = 4(ax+y)$. Now solving for a in $D_a u = 0$, we get $4(u+2a) = 4x$. Then $a = \phi(x,y,u) := (x-u)/2$ and the solution is

$$x^2 = 4\left(\frac{x-u}{2}\right)x+4y$$

which yields $u(x,y) = \frac{x^2+4y}{2x}$.

Example 2.50. Consider the equation $y(u_x^2-u_y^2)+uu_y = 0$ with $u(r^2,0) = 2r$. Note that $F(p,q,z,x,y) = y(p^2-q^2)+zq = 0$. As in example 2.44, we get $u^2(x,y) = (b+x\sqrt{a})^2-ay^2$. Using the initial conditions, we get $4r^2 = (b+r^2\sqrt{a})^2$. Differentiating this w.r.t to r and solving for r, we get

$$r^2 = \frac{1}{\sqrt{a}}\left(\frac{2}{\sqrt{a}}-b\right).$$

Substituting this in the equation of r, we get $b = 1/\sqrt{a}$. This gives

$$u^2 = \left(\frac{1}{\sqrt{a}}+x\sqrt{a}\right)^2-ay^2 = \frac{1}{a}(1+xa)^2-ay^2.$$

Now solving for a in $D_a u = 0$, we get

$$\begin{aligned}
0 &= 2\left(\frac{1}{\sqrt{a}}+x\sqrt{a}\right)\left(-\frac{1}{2a\sqrt{a}}+\frac{x}{2\sqrt{a}}\right)-y^2\\
y^2 &= \left(x^2-\frac{1}{a^2}\right)\\
a &= \phi(x,y) := \frac{1}{\sqrt{x^2-y^2}}.
\end{aligned}$$

We choose the positive root above to keep $a>0$ so that all roots above made sense. Therefore,

$$u^2(x,y) = \sqrt{x^2-y^2}\left(1+\frac{x}{\sqrt{x^2-y^2}}\right)^2-\frac{y^2}{\sqrt{x^2-y^2}}.$$

Example 2.51. Let us find the complete integral, general solution and singular solution of the fully non-linear PDE $u_x^2 + u_y^2 = 1 + 2u$. Since $F(p,q,z,x) = p^2 + q^2 - 1 - 2z$, the ODEs are

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (2p(s), 2q(s)),$$

$$\left(\frac{dp(s)}{ds}, \frac{dp(s)}{ds}\right) = (2p, 2q)$$

Thus, on dividing and integrating, we get $p/q = a$. Using the PDE, we get $(1 + a^2)q^2 = 1 + 2z$. Thus,

$$q = \pm\sqrt{\frac{1+2z}{1+a^2}} \quad p = \pm a\sqrt{\frac{1+2z}{1+a^2}}.$$

Now,

$$\begin{aligned}
\frac{dz(s)}{ds} &= (p(s), q(s)) \cdot \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) \\
&= \pm\sqrt{\frac{1+2z}{1+a^2}}\left(a\frac{dx(s)}{ds} + \frac{dy(s)}{ds}\right) \\
\frac{1}{\sqrt{1+2z}}\frac{dz(s)}{ds} &= \pm\frac{1}{\sqrt{1+a^2}}\left(a\frac{dx(s)}{ds} + \frac{dy(s)}{ds}\right) \\
\sqrt{1+2z} &= \pm\frac{ax+y}{\sqrt{1+a^2}} \pm b.
\end{aligned}$$

Thus,

$$u(x,y) = \frac{1}{2}\left(\frac{ax+y}{\sqrt{1+a^2}} + b\right)^2 - \frac{1}{2}$$

is a complete integral. Note that no choice of a and b will give the constant solution $u = -1/2$. Thus, $u \equiv -1/2$ is called *singular solution*. INCOMPLETE!!!!!!!

Exercise 2.6. (i) Find a complete integral of $uu_xu_y = u_x + u_y$.
(ii) Find a complete integral of $u_x^2 + u_y^2 = xu$.
(iii) Find a complete integral of $x(u_x^2 + u_y^2) = uu_x$ with u given on the curve

(a) $u(2y,y) = 5y$
(b) $u(0,r^2) = 2r$

(iv) Find a complete integral of $4uu_x - u_y^3 = 0$ with u given on the curve $u(0,r) = 4r$.

2.7 Analytic Solutions

In this section, we establish the existence of analytic solutions for first order system of quasilinear PDE with all the initial data being analytic. Consider the system of m

first order quasilinear PDE given as:

$$\begin{cases} \mathbf{u}_t(x,t) = \sum_{j=1}^{n} \mathbf{A}_j(\mathbf{u})\mathbf{u}_{x_j} & \text{in } \mathbb{R}^{n+1} \\ \mathbf{u}(x,0) = \mathbf{g}(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{2.14}$$

where each $\mathbf{A}_j(\mathbf{u})$ is a $m \times m$ matrix, $\mathbf{u}(x) = (u^1, \ldots, u^m)$ and $\mathbf{g}(x) = (g^1, \ldots, g^m)$.

Theorem 2.6. *Let $\mathbf{g}(0) = \mathbf{g}_0$. If, for all j, the entries of the matrix $\mathbf{A}_j$ are analytic at $\mathbf{g}_0$ and $\mathbf{g}$ is analytic at 0 then there exists a unique local analytic solution $\mathbf{u}$ at $(0,0)$ of (2.14).*

Proof. Without loss of generality we assume that $\mathbf{g}_0 = \mathbf{0}$. The proof employs the method of power series due to the analyticity assumptions. The proof involves three major steps: firstly compute the Taylor series of $\mathbf{u}$ at $(0,0)$, secondly establish the convergence of the Taylor series of $\mathbf{u}$ in a neighbourhood of $(0,0)$ and finally to show that the limit satisfies the system (2.14).

First step: By the analyticity hypotheses of $\mathbf{A}_j$ and $\mathbf{g}$, their respective Taylor series around 0 converge to itself. Thus,

$$\mathbf{A}_j^{ik}(\mathbf{u}) = \sum_{|\beta|=0}^{\infty} b_\beta^{ikj} \mathbf{u}^\beta \text{ and } \mathbf{g}^i(x) = \sum_{|\alpha|=0}^{\infty} c_\alpha^i x^\alpha.$$

Let us assume that the solution admits a power series expansion around $(0,0)$, i.e. $u^i(x,t) = \sum_{|\alpha|+j=0}^{\infty} a_{\alpha,j}^i x^\alpha t^j$. We need to compute $a_{\alpha,j}^i$ in order to identify $\mathbf{u}$. For $j = 0$ and $\alpha = 0$, we have $a_{0,0}^i = u^i(0,0) = g^i(0) = 0$ for all i. For $j = 0$ and any α, the corresponding coefficients comes from $u^i(x,0)$. Since $u^i(x,0) = g^i(x)$, for all i, $a_{\alpha,0}^i = c_\alpha^i$. Now, for $j = 1$ and $\alpha = 0$, the corresponding Taylor coefficients comes from $\partial_t u^i(0,0)$. The PDE gives that, for all i,

$$u_t^i(x,0) = \sum_{j=1}^{n} \mathbf{A}_j(\mathbf{g})\mathbf{g}_{x_j}(x).$$

Evaluating above identity at $x = 0$, we get $a_{0,1}^i = \sum_{j=1}^{n} \sum_{k=1}^{m} b_0^{ikj} c_{e_j}^k$. Proceeding this way we can compute all the Taylor coefficients of u^i in terms of coefficients of A_j^{ik} and g^i. For $j = 1$ and $|\alpha| = 1$, differentiate the PDE with respect to x once and equate coefficients. For $j = 2$ and $|\alpha| = 0$, we differentiate the PDE once with respect to t and evaluate at $(0,0)$.

Second Step: Since $\mathbf{A}_j$ are all analytic at 0, there exists constants M and R such that its Taylor coefficients satisfy the estimate, for all i, j, k and β

$$|b_\beta^{ikj}| \leq \frac{M}{R^{|\beta|}}.$$

We define $B(v) = \frac{M}{1 - \frac{v_1 + \ldots + v_m}{R}}$ and set the entries of the matrix $\mathbf{B}_j(v)$ to be $B(v)$, for all j. By construction the matrix B_j admits a power series, i.e.

$$B(v) = M\sum_{k=0}^{\infty} \frac{(v_1+\ldots+v_m)^k}{R^k} = M\sum_{|\beta|=0}^{\infty} \frac{|\beta|!}{\beta!}\frac{v^\beta}{R^{|\beta|}}.$$

Therefore, by construction, the Taylor coefficients of $\mathbf{A}_j$ are bounded by the Taylor coefficients of $\mathbf{B}_j$.

Since g^i are all analytic at 0, there exists constants M and R such that its Taylor coefficients satisfy the estimate, for all i and α

$$|c^i_\alpha| \leq \frac{M}{R^{|\alpha|}}.$$

We define $h(x) = \frac{M(x_1+\ldots+x_n)}{R(1-\frac{x_1+\ldots+x_n}{R})}$ and set the $h^i = h$, for all i. Note that $h(0) = 0$. By construction, h^i admits a power series, i.e.

$$h(x) = M\sum_{k=0}^{\infty} \frac{(x_1+\ldots+x_n)^{k+1}}{R^{k+1}} = M\sum_{|\alpha|=1}^{\infty} \frac{|\alpha|!}{\alpha!}\frac{x^\alpha}{R^{|\alpha|}}.$$

Therefore, by construction, the Taylor coefficients of g^i are bounded by the Taylor coefficients of h^i.

We introduce another first order quasilinear problem

$$\begin{cases} \mathbf{v}_t(x,t) = \sum_{j=1}^n \mathbf{B}_j(\mathbf{v})\mathbf{v}_{x_j} & \text{in } \mathbb{R}^{n+1} \\ \mathbf{v}(x,0) = \mathbf{h}(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{2.15}$$

Due to the choice of $\mathbf{B}_j$ and $\mathbf{h}$, we seek a solution $\mathbf{v} = (v_1,\ldots,v_m)$ such that $v_i = v$ for all i. Then the system of PDE (2.15) reduces to the scalar PDE:

$$\begin{cases} v_t(x,t) = mB(\mathbf{v})\sum_{j=1}^n v_{x_j} & \text{in } \mathbb{R}^{n+1} \\ v(x,0) = h(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

Note that $B(\mathbf{v}) = \frac{M}{1-\frac{mv}{R}}$. Set $z = x_1+\ldots+x_n$ and $w(z,t) := v(x,t)$. The the PDE satisfied by $w(z,t)$ is

$$\begin{cases} w_t(z,t) - A(w)w_z = 0 & \text{in } \mathbb{R}^2 \\ \qquad w(z,0) = G(z) & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

where $A(w) = \frac{RMmn}{R-mw}$ and $G(z) = \frac{Mz}{R-z}$. We solve the above scalar quasilinear PDE using the method of characteristics. Note that $\{(z,0,G(z)\}$ is the known curve on the solution surface of w. We parametrize the initial curve with the r-variable, i.e., $\{(r,0)\}$. The initial curve is non-characteristic because $(A(w),1)\cdot(0,1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dz(r,s)}{ds} = -A(w), \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dw(r,s)}{ds} = 0$$

with initial conditions,

$$z(r,0) = r, \quad t(r,0) = 0, \text{ and } w(r,0) = G(r).$$

Solving the ODE corresponding to w, we get $w(r,s) = c_3(r)$ with initial conditions $w(r,0) = c_3(r) = G(r)$. Thus, $w(r,s) = G(r) = \frac{Mr}{R-r}$. Using this in the ODE of z, we get

$$\frac{dz(r,s)}{ds} = -A(G(r)).$$

Solving the ODE's, we get

$$z(r,s) = -A(G(r))s + c_1(r), \quad t(r,s) = s + c_2(r)$$

with initial conditions

$$z(r,0) = c_1(r) = r \text{ and } t(r,0) = c_2(r) = 0.$$

Therefore,

$$z(r,s) = -A(G(r))s + r, \text{ and } t(r,s) = s.$$

Solving r and s, in terms of z,t and w, we get $s = t$ and $r = z + A(G(r))t$. Therefore, $w(z,t) = \frac{M(z+tA(w))}{R-z-A(w)t}$ is the solution in the implicit form. Observe that the projected characteristic curves are given by $z = G(r)t + r$ passing through $(r,0) \in \Gamma$ and w is constant along these curves. Thus, the characteristics curves do meet but there is a neighbourhood of $(0,0)$ where the curves do not meet and in this neighbourhood an analytic solution w exists and in turn v exists. Further by the construction of $\mathbf{v}$, the Taylor series coefficients of $\mathbf{u}$ are bounded by $\mathbf{v}$ and, hence, the Taylor series of $\mathbf{u}$ converges.

Third Step: In this step we claim that the limit of the convergent Taylor series of $\mathbf{u}$ satisfies (2.14). We have shown that the Taylor series $\sum_{|\alpha|+j=0}^{\infty} a^i_{\alpha,j} x^\alpha t^j$ converges, say to u^i. Let $u^i_k := \sum_{|\alpha|+j=0}^{k} a^i_{\alpha,j} x^\alpha t^j$ be the partial sum of the Taylor series and it uniformly converges to u^i in a neighbourhood of $(0,0)$. Similarly all the partial derivative of u^i_k with respect to (x,t) also uniformly converge to the respective partial derivatives of u^i. We claim that $\mathbf{u} = (u^1, \ldots, u^m)$ satisfies (2.14). Consider

$$\begin{aligned}\mathbf{u}_t - \sum_{j=1}^n \mathbf{A}_j(\mathbf{u})\mathbf{u}_{x_j} &= (\mathbf{u}_t(x,t) - \partial_t \mathbf{u}_k) - \sum_{j=1}^n \mathbf{A}_j(\mathbf{u})\left(\mathbf{u}_{x_j} - \partial_{x_j}\mathbf{u}_k\right) \\ &\quad + \left(\partial_t \mathbf{u}_k - \sum_{j=1}^n \mathbf{A}_j(\mathbf{u})\partial_{x_j}\mathbf{u}_k\right)\end{aligned}$$

The first and second term uniformly converge to zero. Now, the coefficients $a^i_{\alpha,j}$ were defined such that the third term and its derivatives with respect to x and t upto order k vanish at $(0,0)$. Thus the power series expansion of the third term contains $x^\alpha t^j$ only for $|\alpha| + j \geq k+1$, i.e. modulus of the third term is bounded by Cr^{k+1} for $(x,t) \in B_r(0)$. Thus, the third terms also converges uniformly to zero and $\mathbf{u}$ satisfies (2.14).

Remark 2.9. The natural choice of $h(x) = \frac{Mz}{1-z/R}$ was ignored over the specific choice made in the theorem. Of course, $h(0)$ is now non-zero. Further, this choice will not work because the resulting scalar quasilinear PDE satisfies by $w(z,t)$ will be (after normalizing constants)

$$\begin{cases} w_t(z,t) - \frac{1}{1-w} w_z = 0 & \text{in } \mathbb{R}^2 \\ \qquad w(z,0) = \frac{1}{1-z} & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

This PDE does not behave well near $z = 0$ because $w(0,0) = 1$ and the coefficient of w_z is infinite at $(0,0)$.

Remark 2.10. The result of Theorem 2.6 can be extended to the more general system of m first order quasilinear PDE given as:

$$\begin{cases} \mathbf{u}_t(x,t) = \sum_{j=1}^n \mathbf{A}_j(x,t,\mathbf{u})\mathbf{u}_{x_j} + \mathbf{B}(x,t,u) & \text{in } \mathbb{R}^{n+1} \\ \mathbf{u}(x,0) = \mathbf{g}(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{2.16}$$

with analyticity assumptions on all the initial data.

Remark 2.11. The local solution is, in general, not a global solution because

(a) The domains of analyticity of initial data may not be the entire domain.
(b) Even if the initial data are entire the characteristic curves of the scalar quasilinear PDE of w may intersect and the analyticity of the solution of majorant equation (2.15) is not entire.
(c) The size of the analyticity neighbourhood depends on the constants M and R.

Recall that the result Theorem 2.6 was estbalished for a flat initial surface (hyperplane). However, the result could be generalised to non-characteristic hypersurfaces.

Theorem 2.7. *Consider the first order system of linear PDE*

$$\begin{cases} L\mathbf{u}(x) = \mathbf{f}(x) & \text{in } \mathbb{R}^n \\ \mathbf{u}(x) = \mathbf{g}(x) & \text{on } \Gamma \end{cases} \tag{2.17}$$

where $L := \sum_{j=1}^n \mathbf{A}_j(x)\partial_{x_j} + \mathbf{B}(x)$*,* $\mathbf{A}_j$ *and* $\mathbf{B}$ *are* $m \times m$ *matrices, the data and unknown vectors are all m-tuples. If* Γ *is a non-characteristic analytic hypersurface of* $\mathbb{R}^n$ *and all the initial data are analytic in a neighbourhood of* $x - 0$ *then the PDE admits a unique local analytic solution in neighbourhood of* x_0*.*

Proof. The idea of the proof is to flatten the hyperspace to a hyperplane and invoke Theorem 2.6. Let $\{\phi = 0\}$ be the equation of Γ such that $\nabla\phi \neq 0$ on Γ. Without loss of generality, let us assume $\phi_{x_n} \neq 0$. Define the flattening map $(x', x_n) \mapsto (x', \phi(x))$. Thus, the initial hypersurface is mapped to the hyperplane $\{t := \phi(x) = 0\}$. By chain rule, the operator L becomes

$$L := \sum_{j=1}^{n-1} \mathbf{A}_j \partial_{x_j} + \sum_{j=1}^{n} \mathbf{A}_j \phi_{x_j} \partial_t.$$

The non-characteristic condition on Γ implies that the coeeficient of ∂_t, $\sum_{j=1}^n \mathbf{A}_j \phi_{x_j}$, is invertible and the PDE (2.17) transforms to

$$\begin{cases} \mathbf{v}_t + \left(\sum_{j=1}^n \mathbf{A}_j \phi_{x_j}\right)^{-1} \sum_{j=1}^{n-1} \mathbf{A}_j \mathbf{v}_{x_j} = \left(\sum_{j=1}^n \mathbf{A}_j \phi_{x_j}\right)^{-1} \mathbf{f} & \text{in } \mathbb{R}^{n-1} \\ \mathbf{v}(x',t) = \mathbf{g}(x') & \text{on } \{t=0\}. \end{cases}$$

Since Γ is analytic ϕ is analytic in a neighbourhood of x_0 and hence the coefficients of the new PDE are all analytic and, by Theorem 2.6, there exists a local analytic solution.

The Cauchy-Kowalevski theorem establishes the uniqueness of solutions in the class of analytic functions. But there could be non-analytic solutions for analytic data. The Holmgren's result establishes the uniqueness in the C^1 class of solutions for linear system of first order PDE.

Theorem 2.8. *If $u \in C^1(\mathbb{R}^{n+1})$ is a local solution at $(0,0)$ of the linear system of PDE*

$$\begin{cases} \mathbf{u}_t(x,t) = \sum_{j=1}^n \mathbf{A}_j(x,t)\mathbf{u}_{x_j} & \text{in } \mathbb{R}^{n+1} \\ \mathbf{u}(x,0) = \mathbf{0} & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

where entries of $\mathbf{A}_j$ are analytic in a neighbourhood of $(0,0)$ then $u \equiv 0$ in a neighbourhood of $(0,0)$.

Proof. The idea of the proof is to use the equivalence of the uniqueness of solution of the above PDE is same as the existence of solution of the adjoint PDE and the existence of the adjoint is a consequence of the Cauchy-Kowalevski result.

Choose a region Ω bounded by the hyperplane $\{t=0\}$ and some non-characteristic hypersurface Γ with respect to L such that $(0,0) \in \partial\Omega := \Gamma \cup \{t=0\}$. For any polynomial $\mathbf{p}$, define v as the solution of the $L^*\mathbf{v} = \mathbf{p}$ in Ω with $v=0$ on Γ where $(L^*v)_i = -v_t^i + \sum_{j=1}^n (\sum_{k=1}^n A_j^{ik} v_k)$, the adjoint of L. The existence of v follows from Theorem 2.6. Now, by integration by parts,

$$\begin{aligned} \int_\Omega L\mathbf{u}\cdot\mathbf{v} &= \int_\Omega \mathbf{u}_t \cdot \mathbf{v} - \sum_{i,j,k=1}^n \int_\Omega A_j^{ik} u_{x_j}^k v^i \\ &= \int_\Omega L^*\mathbf{v}\cdot\mathbf{u} + \int_{\partial\Omega} \left(\mathbf{u}\cdot\mathbf{v}\nu_t - \sum_{i,j,k=1}^n A_j^{ik} u^i v^k \nu \nu_{x_j} \right). \end{aligned}$$

The integral over $\partial\Omega$ vanishes because u vanishes on $\{t=0\}$ and v vanishes on Γ. Further, $L\mathbf{u} = 0$ and $L^*\mathbf{v} = \mathbf{p}$. Thus, we have the identity $\int_\Omega \mathbf{p}\mathbf{u} = 0$ for all polynomial $\mathbf{p}$. By Weierstrass theorem, choose a sequence of polynomial p_n converging uniformly to $\mathbf{u}$. Thus,

$$\int_\Omega \mathbf{u}^2 = \lim_{n\to\infty} \int_\Omega \mathbf{u} p_n = 0$$

and $u \equiv 0$ in Ω.

Chapter 3
Classification of Higher Order based on Characteristics

3.1 Characteristic Hypersurfaces for k-th Order Quasilinear PDE

Definition 3.1. Let $L := \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ be the k-th order linear partial differential operator defined in Ω. Then the *principal part*, denoted as L_p, of L is defined by

$$L_p := \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha \text{ in } \Omega$$

and the *principal symbol* p is defined by

$$p(x;\xi) := \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Definition 3.2. Let L be the k-th order linear partial differential operator defined in a neighbourhood of $x_0 \in \mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. Then Γ is said to be *non-characteristic* at x_0 if

$$p(x_0;\nu) = \sum_{|\alpha|=k} a_\alpha(x_0)\nu^\alpha(x_0) \neq 0$$

where ν is the normal to Γ at x_0. Otherwise, Γ is said to be *characteristic* at x_0 with respect to L. If Γ is *(non)characteristic* at each of its point then Γ is said to be (non)characteristic.

Theorem 3.1. *For any $f \in C(\Omega)$, Let u be a smooth solution to the k-th order linear Cauchy problem*

$$\begin{cases} \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha u = f & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma \\ \partial^i_\nu u(x) = u_i & \text{on } \Gamma \ \forall i = \{1,2,\ldots,k-1\}. \end{cases}$$

If the Γ is a non-characteristic hypersurface then it is possible to compute all order partial derivatives of u on Γ in terms the initial data viz. the hypersurface Γ, the initial conditions $\{u_i\}$ and the coefficients a_α.

Proof. Since the given hyperspace can be flattened to a hyperplane, we shall given the proof in two steps: first for the hyperplane $\{x_n = 0\}$ and then for any general hypersurface.

First Step: Suppose Γ is a hyperplane $\{x_n = 0\}$ and the n-tuple is written as $x = (x', x_n)$ where x' is the $(n-1)$-tuple. In this situation, the unit normal $\nu = e_n$ and $\partial^i_\nu u = \partial^i_{x_n} u$. Without loss of generality, let us compute all the derivatives $\partial^\alpha u(0)$, for all α, at $0 \in \Gamma$. Similar arguments extend to all other points of Γ. Let us assume that the initial conditions $\{u_i\}_{i=0}^{k-1}$ are all smooth functions. Then $\partial^\alpha_{x'} u(0) = \partial^\alpha_{x'} u_0(0)$ and $\partial^\alpha_{x'} \partial^i_{x_n} u(0) = \partial^\alpha_{x'} u_i(0)$ for all $0 \le |\alpha| < \infty$, where α is a $(n-1)$-tuple, and $i = 1, 2, \ldots, k-1$. Thus, all the derivates upto order k except $\partial^k_{x_n} u$ has been computed. However, using the given PDE, we obtain

$$\partial^k_{x_n} u(0) = \frac{-1}{a_{(0,\ldots,k)}(0)} \left(\sum_{\alpha \neq (0,\ldots,k)} a_\alpha(0) \partial^\alpha u(0) - f(0) \right)$$

whenever

$$a_{(0,\ldots,k)}(0) \neq 0. \tag{3.1}$$

The above condition is precisely the non-characteristic condition corresponding the hyperplane. Thus, one has computed all partial derivatives of u, at $0 \in \Gamma$, up to order k in terms of the given data under the condition (3.1). In fact, one can continue this process to compute any order derivative of u at 0 by differentiating the PDE as many times as required, provided the data and solution are smooth enough. Since the choice of 0, as a point of evaluation, is generic the arguments are valid for any point on Γ.

Second Step: Now, let Γ be a general hyperspace given by the equation $\{\phi = 0\}$ for a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in a neighbourhood of $x_0 \in \Gamma$ with $\nabla \phi \neq 0$. Without loss of generality, we assume $\phi_{x_n}(x_0) \neq 0$. Then, by the implicit function theorem, $\phi = 0$ around x_0 has the form $\phi(x) := x_n - \psi(x')$. Consider the map flattening the hyperspace to hyperplane, i.e. $(x', x_n) \mapsto y := (x', \phi(x))$, then its Jacobian matrix is given by

$$\begin{pmatrix} I_{(n-1)\times(n-1)} & \mathbf{0}_{n-1} \\ \nabla_{x'} \phi & \phi_{x_n} \end{pmatrix}_{n \times n}$$

and its determinant at x_0 is non-zero because $\phi_{x_n}(x_0) \neq 0$. If v is such that $u(x) = v \circ \phi(x)$ then, using the multivariable chain rule, the PDE satisfied by v is

$$\sum_{|\alpha|=k} a_\alpha(x) (\nabla \phi)^\alpha \partial^k_{y_n} v + \text{ terms not involving } \partial^k_{y_n} v = f \circ \phi^{-1}$$

and the initial conditions are given on the hyperplane $\{y_n = 0\}$. Thus, the necessary condition equivalent to (3.1) is

$$\sum_{|\alpha|=k} a_\alpha(x)(\nabla\phi)^\alpha \neq 0.$$

Recall that $\nabla\phi$ is parallel to the normal vector ν of the surface $\{\phi = 0\}$, i.e. the hypersurface Γ. This precisely is the motivation for the notion of non-characteristic hypersurface.

Thus, a Cauchy problem is meaningful only if the hypersurface on which the initial data is prescribed is non-characteristic. Otherwise the Cauchy problem may be ill-posed! Note that the notion of non-characteristic hypersurface is dependent on the differential operator (via its highest order coefficients a_α). It is not necessary that a given hypersurface be non-characteristic to a given differential operator.

However, there are cases when any given hypersurface is non-characteristic to the given differential operator.

Definition 3.3. Let L be the k-th order linear partial differential operator defined in a neighbourhood of $x_0 \in \mathbb{R}^n$. Then L is *elliptic* at x_0 if

$$p(x_0;\xi) = \sum_{|\alpha|=k} a_\alpha(x_0)\xi^\alpha(x_0) \neq 0$$

for all non-zero $\xi \in \mathbb{R}^n$. L is said to be *elliptic* in Ω if it is *elliptic* at every point of Ω.

In other words, a linear differential operator is elliptic if every hypersurface is non-characteristic. The Laplacian operator Δ is elliptic because $p(x_0;\xi) = |\xi|^2 > 0$ for all non-zero $\xi \in \mathbb{R}^n$.

Remark 3.1. A first order semilinear differential operator with real coefficients is never elliptic. Because if

$$Lu := \mathbf{a}(x)\cdot\nabla u - f(x,u) \text{ in } \Omega$$

then $p(x;\xi) = \mathbf{a}(x)\cdot\xi$ for all $x \in \Omega$. For instance, choosing ξ to be a vector normal to $\mathbf{a}$ does not satisfy the ellipticity condition. Thus, L is never elliptic. However, a first order linear differential operator with complex coefficients can be ellitpic. For instance, choose $\mathbf{a} = (\frac{1}{2}, \frac{\iota}{2})$.

The proof of Theorem 3.1 can be generalised to a k-th order quasilinear PDE motivating the following definition for the corresponding notion of non-characteristic hyperspaces.

Definition 3.4. For given a_α and f, let

$$Lu := \sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x),\ldots,u(x),x)\partial^\alpha u + f(D^{k-1}u(x),\ldots,u(x),x)$$

be the k-th order quasilinear partial differential operator defined in a neighbourhood of $x_0 \in \mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. Then Γ is said to be *non-characteristic* at x_0 if

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u_0(x_0),\ldots,u_0(x_0),x_0)\nu^\alpha(x_0) \neq 0$$

where $\nu(x_0)$ is the normal to Γ at x_0. Otherwise, Γ is said to be *characteristic* at x_0 with respect to L. If Γ is *(non)characteristic* at each of its point then Γ is said to be (non)characteristic.

In the linear case, the non-characteristic condition depended only on the initial hypersurface Γ and the coefficients of the highest order partial derivatives. However, in the quasilinear case, it also depends, in addition, on the initial data u_0.

3.2 Cauchy Problem of Second Order PDE in Two Dimension

Let us elaborate the classification of second order PDE in the two dimension. Consider the second order Cauchy problem in two variables (x,y)

$$\begin{cases} F\left(D^2u(x,y),\nabla u(x,y),u(x,y),x,y\right) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{on } \Gamma \\ \partial_\nu u(x) = u_1(x) & \text{on } \Gamma \end{cases} \tag{3.2}$$

where ν is the unit normal vector to the curve Γ and u_0,u_1 are known functions on Γ. If the curve Γ is parametrised by $s \mapsto (x(s),y(s))$ then the directional derivative of u at any point on Γ, along the tangent vector, is $u'(s) = u_x x'(s) + u_y y'(s)$. But $u'(s) = u_0'(s)$ on Γ. Thus, instead of the normal derivative, one can prescribe the partial derivatives u_x and u_y on Γ and reformulate the Cauchy problem (3.2) as

$$\begin{cases} F\left(D^2u,\nabla u,u,x,y\right) = 0 & \text{in } \Omega \\ u(x,y) = u_0(x,y) & \text{on } \Gamma \\ u_x(x,y) = u_{11}(x,y) & \text{on } \Gamma \\ u_y(x,y) = u_{12}(x,y) & \text{on } \Gamma \end{cases}$$

satisfying the *compatibility condition* $u_0'(s) = u_{11}x'(s) + u_{12}y'(s)$. The compatibility condition implies that among u_0,u_{11},u_{12} only two can be assigned independently, as expected for a second order equation. Now, consider the Cauchy problem for a second order quasilinear PDE defined in $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} Au_{xx} + 2Bu_{xy} + Cu_{yy} = f(x,y,u,u_x,u_y) & \text{in } \Omega \\ u(x,y) = u_0(x,y) & \text{on } \Gamma \\ u_x(x,y) = u_{11}(x,y) & \text{on } \Gamma \\ u_y(x,y) = u_{12}(x,y) & \text{on } \Gamma. \end{cases} \tag{3.3}$$

where A,B,C and f may nonlinearly depend on its arguments (x,y,u,u_x,u_y) and the initial data satisfies the compatibility condition $u_0'(s) = u_{11}x\prime(s) + u_{12}y'(s)$. Also, one of the coefficients A,B or C is identically non-zero (else the PDE is not of second order). By computing the second derivatives of u on Γ and considering u_{xx},

u_{yy} and u_{xy} as unknowns, we have the system of three equations in three unknowns on Γ,

$$\begin{array}{rlll} Au_{xx} & +2Bu_{xy} & +Cu_{yy} & = f \\ x'(s)u_{xx} & +y'(s)u_{xy} & & = u_{11}'(s) \\ & x'(s)u_{xy} & +y'(s)u_{yy} & = u_{12}'(s). \end{array}$$

This system of equation is solvable whenever the determinant of the coefficients are non-zero, i.e.,

$$\begin{vmatrix} A & 2B & C \\ x' & y' & 0 \\ 0 & x' & y' \end{vmatrix} \neq 0.$$

Definition 3.5. We say a curve $\Gamma \subset \mathbb{R}^2$ is *characteristic* with respect to (3.3)) if

$$A(y')^2 - 2Bx'y' + C(x')^2 = 0.$$

where $(x(s), y(s))$ is a parametrisation of Γ.

3.3 Classification of Second order Quasilinear PDE

Definition 3.1 *For any given $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ and $f : \mathbb{R}^n \to \mathbb{R}$, the second order quasi-linear partial differential operator is given as:*

$$Lu := A(\nabla u, u, x) : D^2 u - f(\nabla u, u, x) \tag{3.4}$$

where $:$ indicates the dot product in $\mathbb{R}^{n^2}$. A smooth hypersurface is non-characteristic *at $x_0 \in \Gamma$ with respect L if*

$$\sum_{i,j=1}^{n} A_{ij}(\nabla u(x_0), u(x_0), x_0)\nu_i(x_0)\nu_j(x_0) \neq 0.$$

where $\nu(x_0)$ is the normal vector of Γ at x_0.

Example 3.1. Consider the (wave) operator $\partial_t^2 - \Delta$ and the hypersurface $\{|x|^2 = t^2\}$ in $\mathbb{R}^n \times (0, \infty)$. We claim that this hypersurface is characteristic. Note that the equation of the hypersurface is $\{\phi(x,t) := |x|^2 - t^2 = 0\}$. Then $\nabla\phi = 2(x, -t)$ and $p(x_0, t_0; \xi) = -|x_0|^2 + t_0^2 = 0$.

If $\{\phi = 0\}$ is the equation of a hypersurface Γ then the characteristic condition at x_0 means that $\nabla^T \phi(x_0) A(x_0) \nabla\phi(x_0) = 0$. Since we seek smooth solutions then the mixed derivatives of the solution are equal, thus the Hessian matrix D^2u is symmetric. Therefore, without loss generality, one may assume A is symmetric. Because if A is not symmetric, one can replace A with its symmetric part $A^s := \frac{1}{2}(A + A^t)$ in L and L remains unchanged because $A \cdot D^2u = A^s \cdot D^2u$. The coefficient matrix $A(\nabla u(x), u(x), x)$ being a real symmetric matrix will admit n real eigenvalues

$(\lambda_1(x), \ldots, \lambda_n(x))$, at each x, i.e. there is a unitary matrix $U(x)$ such that $U^{-1}AU$ is the diagonal matrix with $\{\lambda_i\}$ as entries. Then the characteristic condition becomes the equation[1]

$$\begin{aligned} 0 = \nabla^T \phi(x_0)A(x_0)\nabla\phi(x_0) &= \nabla^T \phi(x_0)UU^{-1}A(x_0)UU^{-1}\nabla\phi(x_0) \\ &= [U^{-1}\nabla\phi]^T U^{-1}AU[U^{-1}\nabla\phi] = \sum_{i=1}^{n} \lambda_i \phi_{w_i}^2 \end{aligned}$$

where $w := Ux$. Note that if one of the eigenvalues are zero, say $\lambda_j = 0$, then $\phi(x) = \phi(x_j)$ is a solution to the equation. Thus, the characteristic hypersurfaces are hyperplanes $x_j =$ constant. In this situation the PDE is said to be of type *parabolic*. If all the eigen values are non-zero and of the same sign then

$$0 = |\lambda_i \phi_{x_i}^2| \geq \min_{1\leq i\leq n} |\lambda_i||\nabla\phi|^2.$$

The above situation is a contradiction and corresponds to the non-existence of real characteristic hypersurfaces. In this situation the PDE is said to be of type *elliptic*.

Definition 3.6. For each x, let $P(x)$ and $Z(x)$ denote the number of positive and zero eigenvalues of $A(\nabla u(x), u(x), x)$. We say the partial differential operator given in (3.4) is *hyperbolic* at $x \in \Omega$, if $Z(x) = 0$ and either $P(x) = 1$ or $P(x) = n-1$. It is *elliptic*, if $Z(x) = 0$ and either $P(x) = n$ or $P(x) = 0$. If $Z(x) = 0$ and $1 < P(x) < n-1$ then the PDE is said to be *ultra hyperbolic*. It is *parabolic* if $Z(x) > 0$.

In the case of hyperbolic and ultra hyperbolic, all the eigenvalues are non-zero with the positive and negative eigenvalues arranged as $\{\lambda_i\}_1^P$ and $\{\lambda_i\}_{P+1}^n$, respectively. Then the characteristic condition becomes

$$\sum_{i=1}^{P} \lambda_i \phi_{x_i}^2 = \sum_{j=P+1}^{n} |\lambda_j| \phi_{x_j}^2.$$

Thus, the family

$$\phi_\pm(r) = \sum_{i=1}^{P} \sqrt{\lambda_i} r_i \pm \frac{\sum_{i=1}^{P} \lambda_i^2}{\sum_{j=P+1}^{n} \lambda_j^2} \sum_{P+1}^{n} \sqrt{|\lambda_j|} r_j$$

solves the characteristic condition. We have obtained two families of characteristic hypersurfaces of the PDE, i.e. $\phi_\pm =$ constant.

Remark 3.2. In the two dimensions, the hyperbolic second order PDE have two families of real characteristic curves, parabolic has one family of real characteristic curves and elliptic have no real characteristic curves.

Observe that the elliptic case agrees with Definition 3.3 and corresponds to the situation where every hypersurface is non-characteristic, i.e. there are no real characteristics hypersurfaces.

[1] The resulting equation is a first order PDE

Definition 3.7. The operator L defined in (3.4) is said to be *elliptic* if the matrix $A_{ij}(x)$ is positive definite for each x, i.e.

$$0 < \alpha(x)|\xi|^2 \leq \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \leq \beta(x)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

The bounds $\alpha(x)$ and $\beta(x)$ are minimum and maximum eigenvalues, respectively.

Definition 3.8. If β/α is uniformly bounded in U then PDE is *uniformly elliptic*. The interesting thing about uniformly elliptic equation is that they behave very similar to linear elliptic equations.

Remark 3.3 (Two Dimension). If Γ is given by the parametrization (x,y) then $\nu := (-y', x')$ is the normal to Γ at each point and

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} -y' \\ x' \end{pmatrix} \cdot \begin{pmatrix} -y' \\ x' \end{pmatrix} = A(y')^2 - 2Bx'y' + C(x')^2 = 0.$$

If $y = y(x)$ is a representation of the curve Γ (locally, if necessary), we have $x(s) = s$ and $y(s) = y(s)$. Then the characteristic equation reduces as

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0.$$

Therefore, the characteristic curves of (3.3) are given by the graphs whose equation is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

Thus, we have three situations depending on the sign of the *discriminant* $d(x) := B^2 - AC$. A second order quasilinear PDE in two dimension is of

(a) *hyperbolic* type at x if $d(x) > 0$,
(b) *parabolic* type at x if $d(x) = 0$ and
(c) *elliptic* type at x if $d(x) < 0$.

To compare the above classification with the general one given in Definition 3.6, note that the second order coefficent matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

is symmetric and diagonalisable with eigenvalues, say, $\lambda_1(x,y)$ and $\lambda_2(x,y)$ for every $(x,y) \in \Omega$. Then, the discriminant of the diagonal matrix is $d = -\lambda_1\lambda_2$. Thus, the PDE is of hyperbolic type at a point (x,y) if the eigen values have opposite sign. It is ellipic if the eigenvalues have same sign and is parabolic if, at least, one of the eigenvalue is zero.

Example 3.2 (2D Wave Equation). For a given $c \in \mathbb{R}$, $u_{yy} - c^2 u_{xx} = 0$ is hyperbolic. Since $A = -c^2$, $B = 0$ and $C = 1$, we have $d = B^2 - AC = c^2 > 0$. The eigen values of the coefficient matrix are $1, -c^2$ which have opposite sign. Since the equation is hyperbolic, it admits two characteristic curves. Recall that the characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{\pm\sqrt{c^2}}{-c^2} = \frac{\mp 1}{c}.$$

Thus, $cy \pm x =$ a constant is the equation for the two characteristic curves. Note that the characteristic curves $y = \mp x/c + y_0$ are boundary of two cones in $\mathbb{R}^2$ with vertex at $(0, y_0)$.

Example 3.3 (Wave Equation). The wave equation $u_{tt} - \Delta_x u = f(x,t)$ for $(x,t) \in \mathbb{R}^{n+1}$ is hyperbolic because the $(n+1) \times (n+1)$ second order coefficient matrix is

$$A := \begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}$$

has no zero eigenvalue and exactly one positive eigenvalue, where I is the $n \times n$ identity matrix. The principal symbol associated to the wave operator is $p(x,t;\xi,\tau) := \tau^2 - |\xi|^2$. In two dimension $p(x,t;\xi,\tau) =$ a constant represents a hyperbola. Hence, the name hyperbolic! A hypersurface $\{\phi(x,t) = 0\}$ is characteristic to wave equation if $A\nabla\phi \cdot \nabla\phi = 0$. Thus, the hypersurface is such that $\phi_t^2 = |\nabla_x \phi|^2$. Thus, $\phi_\pm := t \pm \frac{1}{n}\sum_{i=1}^n x_i =$ a constant. The cone with vertex at (x_0, t_0), given by the equation $\{(x,t) \mid \phi := |x - x_0|^2 - (t - t_0)^2 = 0\}$ satisfies $A\nabla\phi \cdot \nabla\phi = 0$. Hence, the family cones is the characteristic curves for the above hyperbolic PDE called the *characteristic cones*.

Example 3.4 (2D Heat Equation). For a given $c \in \mathbb{R}$, $u_y - cu_{xx} = 0$ is parabolic. Since $A = -c$, $B = 0$ and $C = 0$, thus $d = B^2 - AC = 0$. The eigen values of the coefficient matrix are $0, -c$ has a zero eigenvalue. Since the PDE is of parabolic type, it admits one characteristic curve. The characteristic curve is given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = 0.$$

Thus, $y =$ a constant is the equation of the characteristic curve. i.e., any horizontal line in $\mathbb{R}^2$ is a charateristic curve.

Example 3.5. The heat equation $u_t - \Delta_x u = f(x,t)$ for $(x,t) \in \mathbb{R}^{n+1}$ is parabolic because the $(n+1) \times (n+1)$ second order coefficient matrix is

$$\begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix}$$

has one zero eigenvalue. The principal symbol associated to the heat operator is $p(x,t;\xi,\tau) := -|\xi|^2$. In two dimension $p(x,t;\xi,\tau) = \tau$ represents a parabola.

Hence, the name parabolic! A hypersurface $\{\phi(x,t) = 0\}$ is characteristic to heat equation if $A\nabla\phi \cdot \nabla\phi = 0$. Thus, the hypersurface is such that $|\nabla_x \phi|^2 = 0$, i.e ϕ is such that $\nabla_x \phi = 0$ and $\phi_t \neq 0$. For instance, $\phi := t =$ a constant is a characteristic hypersurface.

Example 3.6 (2D Laplace equation). $u_{xx} + u_{yy} = 0$ is elliptic. Since $A = 1$, $B = 0$ and $C = 1$, thus $d = B^2 - AC = -1 < 0$. The eigen values of the coefficient matrix are $1, 1$ which have same sign. Since the PDE is of elliptic type, it admits no real characteristics.

Example 3.7. The Laplace equation $\Delta u = f(\nabla u, u, x)$ for $x \in \mathbb{R}^n$ is elliptic because $\Delta u = I \cdot D^2 u(x)$ where I is the $n \times n$ identity matrix. The eigen values are all positive. The principal symbol associated to the Laplacian Δ is $p(x;\xi) := |\xi|^2$. In two dimension $p(x;\xi) =$ a constant represents a ellipse. Hence, the name elliptic!

Example 3.8 (Velocity Potential Equation). In the equation $(1 - M^2)u_{xx} + u_{yy} = 0$, $A = (1-M^2)$, $B = 0$ and $C = 1$. Then $d = B^2 - AC = -(1-M^2)$. The eigen values of the coefficient matrix are $1 - M^2, 1$. Thus, for $|M| > 1$ (opposite sign), the equation is hyperbolic (supersonic flow), for $|M| = 1$ (zero eigenvalue) it is parabolic (sonic flow) and for $|M| < 1$ (same sign) it is elliptic (subsonic flow).

Note that the classification of PDE is dependent on its coefficients. Thus, for constant coefficients the type of PDE remains unchanged throughout the region Ω. However, for variable coefficients, the PDE may change its classification from region to region.

Example 3.9. An example is the *Tricomi* equation , $u_{xx} + xu_{yy} = 0$. The discriminant of the Tricomi equation is $d = -x$. The eigenvalues are $1, x$. Thus, tricomi equation is hyperbolic when $x < 0$, elliptic when $x > 0$ and degenerately parabolic when $x = 0$, i.e., y-axis. Such equations are called *mixed type*. The equation $u_{xx} + xu_{yy} = 0$ is of mixed type. In the region $x > 0$, the characteristic curves are $y \mp 2x^{3/2}/3 =$ a constant.

Example 3.10. Consider the quasilinear PDE $u_{xx} - uu_{yy} = 0$. The discriminant is $d = u$. The eigenvalues are $1, -u(x)$. It is hyperbolic for $\{u > 0\}$[2], elliptic when $\{u < 0\}$ and parabolic when $\{u = 0\}$.

Example 3.11. Consider the quasilinear PDE

$$(c^2 - u_x^2)u_{xx} - 2u_x u_y u_{xy} + (c^2 - u_y^2)u_{yy} = 0$$

where $c > 0$. Then $d = B^2 - AC = c^2(u_x^2 + u_y^2 - c^2) = c^2(|\nabla u|^2 - c^2)$. It is hyperbolic if $|\nabla u| > c$, parabolic if $|\nabla u| = c$ and elliptic if $|\nabla u| < c$.

Example 3.12. Consider the minimal surface equation

[2] The notation $\{u > 0\}$ means $\{x \in \Omega \mid u(x) > 0\}$

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f(x)$$

where the second order coefficients are

$$A_{ij}(x,z,p) = (1+|p|^2)^{-1/2} \left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2} \right)$$

and

$$\alpha(x,z,p) = \frac{1}{(1+|p|^2)^{3/2}} \quad \beta(x,z,p) = \frac{1}{(1+|p|^2)^{1/2}}.$$

Thus, the equation is not uniformly elliptic.

The minimal surface equation and the capillary equation are not uniformly elliptic.

Remark 3.4. The classification of a PDE, in terms of its characteristics, tells us the right amount of Cauchy data that needs to be provided for gthe Cauchy problem to be well-posed. Let us illustrate this in two dimensions, for simplicity. A hyperbolic PDE, which has two families of real characteristics curves, requires as many initial condition as the number of characteristics **emanating from** initial time (Say $\Omega \times \{0\}$) and as many boundary conditions as the number of characteristics that pass **in to** the spatial boundary (Say $\partial\Omega \times (0,T)$). With similar logic, for parabolic PDE, which has exactly one family of real characteristic curves, we need one boundary condition at each point of the spatial boundary and one initial condition at initial time. For elliptic PDE, which admits no real characteristic curves, we need one boundary condition at each point of the spatial boundary. Observe that this is counter intuitive for a second order elliptic PDE.

Remark 3.5 (Ill-Posedness of Elliptic Cauchy Problem). The non-existence of real characteristic curves for an elliptic Cauchy problem makes them ill-posed (q.v. Example 1.19 and Section 4.3). Consider the Cauchy problem

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

Observe that the PDE is elliptic in its domain. One solution is the trivial solution $u(x,t) = 0$. Consider the Cauchy problem with a small change in data,

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 0 \\ u_t(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right) \end{cases}$$

which has the unique solution $u^\varepsilon(x,t) = \varepsilon^2 \sin(x/\varepsilon)\sinh(t/\varepsilon)$. The solution of the Cauchy problem is not stable because the data change is small, i.e.,

$$\sup_x \{|u_t^\varepsilon(x,0) - u_t(x,0)|\} = \varepsilon \sup_x \{|\sin(x/\varepsilon)|\} = \varepsilon$$

and the solution change is not at all small, i.e.,

$$\lim_{t\to\infty}\sup_x\{|u^{\varepsilon}(x,t)-u(x,t)|\} = \lim_{t\to\infty}\varepsilon^2\,|\sinh(t/\varepsilon)| = +\infty.$$

In fact, the solution will not converge in any reasonable metric.

Exercise 3.1. Classify the following second order PDE in terms of the number of characteristics:

(a) $3u_{xx}+u_{xy}+2u_{yy}=0$.
(b) $u_{zz}+u_z+u_{rr}+\frac{1}{r}u_\theta+c=0$.
(c) $u_t+\beta u_x+\alpha u_{xx}=0$.
(d) $4u_{xx}+y^2u_x+xu_x+u_{yy}+4u_{xy}-4xy=0$.
(e) $xu_{xx}+xu_{xy}+yu_{yy}=0$.
(f) $xu_{xx}+yu_{xy}+c=0$.
(g) $x^2yu_{xx}+xyu_{xy}-y^2u_{yy}=0$.
(h) $\sin xu_{xx}+2\cos xu_{xy}+\sin xu_{yy}=0$.
(i) $u_{xx}+4u_{xy}+5u_{yy}+u_x+2u_y=0$.
(j) $u_{xx}-4u_{xy}+4u_{yy}+3u_x+4u=0$.
(k) $u_{xx}+2u_{xy}-3u_{yy}+2u_x+6u_y=0$.
(l) $(1+x)u_{xx}+2xyu_{xy}-y^2u_{yy}=0$.
(m) $2u_{xx}-4u_{xy}+7u_{yy}-u=0$.
(n) $u_{xx}-2\cos xu_{xy}-\sin^2 xu_{yy}=0$.
(o) $yu_{xx}+2(x-1)u_{xy}-(y+2)u_{yy}=0$.
(p) $yu_{xx}+u_{xy}-x^2u_{yy}-u_x-u=0$.

Exercise 3.2. Classify the following second order PDE, in terms of the number of characteristics, and find their characteristics, when it exists:

(a) $u_{xx}+(5+2y^2)u_{xy}+(1+y^2)(4+y^2)u_{yy}=0$.
(b) $yu_{xx}+u_{yy}=0$.
(c) $yu_{xx}=xu_{yy}$.
(d) $u_{yy}-xu_{xy}+yu_x+xu_y=0$.
(e) $y^2u_{xx}+2xyu_{xy}+x^2u_{yy}=0$.
(f) $u_{xx}+2xu_{xy}+(1-y^2)u_{yy}=0$.

3.3.1 Invariance of Discriminant

The classification of second order semilinear PDE is based on the discriminant B^2-AC. In this section, we note that the classification is independent of the choice of coordinate system (to represent a PDE). Consider the two-variable semilinear PDE

$$A(x,y)u_{xx}+2B(x,y)u_{xy}+C(x,y)u_{yy}=D(x,y,u,u_x,u_y)\quad (x,y)\in\Omega \tag{3.5}$$

where the variables (x,y,u,u_x,u_y) may appear non-linearly in D and $\Omega \subset \mathbb{R}^2$. Also, one of the coefficients A,B or C is identically non-zero (else the PDE is not of second order). We shall observe how (3.5) changes under coordinate transformation.

Definition 3.9. For any PDE of the form (3.5) we define its *discriminant* as $B^2 - AC$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the coordinate transformation with components $T = (w,z)$, where $w,z : \mathbb{R}^2 \to \mathbb{R}$. We assume that $w(x,y)$, $z(x,y)$ are such that w,z are both continuous and twice differentiable w.r.t (x,y), and the Jacobian J of T is non-zero,

$$J = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} \neq 0.$$

We compute the derivatives of u in the new variable,

$$\begin{aligned}
u_x &= u_w w_x + u_z z_x, \\
u_y &= u_w w_y + u_z z_y, \\
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
u_{xy} &= u_{ww} w_x w_y + u_{wz}(w_x z_y + w_y z_x) + u_{zz} z_x z_y + u_w w_{xy} + u_z z_{xy}
\end{aligned}$$

Substituting above equations in (3.5), we get

$$a(w,z)u_{ww} + 2b(w,z)u_{wz} + c(w,z)u_{zz} = d(w,z,u,u_w,u_z).$$

where D transforms in to d and

$$a(w,z) = Aw_x^2 + 2Bw_x w_y + Cw_y^2 \tag{3.6}$$
$$b(w,z) = Aw_x z_x + B(w_x z_y + w_y z_x) + Cw_y z_y \tag{3.7}$$
$$c(w,z) = Az_x^2 + 2Bz_x z_y + Cz_y^2. \tag{3.8}$$

Note that the coefficients in the new coordinate system satisfy

$$b^2 - ac = (B^2 - AC)J^2.$$

Since $J \neq 0$, we have $J^2 > 0$. Thus, both $b^2 - ac$ and $B^2 - AC$ have the same sign. Thus, the sign of the *discriminant* is invariant under coordinate transformation. All the above arguments can be carried over to quasilinear and non-linear PDE.

3.3.2 Standard or Canonical Forms

The advantage of above classification helps us in reducing a given PDE into simple forms. Given a PDE, one can compute the sign of the discriminant and depending on its clasification we can choose a coordinate transformation (w,z) such that

(i) For hyperbolic, $a = c = 0$ or $b = 0$ and $a = -c$.
(ii) For parabolic, $c = b = 0$ or $a = b = 0$. We conveniently choose $c = b = 0$ situation so that $a \neq 0$ (so that division by zero is avoided in the equation for characteristic curves).
(iii) For elliptic, $b = 0$ and $a = c$.

If the given second order PDE (3.5) is such that $A = C = 0$, then (3.5) is of hyperbolic type and a division by $2B$ (since $B \neq 0$) gives

$$u_{xy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/2B$. The above form is the *first standard form* of second order hyperbolic equation. If we introduce the linear change of variable $X = x + y$ and $Y = x - y$ in the first standard form, we get the *second standard form* of hyperbolic PDE

$$u_{XX} - u_{YY} = \hat{D}(X, Y, u, u_X, u_Y).$$

If the given second order PDE (3.5) is such that $A = B = 0$, then (3.5) is of parabolic type and a division by C (since $C \neq 0$) gives

$$u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/C$. The above form is the *standard form* of second order parabolic equation.

If the given second order PDE (3.5) is such that $A = C$ and $B = 0$, then (3.5) is of elliptic type and a division by A (since $A \neq 0$) gives

$$u_{xx} + u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/A$. The above form is the *standard form* of second order elliptic equation.

Note that the standard forms of the PDE is an expression with no mixed derivatives.

3.3.3 Reduction to Standard Form

Consider the second order semilinear PDE (3.5) not in standard form. We look for transformation $w = w(x, y)$ and $z = z(x, y)$, with non-vanishing Jacobian, such that the reduced form is the standard form.

If $B^2 - AC > 0$, we have two characteristics. We are looking for the coordinate system w and z such that $a = c = 0$. This implies from equation (3.6) and (3.8) that we need to find w and z such that

$$\frac{w_x}{w_y} = \frac{-B \pm \sqrt{B^2 - AC}}{A} = \frac{z_x}{z_y}.$$

Therefore, we need to find w and z such that along the slopes of the characteristic curves,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{-w_x}{w_y}.$$

This means that, using the parametrisation of the characteristic curves, $w_x \dot{\gamma}_1(s) + w_y \dot{\gamma}_2(s) = 0$ and $\dot{w(s)} = 0$. Similarly for z. Thus, w and z are chosen such that they are constant on the characteristic curves.

The characteristic curves are found by solving

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

and the coordinates are then chosen such that along the characteristic curve $w(x,y) =$ a constant and $z(x,y) =$ a constant. Note that $w_x z_y - w_y z_x = w_y z_y \left(\frac{2}{A} \sqrt{B^2 - AC} \right) \neq 0$.

Example 3.13. Let us reduce the PDE $u_{xx} - c^2 u_{yy} = 0$ to its canonical form. Note that $A = 1$, $B = 0$, $C = -c^2$ and $B^2 - AC = c^2$ and the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm c.$$

Solving we get $y \mp cx =$ a constant. Thus, $w = y + cx$ and $z = y - cx$. Now writing

$$\begin{aligned}
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
&= c^2 (u_{ww} - 2u_{wz} + u_{zz}) \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
&= u_{ww} + 2u_{wz} + u_{zz} \\
-c^2 u_{yy} &= -c^2 (u_{ww} + 2u_{wz} + u_{zz})
\end{aligned}$$

Substituting into the given PDE, we get

$$\begin{aligned}
0 &= 4c^2 u_{wz} \\
&= u_{wz}.
\end{aligned}$$

Example 3.14. Let us reduce the PDE $u_{xx} - x^2 y u_{yy} = 0$ given in the region $\{(x,y) \mid x \in \mathbb{R}, x \neq 0, y > 0\}$ to its canonical form. Note that $A = 1$, $B = 0$, $C = -x^2 y$ and $B^2 - AC = x^2 y$. In the given region $x^2 y > 0$, hence the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm x\sqrt{y}.$$

Solving we get $x^2/2 \mp 2\sqrt{y} =$ a constant. Thus, $w = x^2/2 + 2\sqrt{y}$ and $z = x^2/2 - 2\sqrt{y}$. Now writing

$$\begin{aligned}
u_x &= u_w w_x + u_z z_x = x(u_w + u_z) \\
u_y &= u_w w_y + u_z z_y = \frac{1}{\sqrt{y}}(u_w - u_z) \\
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
&= x^2(u_{ww} + 2u_{wz} + u_{zz}) + u_w + u_z \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
&= \frac{1}{y}(u_{ww} - 2u_{wz} + u_{zz}) - \frac{1}{2y\sqrt{y}}(u_w - u_z) \\
-x^2 y u_{yy} &= -x^2(u_{ww} - 2u_{wz} + u_{zz}) + \frac{x^2}{2\sqrt{y}}(u_w - u_z)
\end{aligned}$$

Substituting into the given PDE, we get

$$\begin{aligned}
0 &= 4x^2 u_{wz} + \frac{2\sqrt{y} + x^2}{2\sqrt{y}} u_w + \frac{2\sqrt{y} - x^2}{2\sqrt{y}} u_z \\
&= 8x^2\sqrt{y} u_{wz} + (x^2 + 2\sqrt{y}) u_w + (2\sqrt{y} - x^2) u_z.
\end{aligned}$$

Note that $w + z = x^2$ and $w - z = 4\sqrt{y}$. Now, substituting x, y in terms of w, z, we get

$$\begin{aligned}
0 &= 2(w^2 - z^2) u_{wz} + \left(w + z + \frac{w - z}{2} \right) u_w + \left(\frac{w - z}{2} - w - z \right) u_z \\
&= u_{wz} + \left(\frac{3w + z}{4(w^2 - z^2)} \right) u_w - \left(\frac{w + 3z}{4(w^2 - z^2)} \right) u_z . \square
\end{aligned}$$

Example 3.15. Let us reduce the PDE $u_{xx} + u_{xy} - 2u_{yy} + 1 = 0$ given in the region $\{(x, y) \mid 0 \le x \le 1, y > 0\}$ to its canonical form. Note that $A = 1$, $B = 1/2$, $C = -2$ and $B^2 - AC = 9/4 > 0$. Hence the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{1}{2} \pm \frac{3}{2} = 2 \text{ or } -1.$$

Solving we get $y - 2x =$ a constant and $y + x =$ a constant. Thus, $w = y - 2x$ and $z = y + x$.

In the parabolic case, $B^2 - AC = 0$, we have a single characteristic. We are looking for a coordinate system such that either $b = c = 0$.

Example 3.16. Let us reduce the PDE $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} = 0$ to its canonical form. Note that $A = e^{2x}$, $B = e^{x+y}$, $C = e^{2y}$ and $B^2 - AC = 0$. The PDE is parabolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B}{A} = \frac{e^y}{e^x}.$$

Solving, we get $e^{-y} - e^{-x} =$ a constant. Thus, $w = e^{-y} - e^{-x}$. Now, we choose z such that $w_x z_y - w_y z_x \neq 0$. For instance, $z = x$ is one such choice. Then

$$\begin{aligned} u_x &= e^{-x} u_w + u_z \\ u_y &= -e^{-y} u_w \\ u_{xx} &= e^{-2x} u_{ww} + 2e^{-x} u_{wz} + u_{zz} - e^{-x} u_w \\ u_{yy} &= e^{-2y} u_{ww} + e^{-y} u_w \\ u_{xy} &= -e^{-y}(e^{-x} u_{ww} - u_{wz}) \end{aligned}$$

Substituting into the given PDE, we get

$$e^x e^{-y} u_{zz} = (e^{-y} - e^{-x}) u_w$$

Replacing x, y in terms of w, z gives

$$u_{zz} = \frac{w}{1 + we^z} u_w.$$

Example 3.17. Let us reduce the PDE $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{1}{xy}(y^3 u_x + x^3 u_y)$ to its canonical form. Note that $A = y^2$, $B = -xy$, $C = x^2$ and $B^2 - AC = 0$. The PDE is parabolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B}{A} = \frac{-x}{y}.$$

Solving, we get $x^2 + y^2 =$ a constant. Thus, $w = x^2 + y^2$. Now, we choose z such that $w_x z_y - w_y z_x \neq 0$. For instance, $z = x$ is one such choice. Then

$$\begin{aligned} u_x &= 2x u_w + u_z \\ u_y &= 2y u_w \end{aligned}$$

In the elliptic case, $B^2 - AC < 0$, we have no real characteristics. Thus, we choose w, z to be the real and imaginary part of the solution of the characteristic equation.

Example 3.18. Let us reduce the PDE $x^2 u_{xx} + y^2 u_{yy} = 0$ given in the region $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ to its canonical form. Note that $A = x^2$, $B = 0$, $C = y^2$ and $B^2 - AC = -x^2 y^2 < 0$. The PDE is elliptic. Solving the characteristic equation

$$\frac{dy}{dx} = \pm \frac{iy}{x}$$

we get $\ln x \pm i \ln y = c$. Let $w = \ln x$ and $z = \ln y$. Then

$$\begin{aligned} u_x &= u_w/x \\ u_y &= u_z/y \\ u_{xx} &= -u_w/x^2 + u_{ww}/x^2 \\ u_{yy} &= -u_z/y^2 + u_{zz}/y^2 \end{aligned}$$

Substituting into the PDE, we get

$$u_{ww} + u_{zz} = u_w + u_z.$$

Example 3.19. Let us reduce the PDE $u_{xx} + 2u_{xy} + 5u_{yy} = xu_x$ to its canonical form. Note that $A = 1$, $B = 1$, $C = 5$ and $B^2 - AC = -4 < 0$. The PDE is elliptic. The characteristic equation is

$$\frac{dy}{dx} = 1 \pm 2i.$$

Solving we get $x - y \pm i2x = c$. Let $w = x - y$ and $z = 2x$. Then

$$\begin{aligned} u_x &= u_w + 2u_z \\ u_y &= -u_w \\ u_{xx} &= u_{ww} + 4u_{wz} + 4u_{zz} \\ u_{yy} &= u_{ww} \\ u_{xy} &= -(u_{ww} + 2u_{wz}) \end{aligned}$$

Substituting into the PDE, we get

$$u_{ww} + u_{zz} = x(u_w + 2u_z)/4.$$

Replacing x, y in terms of w, z gives

$$u_{ww} + u_{zz} = \frac{z}{8}(u_w + 2u_z).$$

Example 3.20. Let us reduce the PDE $u_{xx} + u_{xy} + u_{yy} = 0$ to its canonical form. Note that $A = 1$, $B = 1/2$, $C = 1$ and $B^2 - AC = -3/4 < 0$. The PDE is elliptic. Solving the characteristic equation

$$\frac{dy}{dx} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

we get $2y = x \pm i\sqrt{3}x + c$. Let $w = 2y - x$ and $z = \sqrt{3}x$.

Exercise 3.3. Rewrite the PDE in their canonical forms and solve them.

(a) $u_{xx} + 2\sqrt{3}u_{xy} + u_{yy} = 0$
(b) $x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0$
(c) $u_{xx} - (2\sin x)u_{xy} - (\cos^2 x)u_{yy} - (\cos x)u_y = 0$
(d) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

Chapter 4
The Laplacian

4.1 Historical Introduction

A *field* is a physical quantity associated to each point of space-time. A field can be classified as a *scalar field* or *vector field* depending on whether the value of the field at each point is a scalar or vector, respectively. For example, the gradient of any function u, ∇u, is a vector field. Some well-known examples of field are Newton's gravitational field, Coulomb's electrostatic field and Maxwell's electromagnetic field. Given a vector field V, is there a scalar field u, called *potential*, such that $\nabla u = V$? In gravitation theory, the gravity potential is the potential energy per unit mass, i.e., if E is the potential energy of an object with mass m, then $u = E/m$ and the potential associated with a mass distribution is the superposition of potentials of point masses. The Newtonian gravitation potential can be computed to be

$$u(x) = \frac{1}{4\pi} \int_{\Omega} \frac{\rho(y)}{|x-y|}\, dy$$

where $\rho(y)$ is the density at y of a mass occupying the region $\Omega \subset \mathbb{R}^3$. In 1782, Laplace discovered that the Newton's gravitational potential satisfies the equation:

$$\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

This is the reason the operator Δ is called *Laplacian*. Later, in 1813, Poisson discovered that on Ω the Newtonian potential satisfies the equation:

$$-\Delta u = \rho \quad \text{in } \Omega.$$

Inhomogeneous Laplace equations are called *Poisson* equations.

The equation obtained by Laplace is a consequence of the conservation laws. Green (1828) and Gauss (1839) observed that the Laplace and Poisson equations can be generalised to any scalar potential including electric and magnetic potentials. Let u be a scalar potential such that the vector field $V(x) := \nabla u(x)$ exists. If V satisfies

$\int_{\partial\omega} V(x)\cdot\nu(x)\,d\sigma = 0$, for all closed surfaces $\partial\omega \subset \Omega$ and $\nu(x)$ is the unit outward normal at x on $\partial\omega$. Then, by Gauss divergence theorem (cf. (B.1)),

$$\int_\omega \nabla\cdot V\,dx = 0 \quad \forall\omega\subset\Omega.$$

Thus, $\nabla\cdot V = \mathrm{div}(V) = 0$ on Ω and hence $\Delta u = \nabla\cdot(\nabla u) = \nabla\cdot V = 0$ on Ω. A function whose Laplacian is null in a region is called a *harmonic* function in that region. Thus, any scalar potential is a harmonic function. The study of potentials, in physics, is called *Potential Theory* and, in mathematics, it is called Harmonic Analysis. Note that, for any potential u, its vector field $V = \nabla u$ is *irrotational*, i.e., $\mathrm{curl}(V) = \nabla\times V = 0$.

A general second order linear elliptic equation is of the form

$$\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + c(x)u(x) = f(x)$$

where $A(x) = a_{ij}(x)$ is real, symmetric and positive definite $n\times n$ matrix. If $A(x)$ is a constant matrix (independent of x) then with a suitable transformation T one can rewrite

$$\sum_{i,j=1}^n a_{ij}u_{x_ix_j}(x) = \Delta v(x)$$

where $v(x) := u(Tx)$. We introduced (cf. Chapter 1) Laplacian to be the trace of the Hessain matrix, $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. The Laplace operator usually appears in physical models associated with dissipative effects (except wave equation). The importance of Laplace operator can be realised by its appearance in various physical models. For instance, the heat equation

$$\frac{\partial}{\partial t} - \Delta,$$

the wave equation

$$\frac{\partial^2}{\partial t^2} - \Delta,$$

or the Schrödinger's equation

$$i\frac{\partial}{\partial t} + \Delta.$$

4.2 Properties of Laplacian

In cartesian coordiantes, the n-dimensional Laplacian is given as

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Note that in one dimension ($n = 1$) $\Delta = \frac{d^2}{dx^2}$.

Recall the definition of the function spaces $C^k(\Omega)$ in § 1.3. Let us set $C(\Omega) = C^0(\Omega)$. The Laplacian is a linear operator from $C^2(\Omega) \to C(\Omega)$, i.e., $\Delta(u+v) = \Delta u + \Delta v$ and $\Delta(\lambda u) = \lambda \Delta u$ for any constant $\lambda \in \mathbb{R}$.

For any $a \in \mathbb{R}^n$, the translation operator $T_a : C(\Omega) \to C(\Omega)$ is defined as $(T_a u)(x) = u(x+a)$. The Laplace operator commutes with the translation operator, i.e., $\Delta \circ T_a = T_a \circ \Delta$. Because, for any $u \in C^2(\Omega)$, $(T_a u)_{x_i}(x) = u_{x_i}(x+a)$ and $(T_a u)_{x_i x_i}(x) = u_{x_i x_i}(x+a)$. Thus, $\Delta(T_a u)(x) = \Delta u(x+a)$.

For any orthogonal[1] $n \times n$ matrix O, the rotation operator $R_O : C(\Omega) \to C(\Omega)$ is defined as $R_O u(x) = u(Ox)$. The Laplace operator commutes with rotation operator, i.e., $\Delta \circ R_O = R_O \circ \Delta$. Let $y = Ox$. Then, $y_j = \sum_{i=1}^n O_{ji} x_i$ and, by chain rule,

$$(R_O u)_{x_i} = \sum_{j=1}^n u_{y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^n u_{y_j} O_{ji}.$$

Therefore, $\nabla_x R_O u = O^t \nabla_y u$ and

$$(\Delta \circ R_O) u(x) = \nabla_x R_O u \cdot \nabla_x (R_O u) = O^t \nabla_y u \cdot O^t \nabla_y u = O O^t \nabla_y u \cdot \nabla_y u = \Delta_y u.$$

But $\Delta_y u = (\Delta u)(Ox) = (R_O \circ \Delta) u(x)$. The invariance of Laplacian under rotation implies that the class of all radial functions is mapped to itself. Recall that a radial function is one which is constant on every sphere about the origin.

In polar coordinates (2 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

where r is the magnitude component ($0 \le r < \infty$) and θ is the direction component ($0 \le \theta < 2\pi$). The direction component is also called the azimuth angle or polar angle. This is easily seen by using the relation $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial y}{\partial r} = \sin\theta \text{ and } \frac{\partial u}{\partial r} = \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}.$$

Also,

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta \frac{\partial^2 u}{\partial x^2} + \sin^2\theta \frac{\partial^2 u}{\partial y^2} + 2\cos\theta \sin\theta \frac{\partial^2 u}{\partial x \partial y}.$$

Similarly,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta, \ \frac{\partial u}{\partial \theta} = r\cos\theta \frac{\partial u}{\partial y} - r\sin\theta \frac{\partial u}{\partial x}$$

and

$$\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \sin^2\theta \frac{\partial^2 u}{\partial x^2} + \cos^2\theta \frac{\partial^2 u}{\partial y^2} - 2\cos\theta \sin\theta \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{r} \frac{\partial u}{\partial r}.$$

---

[1] $O^{-1} = O^t$

Therefore,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

and, hence,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Further, in cylindrical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

where $r \in [0,\infty)$, $\theta \in [0,2\pi)$ and $z \in \mathbb{R}$. In spherical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

where $r \in [0,\infty)$, $\phi \in [0,\pi]$ (zenith angle or inclination) and $\theta \in [0,2\pi)$ (azimuth angle).

Theorem 4.1. *Let $n \geq 2$ and u be a radial function, i.e., $u(x) = v(r)$ where $x \in \mathbb{R}^n$ and $r = |x|$, then*

$$\Delta u(x) = \frac{d^2 v(r)}{dr^2} + \frac{(n-1)}{r}\frac{dv(r)}{dr}.$$

Proof. Note that

$$\frac{\partial r}{\partial x_i} = \frac{\partial |x|}{\partial x_i} = \frac{\partial(\sqrt{x_1^2 + \ldots + x_n^2})}{\partial x_i} = \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-1/2}(2x_i) = \frac{x_i}{r}.$$

Thus,

$$
\begin{aligned}
\Delta u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u(x)}{\partial x_i}\right) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{dv(r)}{dr}\frac{x_i}{r}\right) \\
&= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\frac{dv(r)}{dr}\right) + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\frac{d}{dr}\left(\frac{dv(r)}{dr}\frac{1}{r}\right) + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\left\{\frac{1}{r}\frac{d^2v(r)}{dr^2} - \frac{1}{r^2}\frac{dv(r)}{dr}\right\} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{r^2}{r}\left\{\frac{1}{r}\frac{d^2v(r)}{dr^2} - \frac{1}{r^2}\frac{dv(r)}{dr}\right\} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{d^2v(r)}{dr^2} - \frac{1}{r}\frac{dv(r)}{dr} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{d^2v(r)}{dr^2} + \frac{(n-1)}{r}\frac{dv(r)}{dr}.
\end{aligned}
$$

Hence the result proved.

More generally, the Laplacian in $\mathbb{R}^n$ may be written in polar coordinates as

$$\Delta := \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is a second order differential operator in angular variables only. The angular part of Laplacian is called the *Laplace-Beltrami operator* acting on $\mathbb{S}^{n-1}$ (unit sphere of $\mathbb{R}^n$) with Riemannian metric induced by the standard Euclidean metric in $\mathbb{R}^n$.

4.3 Ill-Posedness of Cauchy Problem

Recall that for a second order Cauchy problem we need to know both u and its normal derivative on a data curve Γ contained in Ω. However, the Cauchy problem for Laplacian (more generally for elliptic equations) is not well-posed. In fact, the Cauchy problem for Laplace equation on a bounded domain Ω is over-determined.

Example 4.1 (Hadamard). Consider the Cauchy problem for Laplace equation

$$
\begin{cases}
u_{xx} + u_{yy} &= 0 \\
u(0,y) &= \frac{\cos ky}{k^2} \\
u_x(0,y) &= 0,
\end{cases}
$$

where $k > 0$ is an integer. It is easy to verify that there is a unique solution

$$u_k(x,y) = \frac{\cosh(kx)\cos(ky)}{k^2}$$

of the Cauchy problem. Note that for any $x_0 > 0$,

$$|u_k(x_0, n\pi/k)| = \frac{\cosh(kx_0)}{k^2}.$$

Since, as $k \to \infty$, $n\pi/k \to 0$ and $|u_k(x_0, n\pi/k)| \to \infty$ the Cauchy problem is not stable, and hence not well-posed.

Exercise 4.1. Show that the Cauchy problem for Laplace equation

$$\begin{cases} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= 0 \\ u_y(x,0) &= k^{-1}\sin kx, \end{cases}$$

where $k > 0$, is not well-posed. (Hint: Compute explicit solution using separation of variable. Note that, as $k \to \infty$, the Cauchy data tends to zero uniformly, but the solution does not converge to zero for any $y \neq 0$. Therefore, a small change from zero Cauchy data (with corresponding solution being zero) may induce bigger change in the solution.)

This issue of ill-posedness of the Cauchy problem is very special to second order elliptic equations. In general, any hyperbolic equation Cauchy problem is well-posed, as long as the hyperbolicity is valid in the full neighbourhood of the data curve.

Example 4.2. Consider the Cauchy problem for the second order hyperbolic equation

$$\begin{cases} y^2 u_{xx} - y u_{yy} + \frac{1}{2}u_y &= 0 \quad y > 0 \\ u(x,0) &= f(x) \\ u_y(x,0) &= g(x). \end{cases}$$

The general solution to this problem can be computed as

$$u(x,y) = F\left(x + \frac{2}{3}y^{3/2}\right) + G\left(x - \frac{2}{3}y^{3/2}\right).$$

On $y = 0$ $u(x,0) = F(x) + G(x) = f(x)$. Further,

$$u_y(x,y) = y^{1/2}F'\left(x + \frac{2}{3}y^{3/2}\right) - y^{1/2}G'\left(x - \frac{2}{3}y^{3/2}\right)$$

and $u_y(x,0) = 0$. Thus, the Cauchy problem has no solution unless $g(x) = 0$. If $g \equiv 0$ then the solution is

$$u(x,y) = F\left(x + \frac{2}{3}y^{3/2}\right) - F\left(x - \frac{2}{3}y^{3/2}\right) + f\left(x - \frac{2}{3}y^{3/2}\right)$$

for arbitrary $F \in C^2$. Therefore, when $g \equiv 0$ the solution is not unique. The Cauchy problem is not well-posed because the equation is hyperbolic ($B^2 - AC = y^3$) not in the full neighbourhood of the data curve $\{y = 0\}$.

4.4 Boundary Conditions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary denoted as $\partial\Omega$. To make the over-determined Cauchy problem of an elliptic equation well-posed, it is reasonable to specify one of the following conditions on the boundary $\partial\Omega$:

(i) (Dirichlet condition) $u = g$;
(ii) (Neumann condition) $\nabla u \cdot \nu = g$, where $\nu(x)$ is the unit outward normal of $x \in \partial\Omega$;
(iii) (Robin condition) $\nabla u \cdot \nu + cu = g$ for any $c > 0$.
(iv) (Mixed condition) $u = g$ on Γ_1 and $\nabla u \cdot \nu = h$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The elliptic equation with Neumann boundary condition naturally imposes a compatibility condition. By Guass divergence theorem (cf. Corollary B.4), if u is a solution of the Neumann problem then u satisfies, for every connected component ω of Ω,

$$\begin{aligned} \int_\omega \Delta u &= \int_{\partial\omega} \nabla u \cdot \nu \quad \text{(Using GDT)} \\ -\int_\omega f &= \int_{\partial\omega} g. \end{aligned}$$

The second equality is called the *compatibility* condition. Thus, for an inhomogeneous Laplace equation with Neumann boundary condition, the given data f, g must necessarily satisfy the compatibility condition. Otherwise, the Neumann problem does not make any sense.

The aim of this chapter is to solve, for any open bounded subset $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega \\ \text{one of the above} & \text{inhomogeneous boudary condition on } \partial\Omega. \end{cases}$$

By the linearity of Laplacian, $u = v + w$ where v is a solution of

$$\begin{cases} \Delta v(x) = 0 & \text{in } \Omega \\ \text{one of the above} & \text{inhomogeneous boudary condition on } \partial\Omega, \end{cases}$$

and w is a solution of

$$\begin{cases} -\Delta w(x) = f(x) & \text{in } \Omega \\ \text{one of the above} & \text{homogeneous boudary condition on } \partial\Omega. \end{cases}$$

Therefore, we shall solve for u by solving for v and w separately.

4.5 Dirichlet Principle

The Dirichlet principle (formulated, independently by Gauss, Lord Kelvin and Dirichlet) states that the solution of the Dirichlet problem minimizes the corresponding energy functional.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$ and let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given. For convenience, recall the Dirichlet problem ((4.7)),

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = g & \text{on } \partial\Omega. \end{cases}$$

Any solution u of (4.7) is in $V = \{v \in C^2(\overline{\Omega}) \mid v = g \text{ on } \partial\Omega\}$. The *energy functional* $J : V \to \mathbb{R}$ is defined as

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f v\,dx$$

Theorem 4.2 (Dirichlet's principle). *A $C^2(\overline{\Omega})$ function u solves (4.7) iff u minimises the functional J on V, i.e.,*

$$J(u) \le J(v) \quad \forall v \in V.$$

Proof. Let $u \in C^2(\overline{\Omega})$ be a solution of (4.7). For any $v \in V$, we multiply both sides of (4.7) by $u - v$ and integrating we get,

$$\begin{aligned} \int_\Omega (-\Delta u)(u-v)\,dx &= \int_\Omega f(u-v)\,dx \\ \int_\Omega \nabla u \cdot \nabla(u-v)\,dx &= \int_\Omega f(u-v)\,dx \\ \int_\Omega \left(|\nabla u|^2 - fu\right)\,dx &= \int_\Omega (\nabla u \cdot \nabla v - fv)\,dx \\ &\le \int_\Omega |\nabla u \cdot \nabla v| - \int_\Omega f v\,dx \\ &\le \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\,dx - \int_\Omega f v\,dx \\ &\quad \text{(since } 2ab \le a^2 + b^2 \text{)} \\ J(u) &\le J(v). \end{aligned}$$

Thus, u minimises J in V. Conversely, let u minimise J in V. Thus,

$$\begin{aligned}
&J(u) \le J(v) \quad \forall v \in V \\
&J(u) \le J(u+t\phi) \quad (\text{for any } \phi \in C^2(\Omega) \text{ such that } \phi = 0 \text{ on } \partial\Omega) \\
&\quad 0 \le \frac{1}{t}\left(J(u+t\phi) - J(u)\right) \\
&\quad 0 \le \frac{1}{t}\left(\frac{1}{2}\int_\Omega \left(t^2|\nabla\phi|^2 + 2t\nabla\phi\cdot\nabla u\right) dx - t\int_\Omega f\phi\, dx\right)
\end{aligned}$$

Taking limit $t \to 0$ both sides, we get

$$0 \le \int_\Omega \nabla\phi\cdot\nabla u\, dx - \int_\Omega f\phi\, dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.$$

Choosing $-\phi$ in place of ϕ we get the reverse inequality, and we have equality in the above. Thus,

$$\begin{aligned}
\int_\Omega \nabla u\cdot\nabla\phi\, dx &= \int_\Omega f\phi\, dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega \\
\int_\Omega (-\Delta u - f)\phi\, dx &= 0 \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.
\end{aligned}$$

Thus u solves (4.7).

4.6 Harmonic Functions

The *one dimensional* Laplace equation is an ODE ($\frac{d^2}{dx^2}$) and is solvable with solutions $u(x) = ax+b$ for some constants a and b. But in higher dimensions solving Laplace equation is not so simple. For instance, a *two dimensional* Laplace equation

$$u_{xx} + u_{yy} = 0$$

has the trivial solution, $u(x,y) = ax+by+c$, all one degree polynomials of two variables. In addition, xy, x^2-y^2, x^3-3xy^2, $3x^2y-y^3$, $e^x\sin y$ and $e^x\cos y$ are all solutions to the two variable Laplace equation. In $\mathbb{R}^n$, it is trivial to check that all polynomials up to degree one, i.e.

$$\sum_{|\alpha|\le 1} a_\alpha x^\alpha$$

is a solution to $\Delta u = 0$ in $\mathbb{R}^n$. But we also have functions of higher degree and functions not expressible in terms of elementary functions as solutions to Laplace equation. For instance, note that $u(x) = \prod_{i=1}^n x_i$ is a solution to $\Delta u = 0$ in $\mathbb{R}^n$.

Definition 4.1. Let Ω be an open subset of $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be *harmonic* on Ω if $\Delta u(x) = 0$ in Ω.

We already remarked that every scalar potential is a harmonic function (cf. § **??**). Gauss was the first to deduce some important properties of harmonic functions and thus laid the foundation for Potential theory and Harmonic Analysis. Due to the linearity of Δ, sum of any finite number of harmonic functions is harmonic and a scalar multiple of a harmonic function is harmonic. Moreover, harmonic functions is the null-space (kernel) of the Laplace operator from $C^2(\Omega)$ to $C(\Omega)$. Thus, we denote the class of harmonic functions as $N(\Delta)$.

In two dimension, one associates with a harmonic function $u(x,y)$, a *conjugate* harmonic function, $v(x,y)$ defined as the solution of a first order system of PDE called the *Cauchy-Riemann* equations,

$$u_x(x,y) = v_y(x,y) \quad \text{and } u_y(x,y) = -v_x(x,y).$$

Harmonic functions and holomorphic functions (differentiable complex functions) are related in the sense that, for any pair (u,v), harmonic and its conjugate, gives a holomorphic function $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$. Conversely, for any holomorphic function f, its real part and imaginary part are conjugate harmonic functions. This observation gives us more examples of harmonic functions, for instance, since all complex polynomials $f(z) = z^m$ are holomorphic we have (using the polar coordinates) $u(r,\theta) = r^m \cos m\theta$ and $v(r,\theta) = r^m \sin m\theta$ are harmonic functions in $\mathbb{R}^2$ for all $m \in \mathbb{N}$. Similarly, since $f(z) = \log z = \ln r + i\theta$ is holomorphic in certain region, we have $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ are harmonic in $\mathbb{R}^2 \setminus (0,0)$ and $\mathbb{R}^2 \setminus \{\theta = 0\}$, respectively.

Exercise 4.2. Show that there are infinitely many linearly independent harmonic functions in the vector space $C^2(\mathbb{R}^2)$.

4.6.1 Spherical Harmonics

A polynomial of degree k in n-variables is of the form

$$P_k(x) := \sum_{|\alpha| \le k} a_\alpha x^\alpha.$$

A polynomial P is said to be *homogeneous* of degree k if $P(\lambda x) = \lambda^k P(x)$, for all non-zero $\lambda \in \mathbb{R}$. Note that a homogeneous polynomial of degree k will be of the form

$$\sum_{|\alpha| = k} a_\alpha x^\alpha.$$

The number of possible n-tuples α such that $|\alpha| = k$ is given by $\binom{n+k-1}{k}$. Let $\mathscr{H}_k(\mathbb{R}^n)$ denote the set of all homogeneous harmonic polynomial of degree k in n variables. The set $\mathscr{H}_k(\mathbb{R}^n)$ forms a vector space and $\mathscr{H}_k(\mathbb{R}^n) \subset N(\Delta) \subset C^2(\mathbb{R}^n)$.

4.6.1.1 Two Dimensions

Consider a general homogeneous polynomial

$$P_k(x,y) := \sum_{i=0}^{k} a_i x^i y^{k-i}$$

of degree k in $\mathbb{R}^2$ (two variables). Note that P_k contains $k+1$ coefficients[2]. Then

$$\Delta P_k(x,y) = \sum_{i=2}^{k} a_i i(i-1)x^{i-2}y^{k-i} + \sum_{i=0}^{k-2} a_i (k-i)(k-i-1)x^i y^{k-i-2}$$

is a homogeneous polynomial of degree $k-2$ and, hence, contains $k-1$ coefficients. If $P_k \in \mathscr{H}_k(\mathbb{R}^2)$, i.e. $\Delta P_k(x,y) = 0$, then all the $k-1$ coefficients should vanish. Thus, we have $k-1$ equations relating the $k+1$ coefficients of P_k and, hence, $\mathscr{H}_k(\mathbb{R}^2)$ is of dimension two (since $k+1-(k-1)=2$). Let us now find the basis of the two dimensional space $\mathscr{H}_k(\mathbb{R}^2)$. In polar coordinates, $P_k(r,\theta) = r^k Q_k(\theta)$ where

$$Q_k(\theta) = \sum_{i=0}^{k} a_i (\cos\theta)^i (\sin\theta)^{k-i}.$$

Note that Q_k is the restriction of P_k to $\mathbb{S}^1$ and are called *spherical harmonics*. If $P_k \in \mathscr{H}_k(\mathbb{R}^2)$ then, using the polar form of Laplacian, we get

$$r^{k-2}\left[Q_k''(\theta) + k^2 Q_k(\theta)\right] = 0.$$

Therefore, for all $r > 0$, $Q_k(\theta)$ is a solution to the ODE

$$Q_k''(\theta) + k^2 Q_k(\theta) = 0.$$

Therefore, $Q_k(\theta) = \alpha\cos k\theta + \beta\sin k\theta$ and $P_k(r,\theta) = r^k(\alpha\cos k\theta + \beta\sin k\theta)$. Thus, P_k is a linear combination $r^k\cos k\theta$ and $r^k\sin k\theta$. In fact, if we identify each vector $(x,y) \in \mathbb{R}^2$ with the complex number $z = x+iy$, then we have shown that $\mathrm{Re}(z^k), \mathrm{Im}(z^k)$ are the basis of $\mathscr{H}_k(\mathbb{R}^2)$. If we choose α_1 and β_1 such that $\beta = -\alpha_1\sin\beta_1$ and $\alpha = \alpha_1\cos\beta_1$, then we can rewrite the polynomial as

$$P_k(r,\theta) = \alpha_1 r^k \cos(k\theta + \beta_1).$$

Thus, we immediately see that the zero set of $P_k(r,\theta)$ in $\mathbb{R}^2$ will be a family of k straight lines passing through origin such that between any two consecutive lines the angle is same.

---

[2] $\binom{k+1}{k} = k+1$

4.6.1.2 Three Dimensions

Consider a general homogeneous polynomial

$$P_k(x) := \sum_{|\alpha|=k} a_\alpha x^\alpha$$

of degree k in $\mathbb{R}^3$ (three variables). Note that P_k contains $\binom{k+2}{k} = \frac{(k+2)(k+1)}{2}$ coefficients. Then $\Delta P_k(x)$ is a homogeneous polynomial of degree $k-2$ and, hence, contains $\frac{k(k-1)}{2}$ coefficients. If $P_k \in \mathscr{H}_k(\mathbb{R}^3)$, i.e. $\Delta P_k(x) = 0$, then all the $\frac{k(k-1)}{2}$ coefficients should vanish. Thus, we have $\frac{k(k-1)}{2}$ equations relating the $\frac{(k+2)(k+1)}{2}$ coefficients of P_k and, hence, $\mathscr{H}_k(\mathbb{R}^3)$ is of dimension

$$\frac{(k+2)(k+1)-k(k-1)}{2} = 2k+1.$$

The basis of the $2k+1$ dimensional space $\mathscr{H}_k(\mathbb{R}^3)$ is given in terms of the Legendre functions which we shall describe now. In spherical coordinates, $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\phi$. Thus, $P_k(r,\phi,\theta) = r^k R(\phi)Q(\theta)$ where

$$R(\phi)Q(\theta) = \sum_{|\alpha|=k} a_\alpha (\sin\phi)^{\alpha_1+\alpha_2}(\cos\phi)^{\alpha_3}(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}.$$

The separated variable assumption above is not a issue because differential operator is linear. Note that RQ is the restriction of P_k to $\mathbb{S}^2$ and are called *spherical harmonics*. If $P_k \in \mathscr{H}_k(\mathbb{R}^2)$ then, using the spherical form of Laplacian, we get

$$r^{k-2}\left[k(k+1)\sin^2\phi + \sin^2\phi\frac{R''(\phi)}{R(\phi)} + \sin\phi\cos\phi\frac{R'(\phi)}{R(\phi)} + \frac{Q''(\theta)}{Q(\theta)}\right] = 0.$$

Therefore, for all $r > 0$, we have equality

$$k(k+1)\sin^2\phi + \sin^2\phi\frac{R''(\phi)}{R(\phi)} + \sin\phi\cos\phi\frac{R'(\phi)}{R(\phi)} = -\frac{Q''(\theta)}{Q(\theta)}.$$

Since LHS is a function of ϕ and RHS is a function of θ they must be equal to some constant λ. Then, we have to solve for the eigenvalue problem

$$-Q''(\theta) = \lambda Q(\theta)$$

where Q is 2π-periodic. This has the solution, for all $m \in \mathbb{N}\cup\{0\}$, $\lambda = m^2$ and $Q_m(\theta) = \alpha_m\cos m\theta + \beta_m \sin m\theta$. For $\lambda = m^2$ we solve for $R(\phi)$ in

$$R''(\phi) + \frac{\cos\phi}{\sin\phi}R'(\phi) = R(\phi)\left(\frac{m^2}{\sin^2\phi} - k(k+1)\right) \quad \phi \in (0,\phi).$$

Set $w = \cos\phi$. Then $\frac{dw}{d\phi} = -\sin\phi$.

$$R'(\phi) = -\sin\phi \frac{dR}{dw} \text{ and } R''(\phi) = \sin^2\phi \frac{d^2R}{dw^2} - \cos\phi \frac{dR}{dw}$$

In the new variable w, we get the Legendre equation

$$(1-w^2)R''(w) - 2wR'(w) = \left(\frac{m^2}{1-w^2} - k(k+1)\right)R(w) \quad w \in [-1,1].$$

For each $k \in \mathbb{N} \cup \{0\}$, this has the Legendre polynomials, $R_{k,m}(\cos\phi)$, as its solutions. Therefore, in general,

$$P_k(r,\phi,\theta) = r^k(\alpha\cos m\theta + \beta\sin m\theta)R_{k,m}(\cos\phi).$$

However, we are interested only those $R_{k,m}$ which gives a polynomial of degree k in $\mathbb{R}^3$. Thus, for $m = 0,1,\ldots,k$,

$$R_{k,m}(w) = (1-w^2)^{m/2}\frac{d^{k+m}}{dw^{k+m}}(1-w^2)^k.$$

Note that, for each fixed k and all $1 \le m \le k$, the collection

$$\{R_{k,0}(\cos\phi), \cos m\theta R_{k,m}(\cos\phi), \sin m\theta R_{k,m}(\cos\phi)\} \subset \mathscr{H}_k(\mathbb{R}^3)$$

is $2k+1$ linearly independent homogeneous harmonic polynomials of degree k and forms a basis. Thus, each P_k is a linear combination of these basis elements.

The zero sets of P_k exhibit properties depending on m. For $m = 0$ the harmonic polynomial P_k is a constant multiple of $R_{k,0}(\cos\phi)$. Since $R_{k,0}(w)$ has k distinct zeros in $[-1,1]$ arranged symmetrically about $w = 0$, there are k distince zeros of $R_{k,0}(\cos\phi)$ in $(0,\pi)$ arranged symmetrically about $\pi/2$. Thus on $\mathbb{S}^2$, the unit sphere, the function $R_{k,0}(\cos\phi)$ vanishes on k circles circumscribed in the latitudinal direction. For k odd the circle along equator is also a zero set. The function $R_{k,0}(\cos\phi)$ and its constant multiples are called *zonal harmonics*.

If $0 < m < k$, then the spherical harmonics is of the form

$$(\alpha\cos m\theta + \beta\sin m\theta)\sin^m\phi \frac{d^{k+m}}{dw^{k+m}}(1-w^2)^k.$$

If the first term is zero then $\tan m\theta = -\alpha/\beta$. This corresponds to great circle through the north pole and south pole of $\mathbb{S}^2$ and the angle between the planes containing two consecutive great circle is π/m. The second term vanishes on $\phi = 0$ and $\phi = \pi$ corresponding to the north and south pole, respectively. The third term vanishes on $k-m$ latitude circle. Thus, we have orthogonally intersecting family of circles which form the zero set which are called *tesseral harmonics*.

If $m = k$ then the spherical harmonics is of the form

$$(\alpha\cos k\theta + \beta\sin k\theta)\sin^k\phi$$

and it vanishes for $\phi = 0$, $\phi = \pi$ or $\tan k\theta = -\alpha/\beta$. The first two cases corresponds to the north and south pole, respectively, and the last case corresponds to great circles through the north pole and south pole of $\mathbb{S}^2$ and the angle between the planes containing two consecutive great circle is π/k. Thus, the great circles divide the $\mathbb{S}^2$ in to $2k$ sectors and are called *sectorial harmoics*.

4.6.1.3 Higher Dimensions

Consider a general homogeneous polynomial

$$P_k(x) := \sum_{|\alpha|=k} a_\alpha x^\alpha$$

of degree k in $\mathbb{R}^n$ (n variables). Note that P_k contains $\binom{n+k-1}{k}$ coefficients. Then $\Delta P_k(x)$ is a homogeneous polynomial of degree $k-2$ and, hence, contains $\binom{n+k-3}{k-2}$ coefficients. If $P_k \in \mathscr{H}_k(\mathbb{R}^n)$, i.e. $\Delta P_k(x) = 0$, then all the $\binom{n+k-3}{k-2}$ coefficients should vanish. Thus, we have $\binom{n+k-3}{k-2}$ equations relating $\binom{n+k-1}{k}$ coefficients of P_k and, hence, $\mathscr{H}_k(\mathbb{R}^n)$ is of dimension

$$\ell := \binom{n+k-1}{k} - \binom{n+k-3}{k-2}.$$

In polar form, $P_k(r,\theta) = r^k Q(\theta)$ where $\theta \in S^{n-1}$ and if $P_k(r,\theta) \in \mathscr{H}_k(\mathbb{R}^n)$ then

$$\Delta P_k = \frac{\partial^2 P_k}{\partial r^2} + \frac{n-1}{r}\frac{\partial P_k}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} P_k = 0$$

where $\Delta_{\mathbb{S}^{n-1}}$ is a second order differential operator in angular variables only called the *Laplace-Beltrami operator*. Therefore, we have

$$r^{k-2}\left[\Delta_{\mathbb{S}^{n-1}} Q(\theta) + k(n+k-2)Q(\theta)\right] = 0$$

and for $r > 0$,

$$\Delta_{\mathbb{S}^{n-1}} Q(\theta) + k(n+k-2)Q(\theta) = 0.$$

4.6.2 Properties of Harmonic Functions

In this section we shall study properties of harmonic functions. We shall assume the divergence theorems from multivariable calculus (cf. Appendix B). Also, note that if u is a harmonic function on Ω then, by Gauss divergence theorem (cf. Theorem B.3),

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

Definition 4.2. Let Ω be an open subset of $\mathbb{R}^n$ and $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ (cf. Appendix C) be the surface area of the unit sphere $S_1(0)$ of $\mathbb{R}^n$.

(a) A function $u \in C(\Omega)$ is said to satisfy the *first mean value property* (FMVP) in Ω if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y \quad \text{for any } B_r(x) \subset \Omega.$$

(b) A function $u \in C(\Omega)$ is said to satisfy the *second mean value property* (SMVP) in Ω if

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \quad \text{for any } B_r(x) \subset \Omega.$$

Exercise 4.3. Show that u satisfies the FMVP iff

$$u(x) = \frac{1}{\omega_n} \int_{S_1(0)} u(x+rz)\, d\sigma_z.$$

Similarly, u satisfies SMVP iff

$$u(x) = \frac{n}{\omega_n} \int_{B_1(0)} u(x+rz)\, dz.$$

Exercise 4.4. Show that the FMVP and SMVP are equivalent. That is show that u satisfies (a) iff u satisfies (b).

Owing to the above exercise we shall, henceforth, refer to the FMVP and SMVP as just the mean value property (MVP).

We shall now prove a result on the smoothness of a function satisfying MVP.

Theorem 4.3. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then $u \in C^\infty(\Omega)$.*

Proof. We first consider $u_\varepsilon := \rho_\varepsilon * u$, the convolution of u with mollifiers, as introduced in Theorem **??**. where

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

We shall now show that $u = u_\varepsilon$ for all $\varepsilon > 0$, due to the MVP of u and the radial nature of ρ. Let $x \in \Omega_\varepsilon$. Consider

$$\begin{aligned}
u_\varepsilon(x) &= \int_\Omega \rho_\varepsilon(x-y)u(y)\,dy \\
&= \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)u(y)\,dy \quad (\text{Since } \operatorname{supp}(\rho_\varepsilon) \text{ is in } B_\varepsilon(x)) \\
&= \int_0^\varepsilon \rho_\varepsilon(r)\left(\int_{S_r(x)} u(y)\,d\sigma_y\right)dr \quad (\text{cf. Theorem C.1}) \\
&= u(x)\omega_n \int_0^\varepsilon \rho_\varepsilon(r)r^{n-1}\,dr \quad (\text{Using MVP of } u) \\
&= u(x)\int_0^\varepsilon \rho_\varepsilon(r)\left(\int_{S_r(0)} d\sigma_y\right)dr \\
&= u(x)\int_{B_\varepsilon(0)} \rho_\varepsilon(y)\,dy = u(x).
\end{aligned}$$

Thus $u_\varepsilon(x) = u(x)$ for all $x \in \Omega_\varepsilon$ and for all $\varepsilon > 0$. Since $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ (cf. Theorem **??**), we have $u \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$.

Theorem 4.4. *Let u be a harmonic function on Ω. Then u satisfies the MVP in Ω.*

Proof. Let $B_r(x) \subset \Omega$ be any ball with centre at $x \in \Omega$ and for some $r > 0$. For the given harmonic function u, we set

$$M(u,x;r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\,d\sigma_y.$$

Note that v is not defined at 0, since $r > 0$. We have from Exercise 4.3 that

$$M(u,x;r) = \frac{1}{\omega_n} \int_{S_1(0)} u(x+rz)\,d\sigma_z.$$

Now, differentiating both sides w.r.t r, we get

$$\begin{aligned}
\frac{dM(u,x;r)}{dr} &= \frac{1}{\omega_n} \int_{S_1(0)} \nabla u(x+rz)\cdot z\,d\sigma_z \\
&= \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla u(y)\cdot \frac{(y-x)}{r}\,d\sigma_y
\end{aligned}$$

Since $|x-y| = r$, by setting $\nu := (y-x)/r$ as the unit vector, and applying the Gauss divergence theorem along with the fact that u is harmonic, we get

$$\frac{dM(u,x;r)}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla u(y)\cdot \nu\,d\sigma_y = \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y)\,dy = 0.$$

Thus, v is a constant function of $r > 0$ and hence

$$M(u,x;r) = M(u,x;\varepsilon) \quad \forall \varepsilon > 0.$$

Moreover, since M is continuous (constant function), we have

$$
\begin{aligned}
M(u,x;r) &= \lim_{\varepsilon\to 0} M(u,x;\varepsilon) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\omega_n}\int_{S_1(0)} u(x+\varepsilon z)\,d\sigma_z \\
&= \frac{1}{\omega_n}\int_{S_1(0)} \lim_{\varepsilon\to 0} u(x+\varepsilon z)\,d\sigma_z \quad (u \text{ is continuous on } S_1(0)) \\
&= \frac{1}{\omega_n}\int_{S_1(0)} u(x)\,d\sigma_z \\
&= u(x) \quad (\text{Since } \omega_n \text{ is the surface area of } S_1(0)).
\end{aligned}
$$

Thus, u satisfies FMVP and, hence, SMVP.

Corollary 4.1. *If u is harmonic on Ω, then $u \in C^\infty(\Omega)$.*

The above corollary is a easy consequence of Theorem 4.4 and Theorem 4.3. We shall now prove that any function satisfying MVP is harmonic.

Theorem 4.5. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then u is harmonic in Ω.*

Proof. Since u satisfies MVP, by Theorem 4.3, $u \in C^\infty(\Omega)$. Thus, Δu makes sense. Now, suppose u is not harmonic in Ω, then there is a $x \in \Omega$ such that $\Delta u(x) \neq 0$. Without loss of generality, let's say $\Delta u(x) > 0$. Moreover, since Δu is continuous there is a $s > 0$ such that, for all $y \in B_s(x)$, $\Delta u(y) > 0$. As done previously, we set for $r > 0$ such that $B_r(x) \subset \Omega$,

$$M(u,x;r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\,d\sigma_y.$$

Since u satisifes MVP, we have $M(u,x;r) = u(x)$ for all $r > 0$. Thus, M is a constant function in r and $M'(u,x;r) = 0$. But, at $r = s$,

$$0 = \frac{dM(u,x;s)}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{B_s(x)} \Delta u(y)\,dy > 0$$

is a contradiction. Therefore, u is harmonic in Ω.

Above results leads us to conclude that a function is harmonic iff it satisfies the MVP.

Exercise 4.5. If u_m is a sequence of harmonic functions in Ω converging to u uniformly on compact subsets of Ω, then show that u is harmonic in Ω.

Theorem 4.6 (Strong Maximum Principle). *Let Ω be an open, connected (domain) subset of $\mathbb{R}^n$. Let u be harmonic in Ω and $S := \sup_{y\in\overline{\Omega}} u(y)$. Then*

$$u(x) < S \quad \forall x \in \Omega$$

or $u \equiv S$ is constant in Ω.

Proof. We define a subset X of Ω as follows,

$$X := \{x \in \Omega \mid u(x) = S\}.$$

If $X = \emptyset$, we have $u(x) < S$ for all $x \in \Omega$. Suppose $X \neq \emptyset$. Then X is closed subset of Ω, since u is continuous. Now, for any $x \in X$, by MVP

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\,dy \quad \text{ for every } r \text{ such that } B_r(x) \subset \Omega.$$

Thus, we have

$$S = u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\,dy \leq S$$

Hence equality will hold above only when $u(y) = S$ for all $y \in B_r(x)$. Thus, we have shown that for any $x \in X$, we have $B_r(x) \subset X$. Therefore, X is open. Since Ω is connected, the only open and closed subsets are $\emptyset$ or Ω. Since X was assumed to be non-empty, we should have $X = \Omega$. Thus, $u \equiv S$ is constant in Ω.

Corollary 4.2 (Weak maximum Principle). *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be harmonic in Ω. Then*

$$\max_{y \in \overline{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y).$$

Proof. Let $S := \max_{y \in \overline{\Omega}} u(y)$. If there is a $x \in \Omega$ such that $u(x) = S$, then $u \equiv S$ is constant on the connected component of Ω containing x. Thus, $u = S$ on the boundary of the connected component which is a part of $\partial\Omega$.

Proof (Aliter). Since $\partial\Omega \subset \overline{\Omega}$, we have $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$. It only remains to prove the other equality. For the given harmonic function u and for a fixed $\varepsilon > 0$, we set $v_\varepsilon(x) = u(x) + \varepsilon|x|^2$, for each $x \in \overline{\Omega}$. For each $x \in \Omega$, $\Delta v_\varepsilon = \Delta u + 2n\varepsilon > 0$. Recall that[3] if a function v attains local maximum at a point $x \in \Omega$, then in each direction its second order partial derivative $v_{x_i x_i}(x) \leq 0$, for all $i = 1, 2, \ldots, n$. Therefore $\Delta v(x) \leq 0$. Thus, we argue that v_ε does not attain (even a local) maximum in Ω. But v_ε has to have a maximum in $\overline{\Omega}$, hence it should be attained at some point $x^\star \in \partial\Omega$, on the boundary. For all $x \in \overline{\Omega}$,

$$u(x) \leq v_\varepsilon(x) \leq v_\varepsilon(x^\star) = u(x^\star) + \varepsilon|x^\star|^2 \leq \max_{x \in \partial\Omega} u(x) + \varepsilon \max_{x \in \partial\Omega} |x|^2.$$

The above inequality is true for all $\varepsilon > 0$. Thus, $u(x) \leq \max_{x \in \partial\Omega} u(x)$, for all $x \in \overline{\Omega}$. Therefore, $\max_{\overline{\Omega}} u \leq \max_{x \in \partial\Omega} u(x)$. and hence we have equality.

Theorem 4.7 (Estimates on derivatives). *If u is harmonic in Ω, then*

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{1,B_r(x)} \quad \forall B_r(x) \subset \Omega \text{ and each } |\alpha| = k$$

---

[3] $v \in C^2(a,b)$ has a local maximum at $x \in (a,b)$ then $v'(x) = 0$ and $v''(x) \leq 0$

where the constants $C_0 = \frac{n}{\omega_n}$ *and* $C_k = C_0(2^{n+1}nk)^k$ *for* $k = 1, 2, \ldots$.

Proof. We prove the result by induction on k. Let $k = 0$. Since u is harmonic, by SMVP we have, for any $B_r(x) \subset \Omega$,

$$
\begin{aligned}
|u(x)| &= \frac{n}{\omega_n r^n} \left| \int_{B_r(x)} u(y)\, dy \right| \\
&\leq \frac{n}{\omega_n r^n} \int_{B_r(x)} |u(y)|\, dy \\
&= \frac{n}{\omega_n r^n} \|u\|_{1,B_r(x)} = \frac{C_0}{r^n} \|u\|_{1,B_r(x)}.
\end{aligned}
$$

Now, let $k = 1$. Observe that if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $u_{x_i} := \frac{\partial u}{\partial x_i}$ is harmoic, for all $i = 1, 2, \ldots, n$. Now, by the SMVP of u_{x_i}, we have

$$
\begin{aligned}
|u_{x_i}(x)| &= \frac{n2^n}{\omega_n r^n} \left| \int_{B_{r/2}(x)} u_{x_i}(y)\, dy \right| \\
&= \frac{n2^n}{\omega_n r^n} \left| \int_{S_{r/2}(x)} u \nu_i \, d\sigma_y \right| \quad \text{(by Gauss-Green theorem)} \\
&\leq \frac{2n}{r} \|u\|_{\infty, S_{r/2}(x)}.
\end{aligned}
$$

Thus, it now remains to estimate $\|u\|_{\infty,S_{r/2}(x)}$. Let $z \in S_{r/2}(x)$, then

$$B_{r/2}(z) \subset B_r(x) \subset \Omega.$$

But, using $k = 0$ result, we have

$$|u(z)| \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_{r/2}(z)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}.$$

Therefore, $\|u\|_{\infty,S_{r/2}(x)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}$ and using this in the estimate of u_{x_i}, we get

$$|u_{x_i}(x)| \leq \frac{C_0 n 2^{n+1}}{r^{n+1}} \|u\|_{1,B_r(x)}.$$

Hence

$$|D^\alpha u(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = 1.$$

Let now $k \geq 2$ and α be a multi-index such that $|\alpha| = k$. We assume the induction hypothesis that the estimate to be proved is true for $k - 1$. Note that $D^\alpha u = \frac{\partial D^\beta u}{\partial x_i}$ for some $i \in \{1, 2, \ldots, n\}$ and $|\beta| = k - 1$. Moreover, if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $\frac{\partial D^\beta u}{\partial x_i}$ is harmoic for $i = 1, 2, \ldots, n$. Thus, following an earlier argument, we have

$$|D^\alpha u(x)| = \left|\frac{\partial D^\beta u(x)}{\partial x_i}\right| = \frac{nk^n}{\omega_n r^n}\left|\int_{B_{r/k}(x)} \frac{\partial D^\beta u(y)}{\partial x_i}\,dy\right|$$
$$= \frac{nk^n}{\omega_n r^n}\left|\int_{S_{r/k}(x)} D^\beta u \nu_i \,d\sigma_y\right|$$
$$\le \frac{nk}{r}\|D^\beta u\|_{\infty,S_{r/k}(x)}.$$

It now only remains to estimate $\|D^\beta u\|_{\infty,S_{r/k}(x)}$. Let $z \in S_{r/k}(x)$, then $B_{(k-1)r/k}(z) \subset B_r(x) \subset \Omega$. But, using induction hypothesis for $k-1$, we have

$$|D^\beta u(z)| \le \frac{C_{k-1}k^{n+k-1}}{((k-1)r)^{n+k-1}}\|u\|_{1,B_{(k-1)r/k}(z)} \le \frac{C_{k-1}k^{n+k-1}}{((k-1)r)^{n+k-1}}\|u\|_{1,B_r(x)}.$$

Therefore, using the above estimate for $D^\alpha u$, we get

$$|D^\alpha u(x)| \le \frac{C_{k-1}nk^{n+k}}{(k-1)^{n+k-1}r^{n+k}}\|u\|_{1,B_r(x)}$$
$$= \frac{C_0 2^{(n+1)(k-1)}n^k(k-1)^{k-1}k^{n+k}}{(k-1)^{n+k-1}r^{n+k}}\|u\|_{1,B_r(x)}$$
$$= \frac{C_0(2^{n+1}nk)^k}{r^{n+k}}\left(\frac{k}{k-1}\right)^n\left(\frac{1}{2^{n+1}}\right)\|u\|_{1,B_r(x)}$$
$$= \frac{C_0(2^{n+1}nk)^k}{r^{n+k}}\left(\frac{k}{2(k-1)}\right)^n\left(\frac{1}{2}\right)\|u\|_{1,B_r(x)}$$
$$\le \frac{C_k}{r^{n+k}}\|u\|_{1,B_r(x)} \quad \text{since } \left(\frac{k}{2(k-1)}\right)^n\left(\frac{1}{2}\right) \le 1.$$

Hence
$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}}\|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = k, \forall k \ge 2.$$

Theorem 4.8 (Liouville's Theorem). *If u is bounded and harmonic on $\mathbb{R}^n$, then u is constant.*

Proof. For any $x \in \mathbb{R}^n$ and $r > 0$, we have the estimate on the first derivative as,

$$|\nabla u(x)| \le \frac{C_1}{r^{n+1}}\|u\|_{1,B_r(x)}$$
$$= \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{1,B_r(x)}$$
$$\le \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{\infty,\mathbb{R}^n}\omega_n r^n$$
$$= \frac{2^{n+1}}{r}\|u\|_{\infty,\mathbb{R}^n} \to 0 \text{ as } r \to \infty.$$

Thus, $\nabla u \equiv 0$ in $\mathbb{R}^n$ and hence u is constant.

Exercise 4.6. Show that if u is harmonic in Ω, then u is analytic in Ω. (Hint: Use the estimates on derivatives with Stirling's formula and Taylor expansion).

We end our discussion on the properties of harmonic function with Harnack inequality. The Harnack inequality states that non-negative harmonic functions cannot be very large or very small at any point without being so everywhere in a compact set containing that point.

Theorem 4.9 (Harnack's Inequality). *For each connected open subset $\omega \subset\subset \Omega$, there exists a constant $C > 0$ (depending only on ω and independent of u) such that*

$$\sup_{x\in\omega} u(x) \leq C \inf_{x\in\omega} u(x)$$

for all non-negative harmonic functions in Ω. In particular, for all non-negative harmonic functions and $x, y \in \omega$,

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y).$$

Proof. Set $r := \frac{1}{4}\mathrm{dist}(\omega, \partial\Omega)$. Let $x, y \in \omega$ such that $|x-y| < r$. By SMVP,

$$\begin{aligned} u(x) &= \frac{n}{\omega_n 2^n r^n}\int_{B_{2r}(x)} u(z)\,dz \\ &\geq \frac{n}{\omega_n 2^n r^n}\int_{B_r(y)} u(z)\,dz = \frac{1}{2^n}u(y). \end{aligned}$$

Thus, $1/2^n u(y) \leq u(x)$. Interchanging the role of x and y, we get $1/2^n u(x) \leq u(y)$. Thus, $1/2^n u(y) \leq u(x) \leq 2^n u(y)$ for all $x, y \in \omega$ such that $|x-y| \leq r$.

Now, let $x, y \in \omega$. Since $\overline{\omega}$ is compact and connected in Ω, we can pick points $x = x_0, x_1, \ldots, x_m = y$ such that $\cup_{i=0}^m B_i \supset \overline{\omega}$, where $B_i := B_{r/2}(x_i)$ and are sorted such that $B_i \cap B_{i+1} \neq \emptyset$, for $i = 0, 2, \ldots, m-1$. Hence, note that $|x_{i+1} - x_i| \leq r$. Therefore,

$$u(x) = u(x_0) \geq \frac{1}{2^n}u(x_1) \geq \frac{1}{2^{2n}}u(x_2) \geq \ldots \geq \frac{1}{2^{mn}}u(x_m) = \frac{1}{2^{mn}}u(y).$$

Interchanging the role of x and y, we get $1/2^{mn}u(x) \leq u(y)$. Thus, C can be chosen to be 2^{mn} and the choice of m depends on ω.

The non-negative hypothesis is crucial because for a general harmonic function with $\inf u < 0$ and $\sup u > 0$, the harnack's inequality is trivially false!

4.6.3 Existence and Uniqueness of Solution

A consequence of the maximum principle is the uniqueness of the harmonic functions.

Theorem 4.10 (Uniqueness of Harmonic Functions). *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ in Ω.*

Proof. Note that $u_1 - u_2$ is a harmonic function and hence, by weak maximum principle, should attain its maximum on $\partial\Omega$. But $u_1 - u_2 = 0$ on $\partial\Omega$. Thus $u_1 - u_2 \leq 0$ in Ω. Now, repeat the argument for $u_2 - u_1$, we get $u_2 - u_1 \leq 0$ in Ω. Thus, we get $u_1 - u_2 = 0$ in Ω.

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0 & x \in \Omega \\ u(x) \quad = g(x) & x \in \partial\Omega. \end{cases} \tag{4.1}$$

By the strong maximum principle (cf. Theorem 4.6), if Ω is connected and $g \geq 0$ and $g(x) > 0$ for some $x \in \partial\Omega$ then $u(x) > 0$ for all $x \in \Omega$.

Theorem 4.11. *Let Ω be an open bounded connected subset of $\mathbb{R}^n$ and $g \in C(\partial\Omega)$. Then the Dirichlet problem (4.1) has atmost one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, if u_1 and u_2 are solution to the Dirichlet problem corresponding to g_1 and g_2 in $C(\partial\Omega)$, respectively, then*

(a) (Comparison) $g_1 \geq g_2$ on $\partial\Omega$ and $g_1(x_0) > g_2(x_0)$ for some $x \in \partial\Omega$ implies that $u_1 > u_2$ in Ω.
(b) (Stability) $|u_1(x) - u_2(x)| \leq \max_{y \in \partial\Omega} |g_1(y) - g_2(y)|$ for all $x \in \Omega$.

Proof. The fact that there is atmost one solution to the Dirichlet problem follows from the Theorem 4.10. Let $w = u_1 - u_2$. Then w is harmonic.

(a) Note that $w = g_1 - g_2 \geq 0$ on $\partial\Omega$. Since $g_1(x_0) > g_2(x_0)$ for some $x_0 \in \partial\Omega$, then $w(x) > 0$ for all $x \in \partial\Omega$. This proves the comparison result.
(b) Again, by maximum principle, we have

$$\pm w(x) \leq \max_{y \in \partial\Omega} |g_1(y) - g_2(y)| \forall x \in \Omega.$$

This proves the stability result.

We remark that the uniqueness result is not true for unbounded domains.

Example 4.3. Consider the problem (4.1) with $g \equiv 0$ in the domain $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$. Obviously, $u = 0$ is a solution. But we also have a non-trivial solution

$$u(x) = \begin{cases} \ln|x| & n = 2 \\ |x|^{2-n} - 1 & n \geq 3. \end{cases}$$

Example 4.4. Consider the problem (4.1) with $g \equiv 0$ in the domain $\Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Obviously, $u = 0$ is a solution. But we also have a non-trivial solution $u(x) = x_n$.

We have shown above that if a solution exists for (4.1) then it is unique (cf. Theorem 4.10). So the question that remains to be answered is on the existence of solution of (4.1), for any given domain Ω. In the modern theory, there are three different methods to address this question of existence, viz., Perron's Method, Layer Potential (Integral Equations) and L^2 methods.

Definition 4.3. We say a function $w \in C(\overline{\Omega})$ is a *barrier* at $x_0 \in \partial\Omega$ if there is a neighbourhood U of x_0 such that

1. w is superharmonic in $\Omega \cap U$
2. $w > 0$ in $(\overline{\Omega} \cap U) \setminus \{x_0\}$ and $w(x_0) = 0$.

Definition 4.4. Any point on $\partial\Omega$ is said to be *regular* (w.r.t Laplacian) if there exists a barrier at that point.

A necessary and sufficient condition for the existence of solution to (4.1) is given by the following result:

Theorem 4.12 (Perron's method). *The Dirichlet problem* (4.1) *is solvable for any arbitrary bounded domain Ω and for any arbitrary g on $\partial\Omega$ iff all the points in $\partial\Omega$ are regular.*

Proof. One way is obvious. If (4.1) is solvable and $x_0 \in \partial\Omega$. Then, the solution to

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega, \end{cases}$$

where $g(x) = |x - x_0|$, is a barrier function at x_0. Thus, any $x_0 \in \partial\Omega$ is a regular point. The converse is proved using the Perron's method for subharmonic functions.

Definition 4.5. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *exterior sphere condition* if for every point $x_0 \in \partial\Omega$ there is a ball $B := B_R(y)$ such that $\overline{B} \cap \overline{\Omega} = x_0$.

Lemma 4.1. *If Ω satisfies the exterior sphere condition then all boundary points of Ω are regular.*

Proof. For any $x_0 \in \partial\Omega$, we define the barrier function at $x_0 \in \partial\Omega$ as

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & \text{for } n \geq 3 \\ \ln\left(\frac{|x-y|}{R}\right) & \text{for } n = 2. \end{cases}$$

Theorem 4.13. *Any bounded domain with C^2 boundary satisfies the exterior sphere condition.*

Definition 4.6. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *exterior cone condition* if for every point $x_0 \in \partial\Omega$ there is a finite right circular cone K with vertex at x_0 such that $\overline{K} \cap \overline{\Omega} = x_0$.

Exercise 4.7. Any domain satisfying the exterior sphere condition also satisfies the exterior cone condition.

Exercise 4.8. Every bounded Lipschitz domain satisfies the exterior cone condition.

Lemma 4.2. *If Ω satisfies the exterior cone condition then all boundary points of Ω are regular.*

Example 4.5 (Non-existence of Solutions). In 1912, Lebesgue gave an example of a domain on which the classical Dirichlet problem is not solvable. The domain is

$$\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid r^2 + z^2 < 1; r > e^{-1/2z} \text{ for } z > 0\}.$$

Note that Ω is the unit ball in $\mathbb{R}^3$ with a sharp inward cusp, called *Lebesgue spine*, at the origin $(0,0,0)$. The origin is a not regular point of Ω.

Example 4.6. There are domains with inward cusps for which the classical problem is solvable. For instance, consider

$$\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid r^2 + z^2 < 1; r > z^{2k} \text{ for } z > 0\},$$

for any positive integer k. The proof of this fact involves the theory of capacities.

Remark 4.1 (Characterizing regular points). The *Wiener's criterion* gives the necessary and sufficient condition for the regularity of the boundary points. For $n \geq 3$ and a fixed $\lambda \in (0,1)$, the *Wiener's criterion* states that a point $x_0 \in \partial\Omega$ is regular iff the series

$$\sum_{i=0}^{\infty} \frac{C_i}{\lambda^{i(n-2)}}$$

diverges, where $C_i := \operatorname{cap}_2\{x \notin \Omega \mid |x - x_0| \leq \lambda^i\}$.

Remark 4.2 (Neumann Boundary Condition). The Neumann problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\nu = (\nu_1, \ldots, \nu_n)$ is the outward pointing unit normal vector field of $\partial\Omega$. Thus, the boundary imposed is called the Neumann boundary condition. The solution of a Neumann problem is not necessarily unique. If u is any solution of (4.2), then $u + c$ for any constant c is also a solution of (4.2). More generally, for any v such that v is constant on the connected components of Ω, $u + v$ is a solution of (4.2).

4.6.4 Domains with Simple Geometry

The discussion in the previous section tells us that one can expect solution to the Dirichlet on domains whose boundary is regular. In this section, we give examples of domains for which the solution can be computed.

The *method of separation of variables* was introduced by d'Alembert (1747) and Euler (1748) for the wave equation. This technique was also employed by Laplace (1782) and Legendre (1782) while studying the Laplace equation and also by Fourier while studying the heat equation. The motivation behind the "separation of variable" technique will be highlighted while studying wave equation.

Theorem 4.14 (2D Rectangle). *Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < a$ and $0 < y < b\}$ be a rectangle in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ which vanishes on three sides of the rectangle, i.e., $g(0,y) = g(x,0) = g(a,y) = 0$ and $g(x,b) = h(x)$ where h is a continuous function $h(0) = h(a) = 0$. Then there is a unique solution to (4.1) on this rectangle with given boundary value g.*

Proof. We begin by looking for solution $u(x,y)$ whose variables are separated, i.e., $u(x,y) = v(x)w(y)$. Substituting this form of u in the Laplace equation, we get

$$v''(x)w(y) + v(x)w''(y) = 0.$$

Hence

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}.$$

Since LHS is function of x and RHS is function y, they must equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda.$$

Using the boundary condition on u, $u(0,y) = g(0,y) = g(a,y) = u(a,y) = 0$, we get $v(0)w(y) = v(a)w(y) = 0$. If $w \equiv 0$, then $u \equiv 0$ which is not a solution to (4.1). Hence, $w \not\equiv 0$ and $v(0) = v(a) = 0$. Thus, we need to solve,

$$\begin{cases} v''(x) = \lambda v(x), \; x \in (0,a) \\ v(0) \;= v(a) \quad = 0, \end{cases}$$

the *eigen value problem* for the second order differential operator. Note that the λ can be either zero, positive or negative.

If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(a) = 0$ and $a \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this can not be a solution to (4.1).

If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda} x).$$

Now using $v(a) = 0$, we have $c_2 \sinh \sqrt{\lambda} a = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

If $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve

$$\begin{cases} v''(x) + \omega^2 v(x) = 0 & x \in (0,a) \\ v(0) & = v(a) = 0. \end{cases} \tag{4.3}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using the boundary condition $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(a) = 0$, we have $\beta \sin \omega a = 0$. Thus, either $\beta = 0$ or $\sin \omega a = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega a = k\pi$ or $\omega = k\pi/a$, for all non-zero $k \in \mathbb{Z}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for (4.3), with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{a}\right),$$

for some constant β_k and $\lambda_k = -(k\pi/a)^2$. We now solve w corresponding to each λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$\begin{cases} w_k''(y) & = \left(\frac{k\pi}{a}\right)^2 w_k(y),\ y \in (0,b) \\ w(0) & = 0. \end{cases}$$

Thus, $w_k(y) = c_k \sinh(k\pi y/a)$. Therefore, for each $k \in \mathbb{N}$,

$$u_k = \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right)$$

is a solution to (4.1). The general solution is of the form (principle of superposition) (convergence?)

$$u(x,y) = \sum_{k=1}^{\infty} \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right).$$

The constant δ_k are obtained by using the boundary condition $u(x,b) = h(x)$ which yields

$$h(x) = u(x,b) = \sum_{k=1}^{\infty} \delta_k \sinh\left(\frac{k\pi b}{a}\right) \sin\left(\frac{k\pi x}{a}\right).$$

Since $h(0) = h(a) = 0$, the function h admits a Fourier Sine series. Thus $\delta_k \sinh\left(\frac{k\pi b}{a}\right)$ is the k-th Fourier sine coefficient of h, i.e.,

$$\delta_k = \left(\sinh\left(\frac{k\pi b}{a}\right)\right)^{-1} \frac{2}{a}\int_0^a h(x)\sin\left(\frac{k\pi x}{a}\right).$$

Theorem 4.15 (2D Disk). *Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < R^2\}$ be the disk of radius R in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ is a continuous function. Then there is a unique solution to (4.1) on the unit disk with given boundary value g.*

Proof. Given the nature of the domain, we shall use the Laplace operator in polar coordinates,

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

where r is the magnitude component and θ is the direction component. Then $\partial\Omega$ is the circle of radius one. Then, solving for $u(x,y)$ in the Dirichlet problem is to equivalent to finding $U(r,\theta) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} = 0 & \text{in } \Omega \\ U(r,\theta+2\pi) = U(r,\theta) & \text{in } \Omega \\ U(R,\theta) = G(\theta) & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

where $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, $G : [0,2\pi) \to \mathbb{R}$ is $G(\theta) = g(\cos\theta,\sin\theta)$. Note that both U and G are 2π periodic w.r.t θ. We will look for solution $U(r,\theta)$ whose variables can be separated, i.e., $U(r,\theta) = v(r)w(\theta)$ with both v and w non-zero. Substituting it in the polar form of Laplacian, we get

$$\frac{w}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \frac{v}{r^2}\frac{d^2w}{d\theta^2} = 0$$

and hence

$$\frac{-r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{1}{w}\left(\frac{d^2w}{d\theta^2}\right).$$

Since LHS is a function of r and RHS is a function of θ, they must equal a constant, say λ. We need to solve the eigen value problem,

$$\begin{cases} w''(\theta) - \lambda w(\theta) = 0 & \theta \in \mathbb{R} \\ w(\theta+2\pi) = w(\theta) & \forall\theta. \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $w'' = 0$ and the general solution is $w(\theta) = \alpha\theta + \beta$, for some constants α and β. Using the periodicity of w,

$$\alpha\theta + \beta = w(\theta) = w(\theta+2\pi) = \alpha\theta + 2\alpha\pi + \beta$$

implies that $\alpha = 0$. Thus, the pair $\lambda = 0$ and $w(\theta) = \beta$ is a solution. If $\lambda > 0$, then

$$w(\theta) = \alpha e^{\sqrt{\lambda}\theta} + \beta e^{-\sqrt{\lambda}\theta}.$$

If either of α and β is non-zero, then $w(\theta) \to \pm\infty$ as $\theta \to \infty$, which contradicts the periodicity of w. Thus, $\alpha = \beta = 0$ and $w \equiv 0$, which cannot be a solution. If $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$ and the equation becomes

$$\begin{cases} w''(\theta) + \omega^2 w(\theta) = 0 & \theta \in \mathbb{R} \\ w(\theta + 2\pi) = w(\theta) & \forall\theta \end{cases}$$

Its general solution is

$$w(\theta) = \alpha\cos(\omega\theta) + \beta\sin(\omega\theta).$$

Using the periodicity of w, we get $\omega = k$ where k is an integer. For each $k \in \mathbb{N}$, we have the solution (w_k, λ_k) where

$$\lambda_k = -k^2 \quad \text{and } w_k(\theta) = \alpha_k\cos(k\theta) + \beta_k\sin(k\theta).$$

For the λ_k's, we solve for v_k, for each $k = 0, 1, 2, \ldots$,

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

For $k = 0$, we get $v_0(r) = \alpha\ln r + \beta$. But $\ln r$ blows up as $r \to 0$, but any solution U and, hence v, on the closed unit disk (compact subset) has to be bounded. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

Use the change of variable $r = e^s$. Then $e^s\frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. $v_k(e^s) = \alpha e^{ks} + \beta e^{-ks}$. $v_k(r) = \alpha r^k + \beta r^{-k}$. Since r^{-k} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r, \theta) = a_k r^k\cos(k\theta) + b_k r^k\sin(k\theta).$$

The general solution is

$$U(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k r^k\cos(k\theta) + b_k r^k\sin(k\theta)\right).$$

To find the constants, we must use $U(R, \theta) = G(\theta)$. If $G \in C^1[0, 2\pi]$, then G admits Fourier series expansion. Therefore,

$$G(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[R^k a_k\cos(k\theta) + R^k b_k\sin(k\theta)\right]$$

where

$$a_k = \frac{1}{R^k\pi}\int_{-\pi}^{\pi} G(\theta)\cos(k\theta)\,d\theta,$$

$$b_k = \frac{1}{R^k \pi} \int_{-\pi}^{\pi} G(\theta) \sin(k\theta)\, d\theta.$$

Using this in the formula for U and the uniform convergence of Fourier series, we get

$$\begin{aligned} U(r,\theta) &= \frac{1}{\pi} \int_{-\pi}^{\pi} G(\eta) \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k (\cos k\eta \cos k\theta + \sin k\eta \sin k\theta) \right] d\eta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} G(\eta) \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k \cos k(\eta - \theta) \right] d\eta. \end{aligned}$$

Using the relation

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k \cos k(\eta - \theta) = \mathrm{Re}\left[\sum_{k=1}^{\infty} \left(\frac{r}{R} e^{i(\eta-\theta)}\right)^k \right] &= \mathrm{Re}\left[\frac{\frac{r}{R} e^{i(\eta-\theta)}}{1 - \frac{r}{R} e^{i(\eta-\theta)}} \right] \\ &= \frac{R^2 - rR\cos(\eta - \theta)}{R^2 + r^2 - 2rR\cos(\eta - \theta)} - 1 \\ &= \frac{rR\cos(\eta - \theta) - r^2}{R^2 + r^2 - 2rR\cos(\eta - \theta)} \end{aligned}$$

in $U(r,\theta)$ we get

$$U(r,\theta) = \frac{R^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{G(\eta)}{R^2 + r^2 - 2rR\cos(\eta - \theta)} d\eta.$$

Note that the formula derived above for $U(r,\theta)$ can be rewritten in Cartesian coordinates and will have the form

$$u(x) = \frac{R^2 - |x|^2}{2\pi R} \int_{S_R(0)} \frac{g(y)}{|x-y|^2} dy.$$

This can be easily seen, by setting $y = R(x_0^1 \cos\eta + x_0^2 \sin\eta)$, we get $dy = Rd\eta$ and $|x-y|^2 = R^2 + r^2 - 2rR\cos(\eta - \theta)$. This is called the *Poisson* formula. More generally, the unique solution to the Dirichlet problem on a ball of radius R centred at x_0 in $\mathbb{R}^n$ is given by *Poisson formula*

$$u(x) = \frac{R^2 - |x - x_0|^2}{\omega_n R} \int_{S_R(x_0)} \frac{g(y)}{|x-y|^n} dy.$$

We will derive this general form later (cf. (4.12)).

Theorem 4.16 (3D Sphere). *Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $g : \partial\Omega \to \mathbb{R}$ is a continuous function. Then there is a unique solution to (4.1) on the unit sphere with given boundary value g.*

Proof. Given the nature of domain, the Laplace operator in spherical coordinates,

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.$$

where r is the magnitude component, ϕ is the inclination (zenith or elevation) in the vertical plane and θ is the azimuth angle (in the direction in horizontal plane). Solving for u in (4.1) is equivalent to finding $U(r,\phi,\theta):\Omega\to\mathbb{R}$ such that

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial U}{\partial\phi}\right) & \\ \qquad\qquad + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 U}{\partial\theta^2} = 0 & \text{in } \Omega \\ \qquad\qquad U(1,\phi,\theta) = G(\phi,\theta) & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

where $U(r,\phi,\theta)$ and $G(\phi,\theta)$ are appropriate spherical coordinate function corresponding to u and g. We will look for solution $U(r,\phi,\theta)$ whose variables can be separated, i.e., $U(r,\phi,\theta) = v(r)w(\phi)z(\theta)$ with v,w and z non-zero. Substituting it in the spherical form of Laplacian, we get

$$\frac{wz}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) + \frac{vz}{r^2\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right) + \frac{vw}{r^2\sin^2\phi}\frac{d^2z}{d\theta^2} = 0$$

and hence

$$\frac{1}{v}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) = \frac{-1}{w\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right) - \frac{1}{z\sin^2\phi}\frac{d^2z}{d\theta^2}.$$

Since LHS is a function of r and RHS is a function of (ϕ,θ), they must equal a constant, say λ. If Azimuthal symmetry is present then $z(\theta)$ is constant and hence $\frac{dz}{d\theta} = 0$. We need to solve for w,

$$\sin\phi w''(\phi) + \cos\phi w'(\phi) + \lambda\sin\phi w(\phi) = 0, \quad \phi\in(0,\pi)$$

Set $x = \cos\phi$. Then $\frac{dx}{d\phi} = -\sin\phi$.

$$w'(\phi) = -\sin\phi\frac{dw}{dx} \text{ and } w''(\phi) = \sin^2\phi\frac{d^2w}{dx^2} - \cos\phi\frac{dw}{dx}$$

In the new variable x, we get the Legendre equation

$$(1-x^2)w''(x) - 2xw'(x) + \lambda w(x) = 0 \quad x\in[-1,1].$$

We have already seen that this is a singular problem (while studying S-L problems). For each $k\in\mathbb{N}\cup\{0\}$, we have the solution (w_k,λ_k) where

$$\lambda_k = k(k+1) \quad \text{and } w_k(\phi) = P_k(\cos\phi).$$

For the λ_k's, we solve for v_k, for each $k = 0,1,2,\ldots$,

$$\frac{d}{dr}\left(r^2\frac{dv_k}{dr}\right) = k(k+1)v_k.$$

For $k = 0$, we get $v_0(r) = -\alpha/r + \beta$. But $1/r$ blows up as $r \to 0$ and U must be bounded in the closed sphere. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$\frac{d}{dr}\left(r^2\frac{dv_k}{dr}\right) = k(k+1)v_k.$$

Use the change of variable $r = e^s$. Then $e^s\frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. Solving for m in the quadratic equation $m^2 + m = k(k+1)$. $m_1 = k$ and $m_2 = -k-1$. $v_k(e^s) = \alpha e^{ks} + \beta e^{(-k-1)s}$. $v_k(r) = \alpha r^k + \beta r^{-k-1}$. Since r^{-k-1} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r,\phi,\theta) = a_k r^k P_k(\cos\phi).$$

The general solution is

$$U(r,\phi,\theta) = \sum_{k=0}^{\infty} a_k r^k P_k(\cos\phi).$$

Since we have azimuthal symmetry, $G(\phi,\theta) = G(\phi)$. To find the constants, we use $U(1,\phi,\theta) = G(\phi)$, hence

$$G(\phi) = \sum_{k=0}^{\infty} a_k P_k(\cos\phi).$$

Using the orthogonality of P_k, we have

$$a_k = \frac{2k+1}{2}\int_0^{\pi} G(\phi)P_k(\cos\phi)\sin\phi\, d\phi.$$

4.7 Poisson Equation

In this section, we solve the inhomogeneous Laplace equation (called Poisson equation) with homogeneous Dirichlet boundary conditions. The Poisson equation is the problem of finding u satisfying

$$-\Delta u = f \text{ in } \mathbb{R}^n, \tag{4.6}$$

for any given $f \in C(\mathbb{R}^n)$. Recall the notion of convolution of functions detailed in Appendix **??**. We have already used this notion while deriving the C^∞ properties of harmonic functions. The property of convolution is that the differential operator can be accumulated on either side of the convolution operation. Let's assume the existence of a "function" δ which is the identity of the convolution (binary) operation, i.e., $f * \delta = f$. Suppose there is a function K such that $\Delta K = \delta$, then $u := f * K$ is a solution of (4.6) because $\Delta u = f * \Delta K = f$.

Definition 4.7. A function K is said to be the *fundamental solution* of Δ if ΔK is the identity with respect to the convolution operation.

We caution that the above definition is not mathematically precise because the notion of δ is not precise. If a fundamental solution K exists, then $f * \Delta K = f$, for all $f \in C(\mathbb{R}^n)$. In particular, one can choose $f \equiv 1$. Thus, the necessary condition for a fundamental solution, K, is $1 * \Delta K = 1$, i.e.,

$$\int_{\mathbb{R}^n} \Delta K(x)\, dx = 1.$$

Equivalently, the necessary condition for K is

$$\lim_{r\to\infty} \int_{B_r(0)} \Delta K(x)\, dx = 1,$$

which by Gauss divergence theorem implies that

$$\lim_{r\to\infty} \int_{S_r(0)} \nabla K(y) \cdot \nu(y)\, d\sigma_y = 1.$$

4.7.1 Fundamental Solution of Laplacian

The invariance of Laplacian under rotation motivates us to look for a radial fundamental solution. Recall how Laplacian treats radial functions (cf. Proposition 4.1) and, consequently, we have

Corollary 4.3. *The function $u(x) = ax + b$ solves $\Delta u = 0$ in $\mathbb{R}$. For $n \geq 2$, if u is a radial function on $\mathbb{R}^n$ then $\Delta u = 0$ on $\mathbb{R}^n \setminus \{0\}$ iff*

$$u(x) = \begin{cases} a + b \ln |x| & \text{if } n = 2, \\ a + \frac{b}{2-n} |x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

where a, b are some constants.

Proof. For radial functions $u(x) = v(r)$ where $r = |x|$. Observe that $\Delta u(x) = 0$ iff $v''(r) + \frac{(n-1)}{r} v'(r) = 0$. Now, integrating both sides w.r.t r, we get

$$\begin{aligned} \frac{v''(r)}{v'(r)} &= \frac{(1-n)}{r} \\ \ln v'(r) &= (1-n) \ln r + \ln b \\ v'(r) &= b r^{(1-n)} \end{aligned}$$

Integration both sides, once again, yields

$$v(r) = \begin{cases} b\ln r + a & \text{if } n = 2 \\ \frac{b}{2-n} r^{2-n} + a & \text{if } n \neq 2. \end{cases}$$

The reason to choose the domain of the Laplacian as $\mathbb{R}^n \setminus \{0\}$ is because the operator involves a 'r' in the denominator. However, for one dimensional case we can let zero to be on the domain of Laplacian, since for $n = 1$, the Laplace operator is unchanged. Thus, for $n = 1$, $u(x) = a + bx$ is a harmonic function in $\mathbb{R}^n$.

Note that as $r \to 0$, $v(r) \to \infty$. Thus, u has a singularity at 0. In fact, for any given vector $x_0 \in \mathbb{R}^n$, $\Delta u(x - x_0) = 0$ for all $x \in \mathbb{R}^n \setminus \{x_0\}$. We shall choose a, b such that for every sphere $S_r(0)$ about the origin, we have

$$\int_{S_r(0)} v'(r)\, d\sigma = 1.$$

Thus,

$$1 = \int_{S_r(0)} v'(r)\, d\sigma = \begin{cases} \frac{b}{r}(2\pi r) & \text{for } n = 2 \\ br^{1-n}(r^{n-1}\omega_n) & \text{for } n \geq 3. \end{cases}$$

This is possible only for the choice

$$b = \begin{cases} \frac{1}{2\pi} & \text{for } n = 2 \\ \frac{1}{\omega_n} & \text{for } n \geq 3. \end{cases}$$

The constant a can be chosen arbitrarly, but to keep things simple, we choose $a \equiv 0$ for $n \geq 2$. For convention sake, we shall add minus ("$-$") sign (notice the minus sign in (4.6)).

Definition 4.8. For any fixed $x_0 \in \mathbb{R}^n$ We say $K(x_0, x)$, defined as

$$K(x_0, x) := \begin{cases} -\frac{1}{2\pi} \ln|x - x_0| & (n = 2) \\ \frac{|x-x_0|^{2-n}}{\omega_n(n-2)} & (n \geq 3), \end{cases}$$

is the *fundamental solution* of Δ at any given $x_0 \in \mathbb{R}^n$.

We end this section by emphasising that the notion of fundamental solution has a precise definition in terms of the Dirac measure. The Dirac measure, at a point $x \in \mathbb{R}^n$, is defined as,

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for all measurable subsets E of the measure space $\mathbb{R}^n$. The Dirac measure has the property that

$$\int_E d\delta_x = 1$$

if $x \in E$ and zero if $x \notin E$. Also, for any integrable function f,

$$\int_{\mathbb{R}^n} f(y)\, d\delta_x = f(x).$$

In this new set-up a fundamental solution $K(x_0,\cdot)$ can be defined as the solution corresponding to δ_{x_0}, i.e.,

$$-\Delta K(x_0,x) = \delta_{x_0} \text{ in } \mathbb{R}^n.$$

Note that the above equation, as such, makes no sense because the RHS is a set-function taking subsets of $\mathbb{R}^n$ as arguments, whereas K is a function on $\mathbb{R}^n$. To give meaning to above equation, one needs to view δ_x as a distribution (introduced by L. Schwartz) and the equation should be interpreted in the distributional derivative sense. The Dirac measure is the distributional limit of the sequence of mollifiers, ρ_ε, in the space of distributions.

4.7.2 Existence and Uniqueness of Solution

In this section, we shall give a formula for the solution of the Poisson equation (4.6) in $\mathbb{R}^n$ in terms of the fundamental solution.

Theorem 4.17. *For any given $f \in C_c^2(\mathbb{R}^n)$, $u := K * f$ is a solution to the Poisson equation (4.6).*

Proof. By the property of convolution (cf. proof of Theorem **??**), we know that $D^\alpha u(x) = (K * D^\alpha f)(x)$ for all $|\alpha| \leq 2$. Since $f \in C_c^2(\mathbb{R}^n)$, we have $u \in C^2(\mathbb{R}^n)$. The difficulty arises due to the singularity of K at the origin. Consider, for any fixed $m > 0$,

$$
\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_y f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot \nu\,d\sigma_y \\
&\quad - \int_{\mathbb{R}^n\setminus B_m(0)} \nabla_y K(y)\cdot\nabla_y f(x-y)\,dy \quad \text{(By Corollary B.4)} \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot \nu\,d\sigma_y \\
&\quad + \int_{\mathbb{R}^n\setminus B_m(0)} \Delta_y K(y) f(x-y)\,dy \\
&\quad - \int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot \nu\,d\sigma_y \quad \text{(By Corollary B.4).}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta u(x) &= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot \nu\,d\sigma_y \\
&\quad - \int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot \nu\,d\sigma_y \\
&= I_m(x) + J_m(x) + K_m(x).
\end{aligned}
$$

But, due to the compact support of f, we have

$$|I_m(x)| \le \|D^2 f\|_{\infty,\mathbb{R}^n} \int_{B_m(0)} |K(y)|\,dy.$$

Thus, for $n=2$,

$$|I_m(x)| \le \frac{m^2}{2}\left(\frac{1}{2} + |\ln m|\right)\|D^2 f\|_{\infty,\mathbb{R}^n}$$

and for $n \ge 3$, we have

$$|I_m(x)| \le \frac{m^2}{2(n-2)}\|D^2 f\|_{\infty,\mathbb{R}^n}.$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Similarly,

$$
\begin{aligned}
|J_m(x)| &\le \int_{S_m(0)} |K(y)\nabla_y f(x-y)\cdot \nu|\,d\sigma_y \\
&\le \|\nabla f\|_{\infty,\mathbb{R}^n} \int_{S_m(0)} |K(y)|\,d\sigma_y.
\end{aligned}
$$

Thus, for $n = 2$,
$$|J_m(x)| \le m|\ln m| \|\nabla f\|_{\infty,\mathbb{R}^n}$$
and for $n \ge 3$, we have
$$|J_m(x)| \le \frac{m}{(n-2)} \|\nabla f\|_{\infty,\mathbb{R}^n}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Now, to tackle the last term $K_m(x)$, we note that a simple computation yields that $\nabla_y K(y) = \frac{-1}{\omega_n |y|^n} y$. Since we are in the m radius sphere $|y| = m$. Also the unit vector ν outside of $S_m(0)$, as a boundary of $\mathbb{R}^n \setminus B_m(0)$, is given by $-y/|y| = -y/m$. Therefore,
$$\nabla_y K(y) \cdot \nu = \frac{1}{\omega_n m^{n+1}} y \cdot y = \frac{1}{\omega_n m^{n-1}}.$$

$$\begin{aligned} K_m(x) &= -\int_{S_m(0)} f(x-y) \nabla_y K(y) \cdot \nu \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(0)} f(x-y) \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(x)} f(y) \, d\sigma_y \end{aligned}$$

Since f is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider
$$\begin{aligned} |K_m(x) - (-f(x))| &= \left| f(x) - \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} f(y) \, d\sigma_y \right| \\ &= \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} |f(x) - f(y)| \, d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -f(x)$. Hence, u solves (4.6).

Remark 4.3. Notice that in the proof above, we have used the Green's identity eventhough our domain is not bounded (which is a hypothesis for Green's identity). This can be justified by taking a ball bigger than $B_m(0)$ and working in the annular region, and later letting the bigger ball approach all of $\mathbb{R}^n$.

A natural question at this juncture is: Is every solution of the Poisson equation (4.6) of the form $K * f$. We answer this question in the following theorem.

Theorem 4.18. *Let $f \in C_c^2(\mathbb{R}^n)$ and $n \ge 3$. If u is a solution of (4.6) and u is bounded, then u has the form $u(x) = (K * f)(x) + C$, for any $x \in \mathbb{R}^n$, where C is some constant.*

Proof. We know that (cf. Theorem 4.17) $u'(x) := (K * f)(x)$ solves (4.6), the Poisson equation in $\mathbb{R}^n$. Moreover, u' is bounded for $n \geq 3$, since $K(x) \to 0$ as $|x| \to \infty$ and f has compact support in $\mathbb{R}^n$. Also, since u is given to be a bounded solution of (4.6), $v := u - u'$ is a bounded harmonic function. Hence, by Liouville's theorem, v is constant. Therefore $u = u' + C$, for some constant C.

We turn our attention to studying Poisson equation in proper subsets of $\mathbb{R}^n$. Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$.

Theorem 4.19 (Uniqueness). *Let Ω be an open bounded subset of $\mathbb{R}^n$. For the Poisson equation $\Delta u = f$ with one of Dirichlet, Robin or Mixed conditions on $\partial\Omega$, there exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. In the Neumann problem two solutions differ by a constant.*

Proof. Let u and v be solutions of the Poisson equation with same boundary conditions on $\partial\Omega$. Then $w := u - v$ is a harmonic function, $\Delta w = 0$, with homogeneous boundary condition on $\partial\Omega$. By Green's identity B.4, we have

$$\int_\Omega |\nabla w|^2\, dx = \int_{\partial\Omega} w(\nabla w \cdot \nu)\, d\sigma.$$

For the Drichlet, Neumann and Mixed case, the RHS is zero. For the Robin condition the RHS is negative,

$$\int_{\partial\Omega} w(\nabla w \cdot \nu)\, d\sigma = -c \int_{\partial\Omega} w^2\, d\sigma \leq 0.$$

Thus, in all the four boundary conditions

$$\int_\Omega |\nabla w|^2\, dx \leq 0$$

and $\nabla w = 0$. Therefore, $w = u - v$ is constant in the connected components of Ω. In the case of Dirichlet, mixed and Robin the constant has to be zero, by Maximum principle[4]. Thus, $u = v$ in these three cases.

Lemma 4.3. *Let f be bounded and locally Hölder continuous[5] with exponent $\gamma \leq 1$ in Ω. Then $u := K * f \in C^2(\Omega)$, $-\Delta u = f$ in Ω.*

Theorem 4.20 (Existence). *Let Ω be a bounded domain with all boundary points being regular w.r.t Laplacian. The classical Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad\;\, u = g & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

is solvable (hence, uniquely) for any bounded, locally Hölder continuous function f in Ω and continuous function g on $\partial\Omega$.

---

[4] or, simply, from the fact that a non-zero c will contradict the continuous extension of w to boundary.

[5] Hölder continuous in each compact subset of Ω

Proof. Recall that K is a fundamental solution of $-\Delta$. Set $w(x) := f * K$ in $\mathbb{R}^n$ then $-\Delta w = f$. Set $v = u - w$. Then (4.7) is solvable iff

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \\ \quad v = g - w & \text{on } \partial\Omega \end{cases}$$

is solvable. The equation for v is solvable by Theorem 4.12.

4.7.3 Green's Function

The fundamental solution was defined for entire $\mathbb{R}^n$. The Green's function is an analogue of fundamental solution for proper subsets of $\mathbb{R}^n$. Thus, we shall now attempt to solve the Poisson equation on a proper open subset Ω of $\mathbb{R}^n$. This is done via the Green's function. For any $x \in \Omega$, choose $m > 0$ such that $B_m(x) \subset \Omega$. Set $\Omega_m := \Omega \setminus B_m(x)$. By applying the second identity of Corollary B.4, for any $u \in C^2(\overline{\Omega})$ and $v_x(y) = K(y-x)$, where K is the fundamental solution on $\mathbb{R}^n \setminus \{x\}$, on the domain Ω_m, we get

$$\begin{aligned}
\int_{\Omega_m} u(y)\Delta_y v_x(y)\,dy - \int_{\Omega_m} v_x(y)\Delta_y u(y)\,dy &= \int_{\partial\Omega_m} \left(u(y)\frac{\partial v_x}{\partial \nu} - v_x(y)\frac{\partial u}{\partial \nu} \right) d\sigma_y \\
-\int_{\Omega_m} v_x(y)\Delta_y u(y)\,dy &= \int_{\partial\Omega_m} \left(u(y)\frac{\partial v_x}{\partial \nu} - v_x(y)\frac{\partial u}{\partial \nu} \right) d\sigma_y \\
-\int_{\Omega} v_x(y)\Delta_y u(y)\,dy + \int_{B_m(x)} v_x(y)\Delta_y u(y)\,dy &= \int_{\partial\Omega} \left(u(y)\frac{\partial v_x}{\partial \nu} - v_x(y)\frac{\partial u}{\partial \nu} \right) d\sigma_y \\
&\quad + \int_{S_m(x)} \left(u\frac{\partial v_x}{\partial \nu} - v_x\frac{\partial u}{\partial \nu} \right) d\sigma_y \\
\int_{B_m(x)} v_x(y)\Delta_y u(y)\,dy - \int_{S_m(x)} u(y)\frac{\partial v_x}{\partial \nu}(y)\,d\sigma_y & \\
+ \int_{S_m(x)} v_x(y)\frac{\partial u(y)}{\partial \nu}\,d\sigma_y &= \int_{\partial\Omega} \left(u(y)\frac{\partial v_x}{\partial \nu} - v_x(y)\frac{\partial u}{\partial \nu} \right) d\sigma_y \\
&\quad + \int_{\Omega} v_x(y)\Delta_y u(y)\,dy.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_m(x) + K_m(x) + J_m(x) &= \int_{\partial\Omega} \left(u(y)\frac{\partial K}{\partial \nu}(y-x) - K(y-x)\frac{\partial u(y)}{\partial \nu} \right) d\sigma_y \\
&\quad + \int_{\Omega} K(y-x)\Delta_y u(y)\,dy.
\end{aligned}$$

The LHS is handled exactly as in the proof of Theorem 4.17, since u is a continuous function on the compact set $\overline{\Omega}$ and is bounded. We repeat the arguments below for completeness sake. Consider the term I_m.

$$|I_m(x)| \leq \|D^2u\|_{\infty,\Omega} \int_{B_m(x)} |K(y-x)|\, dy.$$

Thus,

$$|I_m(x)| \leq \begin{cases} \frac{m^2}{2}\left(\frac{1}{2} + |\ln m|\right) \|D^2u\|_{\infty,\Omega} & \text{for } n = 2 \\ \frac{m^2}{2(n-2)} \|D^2u\|_{\infty,\Omega} & \text{for } n \geq 3. \end{cases}$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Next, consider the term $K_m(x)$. Note that $\nabla_y K(y-x) = \frac{-1}{\omega_n |y-x|^n}(y-x)$. Since we are in the m radius sphere $|y-x| = m$. Also the unit vector ν inside of $S_m(x)$, as a boundary of $\Omega \setminus B_m(x)$, is given by $-(y-x)/|y-x| = -(y-x)/m$. Therefore,

$$\nabla_y K(y-x) \cdot \nu = \frac{1}{\omega_n m^{n+1}}(y-x)\cdot(y-x) = \frac{1}{\omega_n m^{n-1}}.$$

Thus,

$$\begin{aligned} K_m(x) &= -\int_{S_m(x)} u(y) \nabla_y K(y-x) \cdot \nu \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(x)} u(y)\, d\sigma_y \end{aligned}$$

Since u is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|u(x) - u(y)| < \varepsilon$ whenever $|x-y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} |K_m(x) - (-u(x))| &= \left| u(x) - \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} u(y)\, d\sigma_y \right| \\ &= \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} |u(x) - u(y)|\, d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -u(x)$. Finally, we consider the term $J_m(x)$,

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(x)} |K(y-x) \nabla_y u(y) \cdot \nu|\, d\sigma_y \\ &\leq \|\nabla_y u\|_{\infty,\Omega} \int_{S_m(x)} |K(y-x)|\, d\sigma_y. \end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq \begin{cases} m|\ln m| \|\nabla_y u\|_{\infty,\Omega} & \text{for } n = 2 \\ |J_m(x)| \leq \frac{m}{(n-2)} \|\nabla_y u\|_{\infty,\Omega} & \text{for } n \geq 3. \end{cases}$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Therefore, letting $m \to 0$, we have the identity

$$u(x) = \int_{\partial\Omega} \left(K(y-x)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial K}{\partial \nu}(y-x) \right) d\sigma_y - \int_{\Omega} K(y-x)\Delta_y u(y)\, dy \quad (4.8)$$

For the Dirichlet problem, Δu is known in Ω and u is known on $\partial\Omega$. Thus, (4.8) gives an expression for the solution u, provided we know the normal derivative $\frac{\partial u(y)}{\partial \nu}$ along $\partial\Omega$. But this quantity is usually an unknown for Dirichlet problem. Thus, we wish to rewrite (4.8) such that the knowledge of the normal derivative is not necessary. To do so, we introduce a function $\psi_x(y)$, for a fixed $x \in \Omega$, as the solution of the boundary-value problem,

$$\begin{cases} \Delta \psi_x(y) = 0 \text{ in } \Omega \\ \psi_x(y) = K(y-x) \text{ on } \partial\Omega. \end{cases} \quad (4.9)$$

Now applying the second identity of Corollary B.4 for any $u \in C^2(\overline{\Omega})$ and $v(y) = \psi_x(y)$, we get

$$\int_{\partial\Omega} \left(u\frac{\partial \psi_x}{\partial \nu} - \psi_x \frac{\partial u}{\partial \nu} \right) d\sigma_y = \int_{\Omega} (u\Delta_y \psi_x - \psi_x \Delta_y u)\, dy.$$

Therefore, substituting the following identity

$$\int_{\partial\Omega} K(y-x)\frac{\partial u(y)}{\partial \nu}\, d\sigma_y = \int_{\Omega} \psi_x(y)\Delta_y u(y)\, dy + \int_{\partial\Omega} u(y)\frac{\partial \psi_x(y)}{\partial \nu}\, d\sigma_y$$

in (4.8), we get

$$u(x) = \int_{\Omega} (\psi_x(y) - K(y-x))\, \Delta_y u\, dy + \int_{\partial\Omega} u\nabla\left(\psi_x(y) - K(y-x)\right) \cdot \nu\, d\sigma_y.$$

The identity above motivates the definition of what is called the *Green's function*.

Definition 4.9. For any given open subset $\Omega \subset \mathbb{R}^n$ and $x, y \in \Omega$ such that $x \neq y$, we define the Green's function as

$$G(x,y) := \psi_x(y) - K(y-x).$$

Rewriting (4.8) in terms of Green's function,we get

$$u(x) = \int_{\Omega} G(x,y)\Delta_y u(y)\, dy + \int_{\partial\Omega} u(y)\frac{\partial G(x,y)}{\partial \nu}\, d\sigma_y.$$

Thus, in the arguments above, we have proved the following theorem.

Theorem 4.21. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Also, given $f \in C(\Omega)$ and $g \in C(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem (4.7), then u has the representation*

$$u(x) = -\int_{\Omega} G(x,y) f(y)\, dy + \int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\, d\sigma_y. \quad (4.10)$$

Observe that we have solved the Dirichlet problem (4.7) provided we know the Green's function. The construction of Green's function depends on the construction of ψ_x for every $x \in \Omega$. In other words, (4.7) is solved if we can solve (4.9). However, computing ψ_x is usually possible when Ω has simple geometry. We shall identify two simple cases of Ω, half-space and ball, where we shall explicitly compute G.

The Green's function is the analogue of the fundamental solution K for the boundary value problem. This is clear by observing that, for a fixed $x \in \Omega$, G satisfies (informally) the equation,

$$\begin{cases} -\Delta G(x,\cdot) = \delta_x \text{ in } \Omega \\ \quad G(x,\cdot) = 0 \text{ on } \partial\Omega, \end{cases}$$

where δ_x is the Dirac measure at x.

Theorem 4.22. *For all $x, y \in \Omega$ such that $x \neq y$, we have $G(x,y) = G(y,x)$, i.e., G is symmetric in x and y.*

Proof. Let us fix $x, y \in \Omega$. For a fixed $m > 0$, set $\Omega_m = \Omega \setminus (B_m(x) \cup B_m(y))$ and applying Green's identity for $v(\cdot) := G(x,\cdot)$ and $w(\cdot) := G(y,\cdot)$, we get

$$\begin{aligned}
\int_{\partial\Omega_m} \left(v(z)\frac{\partial w(z)}{\partial \nu} - w(z)\frac{\partial v(z)}{\partial \nu} \right) d\sigma_z &= \int_{\Omega_m} v(z)\Delta_z w(z)\, dz \\
&\quad - \int_{\Omega_m} w(z)\Delta_z v(z)\, dz \\
\int_{\partial\Omega_m} \left(v(z)\frac{\partial w(z)}{\partial \nu} - w(z)\frac{\partial v(z)}{\partial \nu} \right) d\sigma_z &= 0 \\
\int_{S_m(x)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) d\sigma_z &= \int_{S_m(y)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) d\sigma_z \\
J_m(x) + K_m(x) &= J_m(y) + K_m(y).
\end{aligned}$$

$$\begin{aligned}
|J_m(x)| &\leq \int_{S_m(x)} |v(z)\nabla_z w(z)\cdot \nu|\, d\sigma_z \\
&\leq \|\nabla w\|_{\infty,\Omega} \int_{S_m(x)} |v(z)|\, d\sigma_z \\
&= \|\nabla w\|_{\infty,\Omega} \int_{S_m(x)} |\psi_x(z) - K(z-x)|\, d\sigma_z.
\end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq \left(2\pi m\|\psi_x\|_{\infty,\Omega} + m|\ln m|\right)\|\nabla w\|_{\infty,\Omega}$$

and for $n \geq 3$, we have

$$|J_m(x)| \leq \left(\omega_n m^{n-1}\|\psi_x\|_{\infty,\Omega} + \frac{m}{(n-2)}\right)\|\nabla w\|_{\infty,\Omega}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Now, consider the term $K_m(x)$,

$$\begin{aligned} K_m(x) &= -\int_{S_m(x)} w(z) \frac{\partial v(z)}{\partial \nu}\, d\sigma_z \\ &= \int_{S_m(x)} w(z) \frac{\partial K}{\partial \nu}(z-x)\, d\sigma_z - \int_{S_m(x)} w(z) \frac{\partial \psi_x(z)}{\partial \nu}\, d\sigma_z. \end{aligned}$$

The second term goes to zero by taking the sup-norm outside the integral. To tackle the first term, we note that $\nabla_z K(z-x) = \frac{-1}{\omega_n |z-x|^n}(z-x)$. Since we are in the m radius sphere $|z-x| = m$. Also the unit vector ν outside of $S_m(x)$, as a boundary of $\Omega \setminus B_m(x)$, is given by $-(z-x)/|z-x| = -(z-x)/m$. Therefore,

$$\nabla_z K(z-x) \cdot \nu = \frac{1}{\omega_n m^{n+1}}(z-x) \cdot (z-x) = \frac{1}{\omega_n m^{n-1}}.$$

$$\int_{S_m(x)} w(z) \nabla_z K(z-x) \cdot \nu\, d\sigma_z = \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} w(z)\, d\sigma_z$$

Since w is continuous in $\Omega \setminus \{y\}$, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|w(z) - w(x)| < \varepsilon$ whenever $|x-z| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} &\left| \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} w(z)\, d\sigma_z - w(x) \right| \\ &\qquad = \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} |w(z) - w(x)|\, d\sigma_z < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to w(x)$. Arguing similarly, for $J_m(y)$ and $K_m(y)$, we get $G(y,x) = G(x,y)$.

Remark 4.4. In two dimensions, the Green's function has a nice connection with conformal mapping. Let $w = f(z)$ be a conformal mapping from an open domain (connected) $\Omega \subset \mathbb{R}^2$ onto the interior of the unit circle. The Green's function of Ω is

$$G(z,z_0) = \frac{1}{2\pi} \ln \left| \frac{1 - f(z)\overline{f(z_0)}}{f(z) - f(z_0)} \right|$$

where $z = x_1 + ix_2$ and $z_0 = y_1 + iy_2$.

4.7.4 Green's Function for half-space

In this section, we shall compute explicitly the Green's function for positive half-space. Thus, we shall have

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

and

$$\partial\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n \mid x_n = 0\}.$$

To compute the Green's function, we shall use the *method of reflection*. The reflection technique ensures that the points on the boundary (along which the reflection is done) remains unchanged to respect the imposed Dirichlet condition.

Definition 4.10. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, we define its *reflection* along its boundary $\mathbb{R}^{n-1}$ as $x^\star = (x_1, x_2, \ldots, -x_n)$.

It is obvious from the above definition that, for any $y \in \partial\mathbb{R}^n_+$, $|y - x^\star| = |y - x|$. Given a fixed $x \in \mathbb{R}^n_+$, we need to find a harmonic function ψ_x in $\mathbb{R}^n_+$, as in (4.9). But $K(\cdot - x)$ is harmonic in $\mathbb{R}^n_+ \setminus \{x\}$. Thus, we use the method of reflection to shift the singularity of K from $\mathbb{R}^n_+$ to the negative half-space and define

$$\psi_x(y) = K(y - x^\star).$$

By definition, ψ_x is harmonic in $\mathbb{R}^n_+$ and on the boundary $\psi_x(y) = K(y - x)$. Therefore, we define the Green's function to be $G(x,y) = K(y - x^\star) - K(y - x)$, for all $x, y \in \mathbb{R}^n_+$ and $x \neq y$. It now only remains to compute the normal derivative of G. Recall that $\nabla_y K(y - x) = \frac{-1}{\omega_n |y-x|^n}(y - x)$. Thus,

$$\nabla_y G(x,y) = \frac{-1}{\omega_n}\left(\frac{y - x^\star}{|y - x^\star|^n} - \frac{y - x}{|y - x|^n}\right)$$

Therefore, when $y \in \partial\mathbb{R}^n_+$, we have

$$\nabla_y G(x,y) = \frac{-1}{\omega_n |y - x|^n}(x - x^\star).$$

Since the outward unit normal of $\partial\mathbb{R}^n_+$ is $\nu = (0, 0, \ldots, 0, -1)$, we get

$$\nabla_y G(x,y) \cdot \nu = \frac{2x_n}{\omega_n |y - x|^n}.$$

Definition 4.11. For all $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$, the map

$$P(x,y) := \frac{2x_n}{\omega_n |y - x|^n}$$

is called the *Poisson kernel* for $\mathbb{R}^n_+$.

Now substituing for G in (4.10), we get the *Poisson formula* for u,

$$u(x) = \int_{\mathbb{R}^n_+} [K(y - x) - K(y - x^\star)] f(y)\, dy + \frac{2x_n}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|y - x|^n}\, d\sigma_y. \tag{4.11}$$

It now remains to show that the u as defined above is, indeed, a solution of (4.7) for $\mathbb{R}^n_+$.

Exercise 4.9. Let $f \in C(\mathbb{R}^n_+)$ be given. Let $g \in C(\mathbb{R}^{n-1})$ be bounded. Then u as given in (4.11) is in $C^2(\mathbb{R}^n_+)$ and solves (4.7).

4.7.5 Green's Function for a disk

In this section, we shall compute explicitly the Green's function for a ball of radius $r > 0$ and centred at $a \in \mathbb{R}^n$, $B_r(a)$. As usual, we denote the surface of the disk as $S_r(a)$, the circle of radius r centred at a. We, once again, use the *method of reflection* but, this time reflected along the boundary of the disk.

Definition 4.12. For any $x \in \mathbb{R}^n \setminus \{a\}$, we define its *reflection* along the circle $S_r(a)$ as $x^\star = \frac{r^2(x-a)}{|x-a|^2} + a$.

The idea behind reflection is clear for the unit disk, i.e., when $a = 0$ and $r = 1$, as $x^\star = \frac{x}{|x|^2}$. The above definition is just the shift of origin to a and dilating the unit disk by r.

Now, for any $y \in S_r(a)$ and $x \neq a$, consider

$$\begin{aligned}
|y - x^\star|^2 &= |y-a|^2 - 2(y-a)\cdot(x^\star - a) + |x^\star - a|^2 \\
&= r^2 - 2r^2(y-a)\cdot\left(\frac{x-a}{|x-a|^2}\right) + \left|\frac{r^2(x-a)}{|x-a|^2}\right|^2 \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + r^2) \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + |y-a|^2) \\
&= \frac{r^2}{|x-a|^2}|y-x|^2
\end{aligned}$$

Therefore, $\frac{|x-a|}{r}|y - x^\star| = |y - x|$ for all $y \in S_r(a)$. For each fixed $x \in B_r(a)$, we need to find a harmonic function ψ_x in $B_r(a)$ solving (4.9). Since $K(\cdot - x)$ is harmonic in $B_r(a) \setminus \{x\}$, we use the method of reflection to shift the singularity of K at x to the complement of $B_r(a)$. Thus, we define

$$\psi_x(y) = K\left(\frac{|x-a|}{r}(y - x^\star)\right) \quad x \neq a.$$

For $n \geq 3$, $K\left(\frac{|x-a|}{r}(y - x^\star)\right) = \frac{|x-a|^{2-n}}{r^{2-n}}K(y - x^\star)$. Thus, for $n \geq 3$, ψ_x solves (4.9), for $x \neq a$. For $n = 2$,

$$K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-1}{2\pi}\ln\left(\frac{|x-a|}{r}\right) + K(y-x^\star).$$

Hence ψ_x solves (4.9) for $n=2$. Note that we are yet to identify a harmonic function ψ_a corresponding to $x=a$. We do this by setting ψ_a to be the constant function

$$\psi_a(y) := \begin{cases} -\frac{1}{2\pi}\ln r & (n=2) \\ \frac{r^{2-n}}{\omega_n(n-2)} & (n \geq 3). \end{cases}$$

Thus, ψ_a is harmonic and solves (4.9) for $x=a$. Therefore, we define the Green's function to be

$$G(x,y) := K\left(\frac{|x-a|}{r}(y-x^\star)\right) - K(y-x) \quad \forall x,y \in B_r(a), x \neq a \text{ and } x \neq y$$

and

$$G(a,y) := \begin{cases} -\frac{1}{2\pi}\ln\left(\frac{r}{|y-a|}\right) & (n=2) \\ \frac{1}{\omega_n(n-2)}\left(r^{2-n} - |y-a|^{2-n}\right) & (n \geq 3). \end{cases}$$

We shall now compute the normal derivative of G. Recall that

$$\nabla_y K(y-x) = \frac{-1}{\omega_n |y-x|^n}(y-x)$$

and one can compute $\nabla_y K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-|x-a|^{2-n}}{r^{2-n}\omega_n|y-x^\star|^n}(y-x^\star)$. Therefore,

$$\nabla_y G(x,y) = \frac{-1}{\omega_n}\left[\frac{|x-a|^{2-n}(y-x^\star)}{r^{2-n}|y-x^\star|^n} - \frac{y-x}{|y-x|^n}\right].$$

If $y \in S_r(a)$, we have

$$\begin{aligned} \nabla_y G(x,y) &= \frac{-1}{\omega_n |y-x|^n}\left[\frac{|x-a|^2}{r^2}(y-x^\star) - (y-x)\right] \\ &= \frac{-1}{\omega_n |y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right](y-a) \end{aligned}$$

Since the outward unit normal at any point $y \in S_r(a)$ is $\frac{1}{r}(y-a)$, we have

$$\begin{aligned} \nabla_y G(x,y)\cdot\nu &= \frac{-1}{\omega_n |y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right]\sum_{i=1}^n \frac{1}{r}(y_i-a_i)^2 \\ &= \frac{-r}{\omega_n |y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right]. \end{aligned}$$

Definition 4.13. For all $x \in B_r(a)$ and $y \in S_r(a)$, the map

$$P(x,y) := \frac{r^2 - |x-a|^2}{r\omega_n |y-x|^n}$$

is called the *Poisson kernel* for $B_r(a)$.

Now substituing for G in (4.10), we get the *Poisson formula* for u,

$$u(x) = -\int_{B_r(a)} G(x,y) f(y)\, dy + \frac{r^2 - |x-a|^2}{r\omega_n} \int_{S_r(a)} \frac{g(y)}{|y-x|^n}\, d\sigma_y. \tag{4.12}$$

It now remains to show that the u as defined above is, indeed, a solution of (4.7) for $B_r(a)$.

Exercise 4.10. Let $f \in C(B_r(a))$ be given. Let $g \in C(S_r(a))$ be bounded. Then u as given in (4.12) is in $C^2(B_r(a))$ and solves (4.7).

Chapter 5
Wave Equation

The first PDE was introduced in 1752 by d'Alembert as a model to study vibrating strings. He introduced the one dimensional *wave* equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

This was generalised to

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t),$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, for $n = 2$ and $n = 3$, respectively, by Euler (1759) and D. Bernoulli (1762).

Remark 5.1 (Time Reversibility). The wave equation is invariant under the transformation $t \mapsto -t$, i.e., if $u(x,t)$ is a solution to the wave equation for $t \geq 0$, then $\hat{u}(x,t) := u(x,\tau)$ is a solution of the wave equation for $t < 0$ and $\tau := -t > 0$, because $d\tau/dt = -1$, $u_t(x,t) = -u_\tau(x,\tau)$. This means that wave equation is reversible in time and do not distinguish between past and future.

5.1 Duhamel's Principle

We shall denote Δ as the Laplacian w.r.t. the space variable. The Cauchy initial value problem in higher dimensions is

$$\begin{cases} (\partial_t^2 - c^2\Delta)u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \\ u_t(x,0) = h(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{5.1}$$

Due to the linearity of the wave operator, any solution $u = v + w + z$ where v, w and z are, respectively, solutions of

$$\begin{cases} (\partial_t^2 - c^2\Delta)v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x) & \text{in } \mathbb{R}^n \\ v_t(x,0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{5.2}$$

$$\begin{cases} (\partial_t^2 - c^2\Delta)w = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ w(x,0) = 0 & \text{in } \mathbb{R}^n \\ w_t(x,0) = h(x) & \text{in } \mathbb{R}^n \end{cases} \tag{5.3}$$

and

$$\begin{cases} (\partial_t^2 - c^2\Delta)z = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ z(x,0) = 0 & \text{in } \mathbb{R}^n \\ z_t(x,0) = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{5.4}$$

Theorem 5.1 (Duhamel's Principle). *Let w^h be a solution to (5.3). Then*

$$v(x,t) = w_t^g(x,t)$$

and

$$z(x,t) = \int_0^t w^{f_s}(x,t-s)\,ds,$$

where $f_s = f(\cdot,s)$, are solutions to (5.2) and (5.4).

Proof. Since w^g solves (5.3) with $h = g$, we have

$$(\partial_t^2 - c^2\Delta)v = (\partial_t^2 - c^2\Delta)w_t^g(x,t) = \frac{\partial}{\partial t}[(\partial_t^2 - c^2\Delta)w^g] = 0$$

on $\mathbb{R}^n \times (0,\infty)$. Further, $v(x,0) = w_t^g(x,0) = g(x)$ in $\mathbb{R}^n$ and $v_t(x,0) = w_{tt}^g(x,0) = c^2\Delta w^g(x,0) = 0$. Thus, v solves (5.2). Now, let $w^{f_s}(x,t)$, for $0 < s \le t$, be the solution of (5.3) with $h(\cdot) = f(\cdot,s)$. Then, by Leibniz integral rule (cf. Theorem A.1), we have

$$z_t(x,t) = w^{f_t}(x,0) + \int_0^t w_t^{f_s}(x,t-s)\,ds = \int_0^t w_t^{f_s}(x,t-s)\,ds$$

and

$$\begin{aligned} z_{tt}(x,t) &= w_t^{f_t}(x,0) + \int_0^t w_{tt}^{f_s}(x,t-s)\,ds \\ &= f(x,t) + c^2\Delta \int_0^t w^{f_s}(x,t-s)\,ds \\ &= f(x,t) + c^2\Delta z. \end{aligned}$$

Therefore, z solves (5.4).

The Duhamel's principle can be viewed as a generalization of the method of variations of constants in ODE. Owing to the above theorem it is enough to solve for w in (5.3). To do so, we shall employ the method of spherical means which reduces such problem to one dimensional framework.

5.2 Cauchy Problem for Domains with Boundary

We have already elaborated on the way to choose boundary conditions (q.v. Section 2.3.3 and Remark 3.4). In view of those discussions, let Ω be an open subset of $\mathbb{R}^n$ with a non-empty boundary $\partial\Omega$. The Cauchy problem for wave equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & \text{in } \Omega \times (0,T) \\ u(x,0) = g(x) & \text{in } \overline{\Omega} \times \{0\} \\ u_t(x,0) = h(x) & \text{in } \overline{\Omega} \times \{0\} \end{cases}$$

is well-posed if we choose to specify one of the following conditions on $\partial\Omega \times [0,T)$:

(i) (Dirichlet condition) $u(x,t) = h(x,t)$;
(ii) (Neumann condition) $\nabla_x u(x,t) \cdot \nu(x) = h(x,t)$, where $\nu(x)$ is the unit outward normal of $(x,t) \in \partial\Omega \times (0,T)$;
(iii) (Robin condition) $\nabla_x u(x,t) \cdot \nu + cu(x,t) = h(x,t)$ for any $c > 0$.
(iv) (Mixed condition) $u(x,t) = h(x,t)$ on Γ_1 and $\nabla_x u(x,t) \cdot \nu = h(x,t)$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega \times (0,T)$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Theorem 5.2 (At most one solution). *For any open connected subset $\Omega \subseteq \mathbb{R}^n$, there exists at most one solution $u \in C^2(\Omega \times (0,T))$ of the wave equation*

$$\begin{cases} (\partial_t^2 - \Delta)u = f(x,t) & \text{in } \Omega \times (0,T] \\ u(x,0) = g(x) & \text{in } \Omega \times \{0\} \\ u_t(x,0) = h(x) & \text{in } \Omega \times \{0\} \\ u(x,t) = \phi(x,t) & \text{in } \partial\Omega \times [0,T). \end{cases} \tag{5.5}$$

Proof. If u and v are two solutions of the above wave equations then, by linearity, we have $w := u - v$ is a solution of the homogeneous wave equation

$$\begin{cases} (\partial_t^2 - \Delta)w = 0 & \text{in } \Omega \times (0,T] \\ w(x,0) = 0 & \text{in } \Omega \times \{0\} \\ w_t(x,0) = 0 & \text{in } \Omega \times \{0\} \\ w(x,t) = 0 & \text{in } \partial\Omega \times [0,T). \end{cases}$$

Multiplying w_t both sides of the PDE and integrate over Ω, for each $t \in (0,T]$ to obtain

$$\begin{aligned} 0 &= \int_\Omega (w_{tt} w_t - w_t \Delta w)\, dx \\ &= \frac{1}{2} \int_\Omega \frac{d}{dt}(w_t^2)\, dx + \int_\Omega \nabla w \cdot \nabla w_t \, dx + \int_{\partial\Omega} w_t (\nabla w \cdot \nu)\, d\sigma \\ &= \frac{1}{2} \frac{d}{dt} \int_\Omega (w_t^2)\, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla w|^2 \, dx \\ &= \frac{d}{dt} \left[\frac{1}{2} \int_\Omega \left(w_t^2 + |\nabla w|^2 \right) dx \right] =: \frac{d}{dt} E(t). \end{aligned}$$

Thus, E is a constant function in time. Since $E(0) = 0 = E(T)$, we have $E \equiv 0$, which implies that $w_t = 0 = \nabla w$ in $\Omega \times (0,T]$. Hence, w is constant zero function, from Cauchy data, and $u = v$.

5.3 One Dimensional Wave Equation

The one dimensional wave equation is the first ever partial differential equation (PDE) to be studied, introduced in 1752 by d'Alembert as a model to study vibrating strings. He introduced the one dimensional *wave* equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

d'Alembert used the travelling wave technique to solve the wave equation. In this chapter, we shall explain the technique of d'Alembert and also give the standing wave technique which motivates the idea of separation of variable, and in turn, the evolution of Fourier series. The one dimension wave equation can also describe the propogation of sound waves in tubes. The wave equations were generalised to two and three dimensions by Euler (1759) and D. Bernoulli (1762), respectively.

5.3.1 Derivation

Let us consider a homogeneous string of length L, stretched along the x-axis, with one end fixed at $x = 0$ and the other end fixed at $x = L$. We assume that the string is free to move only in the vertical direction. Let $\rho > 0$ denote the density of the string and $T > 0$ denote the coefficient of tension of the string. Let $u(x,t)$ denote the vertical displacement of the string at the point x and time t.

We shall imagine the string of length L as system of N objects, for N sufficiently large. Think of N objects sitting on the string L at equidistant (uniformly distributed). The position of the n-th object on the string is given by $x_n = nL/N$. One can think of the vibrating string as the harmonic oscillator of N objects governed by the tension on the string (which behaves like the spring). Let $y_n(t) = u(x_n,t)$ denote the displacement of the object x_n at time t. The distance between any two successive objects is $h = x_{n+1} - x_n = L/N$. Then mass of each of the N object is mass of the string divided by N. Since mass of the string is $\rho \times L$, mass of each of the object x_n, $n = 1,2,\ldots,N$, is ρh. Thus, by Newton's second law, $\rho h y_n''(t)$ is same as the force acting on the n-th object. The force on x_n is coming both from left (x_{n-1}) and right (x_{n+1}) side. The force from left and right is given as $T(y_{n-1} - y_n)/h$ and $T(y_{n+1} - y_n)/h$, respectively. Therefore,

$$\begin{aligned}\rho h y_n''(t) &= \frac{T}{h}\{y_{n+1}(t)+y_{n-1}(t)-2y_n(t)\}\\ &= \frac{T}{h}\{u(x_n+h,t)+u(x_n-h,t)-2u(x_n,t)\}\\ y_n''(t) &= \frac{T}{\rho}\left(\frac{u(x_n+h,t)+u(x_n-h,t)-2u(x_n,t)}{h^2}\right)\end{aligned}$$

Note that assuming u is twice differentiable w.r.t the x variable, the term on RHS is same as

$$\frac{T}{\rho}\frac{1}{h}\left(\frac{u(x_n+h,t)-u(x_n,t)}{h}+\frac{u(x_n-h,t)-u(x_n,t)}{h}\right)$$

which converges to the second partial derivative of u w.r.t x as $h \to 0$. The $h \to 0$ is the limit case of the N objects we started with. Therefore the vibrating string system is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

where T is the *tension* and ρ is the *density* of the string. Equivalently,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{5.6}$$

where $c^2 = T/\rho$, $c > 0$, on $x \in (0,L)$ and $t > 0$.

Remark 5.2. The wave equation (5.6) can be rewritten as

$$u_{zz} = u_{ww} \text{ in } (w,z) \in (0,L)\times(0,\infty).$$

under a new coordinate system (w,z). Set $w = x/a$ and $z = t/b$, where a and b will be chosen appropriately. Then, $w_x = 1/a$ and $z_t = 1/b$. Therefore, $u_x = u_w/a$, $u_t = u_z/b$, $a^2u_{xx} = u_{ww}$ and $b^2u_{tt} = u_{zz}$. Choosing $a = 1$ and $b = 1/c$. One may, in fact choose coordinate such that the string is fixed between $(0,\pi)$. Choosing $a = L/\pi$ and $b = L/c\pi$ makes the domain $(0,\pi)$.

5.3.2 Travelling Waves

Consider the wave equation $u_{tt} = c^2u_{xx}$ on $\mathbb{R}\times(0,\infty)$, describing the vibration of an infinite string. We have already seen in Section 3.3 that the equation is hyperbolic and has the two characteristics $x \pm ct$= a constant. Introduce the new coordinates $w = x+ct$, $z = x-ct$ and set $u(w,z) = u(x,t)$. Thus, we have the following relations, using chain rule:

$$\begin{aligned} u_x &= u_w w_x + u_z z_x = u_w + u_z \\ u_t &= u_w w_t + u_z z_t = c(u_w - u_z) \\ u_{xx} &= u_{ww} + 2u_{zw} + u_{zz} \\ u_{tt} &= c^2(u_{ww} - 2u_{zw} + u_{zz}) \end{aligned}$$

In the new coordinates, the wave equation satisfies $u_{wz} = 0$. Integrating[1] this twice, we have $u(w,z) = F(w) + G(z)$, for some arbitrary functions F and G. Thus, $u(x,t) = F(x+ct) + G(x-ct)$ is a *general solution* of the wave equation.

Consider the case where G is chosen to be zero function. Then $u(x,t) = F(x+ct)$ solves the wave equation. At $t = 0$, the solution is simply the graph of F and at $t = t_0$ the solution is the graph of F with origin translated to the left by ct_0. Similarly, choosing $F = 0$ and $G = F$, we have $u(x,t) = F(x-ct)$ also solves wave equation and at time t is the translation to the right of the graph of F by ct. This motivates the name "travelling waves" and "wave equation". The graph of F is shifted to right or left with a speed of c.

Now that we have derived the general form of the solution of wave equation, we return to understand the physical system of a vibrating infinite string. The initial shape (position at initial time $t = 0$) of the string is given as $u(x,0) = g(x)$, where the graph of g on $\mathbb{R}^2$ describes the shape of the string. Since we need one more data to identify the arbitrary functions, we also prescribe the initial velocity of the string, $u_t(x,0) = h(x)$.

Remark 5.3. Another interesting property that follows from the general solution is that for any four points A, B, C and D that form a parallelogram bounded by characteristic curves in $\mathbb{R} \times \mathbb{R}^+$, $u(A) + u(C) = u(B) + u(D)$ because $u(A) = F(\alpha) + G(\beta)$, $u(C) = F(\gamma) + G(\delta)$, $u(B) = F(\alpha) + G(\delta)$ and $u(D) = F(\gamma) + G(\beta)$ (see fig 5.1).

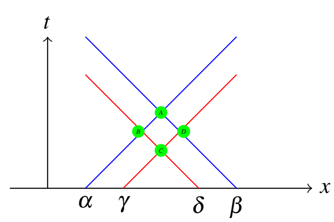

Fig. 5.1 Parallelogram Property

Theorem 5.3 (Infinite Length). *Given $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, there is a unique C^2 solution u of the Cauchy initial value problem (IVP) of the wave equation,*

[1] We are assuming the function is integrable, which may be false

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \\ u_t(x,0) = u_1(x) & \text{in } \mathbb{R}, \end{cases} \tag{5.7}$$

which is given by the d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy. \tag{5.8}$$

Proof. The general solution is $u(x,t) = F(x+ct) + G(x-ct)$ with $F, G \in C^2(\mathbb{R})$. Using the initial position we get

$$F(x) + G(x) = u_0(x).$$

Thus, u_0 should be $C^2(\mathbb{R})$. Now, $u_t(x,t) = c\,(F'(w) - G'(z))$ and putting $t = 0$, we get

$$F'(x) - G'(x) = \frac{1}{c}u_1(x).$$

Thus, u_1 should be $C^1(\mathbb{R})$. Now solving for F' and G', we get $2F'(x) = u_0'(x) + u_1(x)/c$. Similarly, $2G'(x) = u_0'(x) - u_1(x)/c$. Integrating[2] both these equations, we get

$$F(x) = \frac{1}{2}\left(u_0(x) + \frac{1}{c}\int_0^x u_1(y)\,dy\right) + F(0) - \frac{u_0(0)}{2}$$

and

$$G(x) = \frac{1}{2}\left(u_0(x) - \frac{1}{c}\int_0^x u_1(y)\,dy\right) + G(0) - \frac{u_0(0)}{2}.$$

Since $F(x) + G(x) = u_0(x)$, we get $F(0) + G(0) - u_0(0) = 0$. Therefore, the solution to the wave equation is given by (5.8).

Proof (Aliter). Let us derive the d'Alembert's formula in an alternate way. Note that the wave equation can be factored as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = u_{tt} - c^2 u_{xx} = 0.$$

We set $v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t)$ and hence

$$v_t(x,t) + cv_x(x,t) = 0 \quad \text{in } \mathbb{R} \times (0,\infty).$$

and $v(x,0) = u_t(x,0) - cu_x(x,0) = u_1(x) - cu_0'(x)$. Notice that v is solution to the homogeneous linear transport equation (cf. (2.8)), and is given by

$$v(x,t) = u_1(x-ct) - cu_0'(x-ct).$$

[2] assuming they are integrable and the integral of their derivatives is itself

Using v in the original equation, we get the inhomogeneous linear transport equation,

$$u_t(x,t) - cu_x(x,t) = u_1(x-ct) - cu_0'(x-ct)$$

with $u(x,0) = u_0(x)$ whose solution is (cf. (2.9))

$$u(x,t) = u_0(x+ct) + \int_0^t \left[u_1(x+ct-2cs) - cu_0'(x+ct-2cs)\right] ds.$$

Thus,

$$\begin{aligned}
u(x,t) &= u_0(x+ct) + \frac{-1}{2c}\int_{x+ct}^{x-ct} \left[u_1(y) - cu_0'(y)\right] dy \\
&= u_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left(u_1(y) - cu_0'(y)\right) dy \\
&= u_0(x+ct) + \frac{1}{2}\left(u_0(x-ct) - u_0(x+ct)\right) \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\, dy \\
&= \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\, dy
\end{aligned}$$

The solution to the Cauchy Problem 5.7 is unique which follows from the uniqueness of the transport equation. A useful observation from the d'Alembert's formula is that the regularity of u is same as the regularity of its initial value u_0. Note that the solution $u(x,t)$ depends only on the interval $[x-ct, x+ct]$, called the *domain of dependence* for (x,t) because u_0 takes values only on the end-points of this interval and u_1 takes values between this interval. The two characteristic curves that pass through (x,t) intersects x-axis at $x-ct$ and $x+ct$, respectively, the endpoints of the domain of dependence (see fig 5.2).

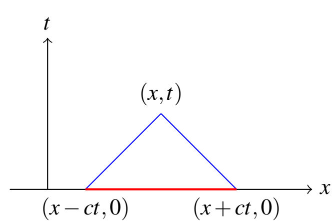

Fig. 5.2 Domain of Dependence

Conversely, given a point p on the initial curve x-axis, the region of the xt-plane where the values of u depend on the value of $u_0(p)$ and $u_1(p)$ is the cone with vertex at p and is called the *range of influence*. The range of influence is the region bounded by the two characteristic curves intersecting at p (see fig 5.3).

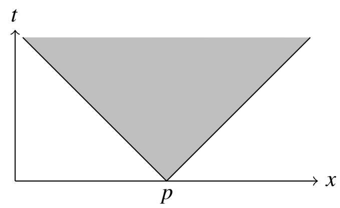

Fig. 5.3 Range of Influence

If the initial data u_0 and u_1 are supported in the interval $B_{x_0}(R)$ then the solution u at (x,t) is supported in the region $B_{x_0}(R+ct)$. Consequently, if u_0 and u_1 have compact support then the solution u has compact support in $\mathbb{R}$ for all time $t>0$. This phenomenon is called the *finite speed of propagation*.

Theorem 5.4 (Inhomogeneous). *Given $u_0 \in C^2(\mathbb{R})$, $u_1 \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R} \times [0,\infty)$, there is a unique C^2 solution u of the inhomogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \\ u_t(x,0) = u_1(x) & \text{in } \mathbb{R}, \end{cases} \tag{5.9}$$

given by the formula

$$\frac{1}{2}\left[u_0(x+ct)+u_0(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} u_1(y)\,dy + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\right].$$

Proof. Fix $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Consider the open triangle in $\mathbb{R} \times \mathbb{R}^+$ with vertices $(x,t),(x-ct,0)$ and $(x+ct,0)$, and denote it by $T(x,t)$. Thus,

$$T(x,t) := \{(y,s) \in \mathbb{R} \times (0,t) \mid |y-x| < c(t-s)\}.$$

The boundary of the triangle $\partial T(x,t)$ consists of three parts

$$T_0 := \{(y,0) \mid x-ct < y < x+ct\},$$

$$T_+ := \{(y,s) \in \mathbb{R} \times (0,t) \mid y+cs = x+ct\}$$

and

$$T_- := \{(y,s) \in \mathbb{R} \times (0,t) \mid y-cs = x-ct\}.$$

Integrating both sides of the wave equation in the Triangle $T(x,t)$ and using Gauss divergence theorem (cf. (B.1)), we get

$$\begin{aligned}
\int_{T(x,t)} f((y,s)\,dy\,ds &= \int_{T(x,t)} \left[u_{tt}(y,s) - c^2 u_{xx}(y,s)\right]\,dy\,ds \\
&= \int_{T(x,t)} \operatorname{div}(-c^2 u_x, u_t)(y,s)\,dy\,ds \\
&= \int_{\partial T(x,t)} \left[u_t \nu_2 - c^2 u_x \nu_1\right]\,d\sigma \\
&= \int_{T_0} \left[u_t \nu_2 - c^2 u_x \nu_1\right]\,d\sigma + \int_{T_+} \left[u_t \nu_2 - c^2 u_x \nu_1\right]\,d\sigma \\
&\quad + \int_{T_-} \left[u_t \nu_2 - c^2 u_x \nu_1\right]\,d\sigma
\end{aligned}$$

The parametrisation (with orientation) of T_0 is $(x-ct+2cs,0)$, T_+ is $(x+ct-cs,s)$ and T_- is $(x-cs,t-s)$ with tangent vectors of T_+ and T_- is $(-c,1)$ and $(-c,-1)$, respectively. The unit outward normal at each point of the boundary $\partial T(x,t)$ is given by $\nu = (\nu_1,\nu_2)$ defined as

$$\nu(y,s) = \begin{cases} (0,-1) & (y,s) \in T_0 \\ \frac{1}{\sqrt{1+c^2}}(1,c) & (y,s) \in T_+ \\ \frac{1}{\sqrt{1+c^2}}(-1,c) & (y,s) \in T_-. \end{cases}$$

Using the normal vector informations, we get

$$\begin{aligned}
\int_{T(x,t)} f((y,s)\,dy\,ds = &-\int_{x-ct}^{x+ct} u_t(y,0)\,dy + \frac{c}{\sqrt{1+c^2}} \int_{T_+} [u_t - cu_x]\,d\sigma \\
&+ \frac{c}{\sqrt{1+c^2}} \int_{T_-} [u_t + cu_x]\,d\sigma
\end{aligned}$$

Note that the second and third integral are the directional derivatives of u along the tangential direction $(-c,1)$ and reverse oriented tangential direction $(c,1)$ integrated on the line T_+ and T_-, respectively. Thus, the line integrals can be written in its parametrised integral form. Therefore, we have

$$\begin{aligned}
\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds &= -\int_{x-ct}^{x+ct} u_1(y)\,dy \\
&\quad + \frac{c}{\sqrt{1+c^2}} \int_0^t u'(x+ct-cs,s)\sqrt{c^2+1}\,ds \\
&\quad - \frac{c}{\sqrt{1+c^2}} \int_0^t u'(x-cs,t-s)\sqrt{c^2+1}\,ds \\
&= -\int_{x-ct}^{x+ct} u_1(y)\,dy + c[u(x,t)-u(x+ct,0)] \\
&\quad + c[u(x,t)-u(x-ct,0)] \\
u(x,t) &= \frac{1}{2}[u_0(x+ct)+u_0(x-ct)] \\
&\quad + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} u_1(y)\,dy + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\right].
\end{aligned}$$

Proof (Aliter). We introduce a new function v defined as $v(x,t) = u_t(x,t)$ and rewrite (5.9) as

$$U'(x,t) + AU(x,t) = F(x,t)$$

where $U = (u,v)$, $F = (0,f)$ and

$$A = \begin{pmatrix} 0 & -1 \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

with the initial condition $G(x) := U(x,0) = (u_0(x), u_1(x))$. The solution $U(x,t)$ is given as (cf. Appendix **??**)

$$U(x,t) = S(t)G(x) + \int_0^t S(t-s)F(s)\,ds$$

where $S(t)$ is a solution operator of the homogeneous system of first order PDE. Therefore, by d'Alembert's formula,

$$S(t)(u_0,u_1) = \begin{pmatrix} \frac{1}{2}[u_0(x+ct)+u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy \\ \frac{c}{2}[u_0'(x+ct)-u_0'(x-ct)] + \frac{1}{2}[u_1(x+ct)+u_1(x-ct)] \end{pmatrix}$$

and, hence, $u(x,t)$ has the required represenation.

Theorem 5.5 (Dirichlet Condition). *Given $u_0 \in C^2[0,\infty)$, $u_1 \in C^1[0,\infty)$ and $u_2 \in C^2(0,\infty)$, there is a unique C^2 solution u of the homogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } (0,\infty)\times(0,\infty) \\ u(x,0) = u_0(x) & \text{in } [0,\infty) \\ u_t(x,0) = u_1(x) & \text{in } [0,\infty) \\ u(0,t) = u_2(t) & \text{in } (0,\infty), \end{cases} \tag{5.10}$$

where u_0, u_1, u_2 satisfies the compatibility condition

$$u_2(0) = u_0(0), u_2'(0) = u_1(0), u_2''(0) = u_0''(0).$$

Proof. If $u_2 \equiv 0$, i.e. the case of homogeneous Dirichlet conditions, then we can extend u_0 and u_1 as an odd function on $(-\infty, \infty)$ by setting, for $i = 0, 1$,

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & x \geq 0 \\ -u_i(-x) & x < 0. \end{cases}$$

Then, we have a unique solution $\tilde{u} \in C^2(\mathbb{R} \times (0, \infty))$ solving the Cauchy problem of wave equation in the upper half-plane $\mathbb{R} \times (0, \infty)$. But note that $v(x,t) := -\tilde{u}(-x,t)$ is also a solution to the Cauchy problem of wave equation in $\mathbb{R} \times (0, \infty)$. By uniqueness of the Cauchy problem of wave equation, we obtain that $\tilde{u}$ is an odd function in the first variable, i.e. $\tilde{u}(x,t) = -\tilde{u}(-x,t)$. Thus, $\tilde{u}(0,t) = 0$ and the restriction of $\tilde{u}$ to $(0, \infty) \times (0, \infty)$ is a solution to the homogeneous Dirichlet problem of wave equation.

More generally, when u_2 is not necessarily the constant zero function, then the line $ct = x$ divides the domain $(0, \infty) \times (0, \infty)$ in to two domains

$$\Omega_r := \{(x,t) \mid x > ct > 0\} \text{ and } \Omega_\ell := \{(x,t) \mid 0 < x < ct\}.$$

For $(x,t) \in \Omega_r$, the solution is

$$U_r(x,t) := \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy.$$

On the line $x = ct$, we get

$$\chi(x) := U_r(x, x/c) = \frac{1}{2}\left(u_0(0) + u_0(2x)\right) + \frac{1}{2c}\int_0^{2x} u_1(y)\,dy.$$

Let U_ℓ be the solution in Ω_ℓ of

$$\begin{cases} \partial_t^2 U_\ell(x,t) - c^2 \partial_x^2 U_\ell(x,t) = 0 & \text{in } \Omega_\ell \\ U_\ell(x, x/c) = \chi(x) & \text{in } \{x = ct\} \\ U_\ell(0,t) = u_2(t) & \text{in } (0, \infty). \end{cases}$$

Fix $A := (x,t) \in \Omega_\ell$. One of the characteristic curve through A intersects t-axis at $B := (0, t - x/c)$. The other characteristic curve intersects the line $ct = x$ at $C := \frac{1}{2}(ct + x, t + x/c)$. The characteristic curve through B intersects $ct = x$ at $D := \frac{1}{2}(ct - x, t - x/c)$. The four points form a parallelogram in Ω_ℓ. Therefore, by Remark 5.3, we have

$$U_\ell(x,t) + U_\ell\left(\frac{1}{2}(ct - x, t - x/c)\right) = U_\ell(0, t - x/c) + U_\ell\left(\frac{1}{2}(ct + x, t + x/c)\right).$$

Thus,

$$\begin{aligned}
U_\ell(x,t) &= u_2(t-x/c) + \chi\left(\frac{1}{2}(ct+x)\right) - \chi\left(\frac{1}{2}(ct-x)\right) \\
&= u_2(t-x/c) + \frac{1}{2}\left[u_0(0)+u_0(ct+x)\right] + \frac{1}{2c}\int_0^{ct+x} u_1(y)\,dy \\
&\quad -\frac{1}{2}\left[u_0(0)+u_0(ct-x)\right] - \frac{1}{2c}\int_0^{ct-x} u_1(y)\,dy
\end{aligned}$$

and

$$U_\ell(x,t) = u_2(t-x/c) + \frac{1}{2}\left[u_0(ct+x)-u_0(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} u_1(y)\,dy.$$

By setting

$$u(x,t) = \begin{cases} U_r(x,t) & \text{for } x \geq ct \geq 0 \\ U_\ell(x,t) & \text{for } 0 \leq x \leq ct. \end{cases} \tag{5.11}$$

The compatibility condition on u_0, u_1 and u_2 implies that all derivatives of u are continuous across the line $ct = x$. Thus, u is a solution to (5.10).

Corollary 5.1 (Dirichlet Condition). *Given $u_0 \in C^2[0,L]$, $u_1 \in C^1[0,L]$ and $u_2, u_3 \in C^2(0,\infty)$, there is a unique C^2-solution u of the homogeneous Cauchy initial value problem of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,\infty) \\ u(x,0) = u_0(x) & \text{in } [0,L] \\ u_t(x,0) = u_1(x) & \text{in } [0,L] \\ u(0,t) = u_2(t) & \text{in } (0,\infty) \\ u(L,t) = u_3(t) & \text{in } (0,\infty), \end{cases} \tag{5.12}$$

where u_0, u_1, u_2, u_3 satisfies the compatibility condition

$$u_2(0) = u_0(0), = u_2'(0) = u_1(0), u_2''(0) = u_0''(0)$$

and

$$u_3(L) = u_0(0), u_3'(0) = u_1(L), u_3''(0) = u_0''(L).$$

Proof. We first consider the case $u_2 = u_3 \equiv 0$, i.e. the case of homogeneous Dirichlet conditions. We extend u_0 and u_1 as an odd function on $[-L,L]$ and extend it to all of $\mathbb{R}$ as a $2L$-periodic function. The extended initial data u_0 and u_1 are in $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$, respectively. Thus, we have a unique solution $u \in C^2(\mathbb{R}\times(0,\infty))$ solving the Cauchy problem for wave equation in $\mathbb{R}\times(0,\infty)$. But note that $v(x,t) = -u(-x,t)$ is also a solution to the Cauchy problem in $\mathbb{R}\times(0,\infty)$. By uniqueness, $u(x,t) = -u(-x,t)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ and $u(0,t) = 0$. Similarly, $w(x,t) = -u(2L-x,t)$ is also a solution to the Cauchy problem in $\mathbb{R}\times(0,\infty)$. Thus, $u(x,t) = -u(2L-x,t)$ which implies $u(L,t) = 0$. Thus, the restriction of u to $(0,L)\times(0,\infty)$ is a required solution to the homogeneous Dirichlet case.

Consider the lines $ct = x$ and $ct = -x + cL$, then we will obtain u in the four regions as v_1, v_2, v_3 and v_4. Then follow the idea similar to the proof in above theorem.

Theorem 5.6 (Neumann Condition). *Given $u_0 \in C^2[0,\infty)$, $u_1 \in C^1[0,\infty)$ and $u_2 \in C^2(0,\infty)$, there is a unique C^2 solution u of the homogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } (0,\infty) \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } [0,\infty) \\ u_t(x,0) = u_1(x) & \text{in } [0,\infty) \\ u_x(0,t) = u_2(t) & \text{in } (0,\infty), \end{cases} \tag{5.13}$$

where u_0, u_1, u_2 satisfies the compatibility condition

$$u_0'(0) = u_2(0), u_0''(0) = u_2'(0), u_1(0) = u_2'(0).$$

Corollary 5.2 (Neumann Condition). *Given $u_0 \in C^2[0,L]$, $u_1 \in C^1[0,L]$ and $u_2, u_3 \in C^2(0,\infty)$, there is a unique C^2-solution u of the homogeneous Cauchy initial value problem of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } (0,L) \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } [0,L] \\ u_t(x,0) = u_1(x) & \text{in } [0,L] \\ u_x(0,t) = u_2(t) & \text{in } (0,\infty) \\ u_x(L,t) = u_3(t) & \text{in } (0,\infty), \end{cases} \tag{5.14}$$

where u_0, u_1, u_2, u_3 satisfies the compatibility condition

$$u_0'(0) = u_2(0), u_0''(0) = u_2'(0), u_1(0) = u_2'(0)$$

and

$$u_0'(L) = u_3(0), u_0''(L) = u_3'(0), u_1(L) = u_3'(0).$$

5.3.3 Standing Waves: Separation of Variable

Recall the set-up of the vibrating string given by the equation $u_{tt} = c^2 u_{xx}$. Initially at time t, let us say the string has the shape of the graph of v, i.e., $u(x,0) = v(x)$. The snapshot of the vibrating string at each time are called the "standing waves". The shape of the string at time t_0 can be thought of as some factor (depending on time) of v. This observation motivates the idea of "separation of variable", i.e., $u(x,t) = v(x)w(t)$, where $w(t)$ is the factor depending on time, which scales v at time t to fit with the shape of $u(x,t)$.

The fact that endpoints are fixed is given by the boundary condition

$$u(0,t) = u(L,t) = 0.$$

We are also given the initial position $u(x,0) = u_0(x)$ (at time $t = 0$) and initial velocity of the string at time $t = 0$, $u_t(x,0) = u_1(x)$. Given $u_0, u_1 : [0,L] \to \mathbb{R}$ such that $u_0(0) = u_0(L) = 0$ and $u_1(0) = u_1(L)$, we need to solve the initial value problem (5.12) with $u_2 = u_3 \equiv 0$.

Let us seek for solutions $u(x,t)$ whose variables can be separated. Let $u(x,t) = v(x)w(t)$. Differentiating and substituting in the wave equation, we get

$$v(x)w''(t) = c^2 v''(x)w(t)$$

Hence

$$\frac{w''(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since RHS is a function of x and LHS is a function t, they must equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t)} = \lambda.$$

Using the boundary condition $u(0,t) = u(L,t) = 0$, we get

$$v(0)w(t) = v(L)w(t) = 0.$$

If $w \equiv 0$, then $u \equiv 0$ and this cannot be a solution to (5.12). Hence, $w \not\equiv 0$ and $v(0) = v(L) = 0$. Thus, we need to solve the *eigen value problem* for the second order differential operator.

$$\begin{cases} v''(x) = \lambda v(x), \ x \in (0,L) \\ v(0) = v(L) \quad = 0, \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(L) = 0$ and $L \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this cannot be a solution to (5.12).

If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda}x).$$

Now using $v(L) = 0$, we have $c_2 \sinh \sqrt{\lambda}L = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

Finally, if $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve the simple harmonic oscillator problem

$$\begin{cases} v''(x) + \omega^2 v(x) = 0 & x \in (0, L) \\ v(0) = v(L) = 0. \end{cases}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(L) = 0$, we have $\beta \sin \omega L = 0$. Thus, either $\beta = 0$ or $\sin \omega L = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega L = k\pi$ or $\omega = k\pi/L$, for all non-zero $k \in \mathbb{Z}$. Since $\omega > 0$, we can consider only $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for the eigen value problem with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{L}\right),$$

for some constant b_k and $\lambda_k = -(k\pi/L)^2$. It now remains to solve w for each of these λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$w_k''(t) + (ck\pi/L)^2 w_k(t) = 0.$$

The general solution is

$$w_k(t) = a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right).$$

For each $k \in \mathbb{N}$, we have

$$u_k(x,t) = \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right)$$

for some constants a_k and b_k. The situation corresponding to $k = 1$ is called the *fundamental mode* and the *frequency* of the fundamental mode is

$$\frac{c\sqrt{-\lambda_1}}{2\pi} = \frac{1}{2\pi}\frac{c\pi}{L} = \frac{c}{2L} = \frac{\sqrt{T/\rho}}{2L}.$$

The frequency of higher modes are integer multiples of the fundamental frequency. Note that the frequency of the vibration is related to eigenvalues of the second order differential operator.

The general solution of (5.12), by principle of superposition, is

$$u(x,t) = \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right).$$

Note that the solution is expressed as series, which raises the question of convergence of the series. Another concern is whether all solutions of (5.12) have this form. We ignore these two concerns at this moment.

Since we know the initial position of the string as the graph of u_0, we get

$$u_0(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right).$$

This expression is again troubling and rises the question: Can any arbitrary function u_0 be expressed as an infinite sum of trigonometric functions? Answering this question led to the study of "Fourier series". Let us also, as usual, ignore this concern for time being. Then, can we find the the constants a_k with knowledge of u_0. By multiplying $\sin\left(\frac{l\pi x}{L}\right)$ both sides of the expression of u_0 and integrating from 0 to L, we get

$$\begin{aligned}\int_0^L u_0(x)\sin\left(\frac{l\pi x}{L}\right)dx &= \int_0^L \left[\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right)\right]\sin\left(\frac{l\pi x}{L}\right)dx \\ &= \sum_{k=1}^{\infty} a_k \int_0^L \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi x}{L}\right)dx\end{aligned}$$

Therefore, the constants a_k are given as

$$a_k = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{k\pi x}{L}\right).$$

Finally, by differentiating u w.r.t t, we get

$$u_t(x,t) = \sum_{k=1}^{\infty} \frac{ck\pi}{L}\left[b_k \cos\frac{ck\pi t}{L} - a_k \sin\frac{ck\pi t}{L}\right]\sin\left(\frac{k\pi x}{L}\right).$$

Employing similar arguments and using $u_t(x,0) = u_1(x)$, we get

$$u_1(x) = u_t(x,0) = \sum_{k=1}^{\infty} \frac{b_k kc\pi}{L}\sin\left(\frac{k\pi x}{L}\right)$$

and hence

$$b_k = \frac{2}{kc\pi}\int_0^L u_1(x)\sin\left(\frac{k\pi x}{L}\right).$$

5.4 Method of Spherical Means

More generally, in this section we solve for $v+w$ which is the solution to the wave equation

$$\begin{cases} (\partial_t^2 - c^2\Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \\ u_t(x,0) = h(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{5.15}$$

For a fixed $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, the spherical mean of a $u \in C^2(\mathbb{R}^n \times (0,\infty))$, denoted as $M(u,x;\cdot,t) : (0,\infty) \to \mathbb{R}$, is defiend as

$$M(u,x;r,t) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y,t)\,d\sigma_y$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, i.e. $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ (cf. Appendix C). Equivalently, after setting $z = (y-x)/r$,

$$M(u,x;r,t) := \frac{1}{\omega_n} \int_{S_1(0)} u(x+rz,t)\,d\sigma_z.$$

We note from the above form of $M(u,x;r,t)$ that $M(u,x;\cdot,t)$ can be extended as an even function to the negative real line. Thus, $M(u,x;\cdot,t)$ is defined for all $r \in \mathbb{R}$ with $M(u,x;0,t) = u(x,t)$. Then

$$\begin{aligned} M_r(u,x;r,t) &= \frac{1}{\omega_n} \int_{S_1(0)} \sum_{i=1}^{n} u_{y_i}(x+rz,t) z_i\,d\sigma_z \\ &= \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla_y u(y,t)\cdot z\,d\sigma_y \\ &= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta_y u(y,t)\,dy. \end{aligned}$$

If $u(x,t)$ is a solution to (5.15) then

$$\begin{aligned} r^{n-1} M_r(u,x;r,t) &= \frac{1}{c^2\omega_n} \int_{B_r(x)} u_{tt}(y,t)\,dy \\ &= \frac{1}{c^2\omega_n} \int_0^r \int_{S_s(x)} u_{tt}(y,t)\,d\sigma_y\,ds \\ \frac{d}{dr}\left(r^{n-1} M_r(u,x;r,t)\right) &= \frac{1}{c^2\omega_n} \int_{S_r(x)} u_{tt}(y,t)\,d\sigma_y \\ &= \frac{r^{n-1}}{c^2} \partial_t^2 \left(\frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y,t)\,d\sigma_y \right) = \frac{r^{n-1}}{c^2} M_{tt}. \end{aligned}$$

Thus, the spherical means of u satisfies the one space variable PDE

$$r^{1-n} \frac{d}{dr}(r^{n-1} M_r) = c^{-2} M_{tt}$$

or

$$M_{rr} + \frac{n-1}{r} M_r = c^{-2} M_{tt},$$

called the *Euler-Poisson-Darboux* equation, a one dimensional hyperbolic equation. Also, if $u(x,t)$ is a solution to (5.15) then using the initial condition, we get

$$G(x;r) := M(u,x;r,0) = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} g(y)\,d\sigma_y$$

and

$$H(x;r) := M_t(u,x;r,0) = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} h(y)\, d\sigma_y.$$

In the following results, we shall illustrate solving the Euler-Poisson-Darboux equation in three and two dimensions. For higher dimensions solving E-P-D equation is tedious.

Theorem 5.7 (Three dimensions). *Given $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$ there exists a unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ of (5.15) given by the Poisson's formula*

$$u(x,t) = \frac{1}{4\pi c^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S_{ct}(x)} g(y)\, d\sigma_y\right) + \frac{1}{t}\int_{S_{ct}(x)} h(y)\, d\sigma_y\right].$$

Proof. For the Euclidean dimension three, the Euler-Poisson-Darboux equation can be rewritten as $c^2(rM)_{rr} = (rM)_{tt}$. Thus, rM is a solution to the one dimensional Cauchy problem

$$\begin{cases} c^2(rM)_{rr}(u,x;r,t) = (rM)_{tt}(u,x;r,t) & \text{in } (0,\infty)\times(0,\infty) \\ rM(u,x;r,0) = rG(x;r) & \text{on } (0,\infty) \\ rM_t(u,x;r,0) = rH(x;r) & \text{on } (0,\infty) \\ rM(u,x;0,t) = 0 & \text{on } (0,\infty). \end{cases}$$

By (5.11), for $0 < r < ct$, we have the formula

$$rM(u,x;r,t) = \frac{1}{2}\left((r+ct)G(x;r+ct) - (ct-r)G(x;ct-r)\right) + \frac{1}{2c}\int_{ct-r}^{ct+r} yH(x;y)\, dy.$$

Using the Leibniz intergal rule (cf. Theorem A.1), we can rewrite above formula as

$$rM(u,x;r,t) = \frac{1}{2c}\left[\partial_t\left(\int_{ct-r}^{ct+r} yG(x;y)\, dy\right) + \int_{ct-r}^{ct+r} yH(x;y)\, dy\right].$$

Then

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} M(u,x;r,t) = \lim_{r\to 0}\frac{rM(u,x;r,t)}{r} = \lim_{r\to 0}\frac{rM(u,x;r,t) - 0M(u,x;0,t)}{r} \\ &= \partial_r(rM(u,x;r,t))\,|_{r=0} \\ &= \frac{1}{2c}\partial_t\left[(ct+r)G(x;ct+r) + (ct-r)G(x;ct-r)\right]|_{r=0} \\ &\quad + \frac{1}{2c}\left[(ct+r)H(x;ct+r) + (ct-r)H(x;ct-r)\right]|_{r=0} \\ &= \partial_t\left[tG(x;ct)\right] + \left[tH(x;ct)\right]. \end{aligned}$$

Using the expression for G and H in terms of g and h, respectively, and the $\omega_3 = 4\pi$, we get the required formula.

The domain of dependence of (x,t_0) for the three dimensional wave equation is the boundary of the three dimensional sphere with radius ct.

The Hadamard's *method of descent* is the technique of finding a solution of the two dimensional wave equation using the three dimensional wave equation.

Theorem 5.8 (Method of Descent). *Given $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$ there exists a unique solution $u \in C^2(\mathbb{R}^2 \times [0,\infty)$ of (5.15) given by the Poisson's formula*

$$\frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\left(\int_{B_{ct}(x,y)} \frac{g(\xi,\eta)}{\sqrt{c^2t^2-\rho^2}}\,d\xi\,d\eta\right) + \int_{B_{ct}(x,y)} \frac{h(\xi,\eta)}{c^2t^2-\rho^2}\,d\xi\,d\eta\right]$$

where

$$\rho = \sqrt{(\xi-x)^2+(\eta-y)^2}.$$

Proof. Let v be a solution of (5.15) in two dimensions with $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then

$$u(x,y,z,t) := v(x,y,t)$$

is solution to (5.15) in three dimensions where g and h are given independent of z. Since $u(x,y,z,t) = u(x,y,0,t) + zu_z(x,y,\varepsilon z,t)$ for $0 < \varepsilon < 1$ and $u_z = 0$, we have $v(x,y,t) = u(x,y,0,t)$. Therefore, using the poisson formula in three dimensions, we get

$$v(x,y,t) = \frac{1}{4\pi c^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S_{ct}(x,y,0)} g(\xi,\eta,\zeta)\,d\sigma\right) + \frac{1}{t}\int_{S_{ct}(x,y,0)} h(\xi,\eta,\zeta)\,d\sigma\right].$$

Recall that g and h are independent of ζ, therefore $g(\xi,\eta,\zeta) = g(\xi,\eta)$ and $h(\xi,\eta,\zeta) = h(\xi,\eta)$. The sphere can be parametrised as th deformation of each point of the two dimensional disk $B_{ct}(x,y)$, i.e., the upper and lower hemi-sphere, denoted by S^+ and S^-, are given by the set of all ζ such that $\zeta = \sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}$ and $\zeta = -\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}$, respectively. The surface element (measure) becomes

$$d\sigma = \left(1+\zeta_\xi^2+\zeta_\eta^2\right)^{1/2} d\xi\,d\eta = \frac{ct}{\zeta}d\xi\,d\eta$$

with the positive sign applying when $\zeta > 0$ in S^+ and negative sign applying when $\zeta < 0$ in S^-. Set $\rho = \sqrt{(\xi-x)^2+(\eta-y)^2}$. We obtain the formula for the solution by observing that

$$\int_{S_{ct}(x,y,0)} d\sigma = \int_{S^+} + \int_{S^-} = 2\int_{S^+} d\sigma = 2\int B_{ct}(x,y)\frac{ct}{\zeta}d\xi\,d\eta.$$

In the two dimensions, the domain of dependence is the entire disk $B_{ct_0}(x_0,y_0)$ in contrast to three dimensions which had only the boundary of the sphere as domain of dependence.

5.4.1 Odd Dimension

One can copy the idea of three dimension to any odd dimension, if we rewrite the Euler-Poisson-Darboux equation in approriate form.

Exercise 5.1. If n is odd, show that the correct form of M that satisfies the one dimensional wave equation is

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}(r^{n-2}M(r,t)).$$

For instance, when $n=5$, r^2M_r+3rM satisfies the one dimensional wave equation.

We have already noted that the solution at a given point is determined by the value of intial data in a subset of the initial hypersurface. Consequently, changing initial value outside the domain of dependence does not change values of solutions.

Also, it takes time for the initial data to make influence. Suppose g and h have their support in $B_r(x_0)$. Then the support of $u(\cdot,t)$ is contained in $\cup_{y\in B_r(x_0)}B_t(y)=B_{r+ct}(x_0)$. The support of u spreads at a finite speed and is called the *finite speed propagation*.

5.4.2 Inhomogeneous Wave equation

We have already derived in Theorem 5.1 the formula for inhomogeneous equation (5.4).

Theorem 5.9. *For any $f\in C^1$, the solution $u(x,t)$ of (5.4) is given as*

$$u(x,t)=\begin{cases}\frac{1}{2c}\int_0^t\left(\int_{x-c(t-s)}^{x+c(t-s)}f(y,s)dy\right)ds & n=1\\ \frac{1}{4\pi c}\int_0^t\left(\int_{B_{c(t-s)}(x)}\frac{f(y,s)}{\sqrt{c^2(t-s)^2-r^2}}dy\right)ds & n=2\\ \frac{1}{4\pi c^2}\int_{B_{ct}(x)}\frac{f(y,t-\frac{|x-y|}{c})}{|x-y|}dy & n=3.\end{cases}$$

Proof. The Poisson's formula corresponding to the three dimension case gives the formula for

$$w^{f_s}(x,t-s)=\frac{1}{4\pi c^2(t-s)}\int_{S_{c(t-s)}(x)}f(y,s)\,d\sigma_y.$$

and

$$\begin{aligned}
u(x,t) &= \int_0^t w^{f_s}(x,t-s)\,ds = \frac{1}{4\pi c^2}\int_0^t \int_{S_{c(t-s)}(x)} \frac{f(y,s)}{t-s}\,d\sigma_y\,ds \\
&= \frac{1}{4\pi c^2}\int_0^{ct}\int_{S_\tau(x)} \frac{f(y,t-\tau/c)}{\tau}\,d\sigma_y\,d\tau \quad [\text{using } \tau = c(t-s)] \\
&= \frac{1}{4\pi c^2}\int_{B_{ct}(x)} \frac{f(y,t-|x-y|/c)}{|x-y|}\,dy.
\end{aligned}$$

Similarly, one can derive the formulae for one and two dimensions.

Note that in the three dimensional case the integrand is not taken at time t, but at an earlier time $t-\frac{|x-y|}{c}$. Thus, the integrand in this case is called *retarded potential*.

Example 5.1. Consider the wave equation

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = \sin 3x & \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{in } (0,\infty) \\ u(x,0) = u_t(x,0) = 0 & \text{in } (0,\pi). \end{cases}$$

We look for the solution of the homogeneous wave equation

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0 & \text{in } (0,\pi)\times(0,\infty) \\ w(0,t) = w(\pi,t) = 0 & \text{in } (0,\infty) \\ w(x,0) = 0 & \text{in } (0,\pi) \\ w_t(x,0) = \sin 3x & \text{in } (0,\pi). \end{cases}$$

By separation of variable technique, we know that the general solution of w is

$$w(x,t) = \sum_{k=1}^{\infty} [a_k \cos(kct) + b_k \sin(kct)] \sin(kx)$$

and

$$w(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx) = 0.$$

Thus, $a_k = 0$, for all $k \in \mathbb{N}$. Also,

$$w_t(x,0) = \sum_{k=1}^{\infty} b_k ck \sin(kx) = \sin 3x.$$

Hence, b_k's are all zeroes, except for $k = 3$ and $b_3 = 1/3c$. Thus,

$$w(x,t) = \frac{1}{3c}\sin(3ct)\sin(3x)$$

and

$$\begin{aligned}
u(x,t) &= \int_0^t w(x,t-s)\,ds = \frac{1}{3c}\int_0^t \sin(3c(t-s))\sin 3x\,ds \\
&= \frac{\sin 3x}{3c}\int_0^t \sin(3c(t-s))\,ds = \frac{\sin 3x}{3c}\frac{\cos(3c(t-s))}{3c}\Big|_0^t \\
&= \frac{\sin 3x}{9c^2}\left(1-\cos 3ct\right).
\end{aligned}$$

5.5 Eigenvalue Problem of Laplacian

The separation of variable technique can be used for studying wave equation on 2D Rectangle and 2D Disk etc. This leads to studying the eigen value problem of the Laplacian. For a given open bounded subset $\Omega \subset \mathbb{R}^2$, the Dirichlet eigenvalue problem,

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y) & (x,y) \in \Omega \\ \quad u(x,y) = 0 & (x,y) \in \partial\Omega. \end{cases}$$

Note that, for all $\lambda \in \mathbb{R}$, zero is a trivial solution of the Laplacian. Thus, we are interested in non-zero λ's for which the Laplacian has non-trivial solutions. Such an λ is called the *eigenvalue* and corresponding solution u_λ is called the *eigen function*.

Note that if u_λ is an eigen function corresponding to λ, then αu_λ, for all $\alpha \in \mathbb{R}$, is also an eigen function corresponding to λ. Let W be the real vector space of all $u : \Omega \to \mathbb{R}$ continuous (smooth, as required) functions such that $u(x,y) = 0$ on $\partial\Omega$. For each eigenvalue λ of the Laplacian, we define the subspace of W as

$$W_\lambda = \{u \in W \mid u \text{ solves Dirichlet EVP for given } \lambda\}.$$

Theorem 5.10. *There exists an increasing sequence of positive numbers $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$ with $\lambda_n \to \infty$ which are eigenvalues of the Laplacian and $W_n = W_{\lambda_n}$ is finite dimensional. Conversely, any solution u of the Laplacian is in W_n, for some n.*

Though the above theorem assures the existence of eigenvalues for Laplacian, it is usually difficult to compute them for a given Ω. In this course, we shall compute the eigenvalues when Ω is a 2D-rectangle and a 2D-disk.

5.5.1 In Rectangle

Let the rectangle be $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < a, 0 < y < b\}$. We wish to solve the Dirichlet EVP in the rectangle Ω

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y) & (x,y) \in \Omega \\ \quad u(x,y) = 0 & (x,y) \in \partial\Omega. \end{cases}$$

The boundary condition amounts to saying

$$u(x,0) = u(a,y) = u(x,b) = u(0,y) = 0.$$

We look for solutions of the form $u(x,y) = v(x)w(y)$ (variable separated). Substituting u in separated form in the equation, we get

$$-v''(x)w(y) - v(x)w''(y) = \lambda v(x)w(y).$$

Hence

$$-\frac{v''(x)}{v(x)} = \lambda + \frac{w''(y)}{w(y)}.$$

Since LHS is function of x and RHS is function y and are equal they must be some constant, say μ. We need to solve the EVP's

$$-v''(x) = \mu v(x) \quad \text{and } -w''(y) = (\lambda - \mu)w(y)$$

under the boundary conditions $v(0) = v(a) = 0$ and $w(0) = w(b) = 0$.

As seen before, while solving for v, we have trivial solutions for $\mu \leq 0$. If $\mu > 0$, then $v(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x)$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$. Now using $v(a) = 0$, we have $c_2 \sin\sqrt{\mu}a = 0$. Thus, either $c_2 = 0$ or $\sin\sqrt{\mu}a = 0$. We have non-trivial solution, if $c_2 \neq 0$, then $\sqrt{\mu}a = k\pi$ or $\sqrt{\mu} = k\pi/a$, for $k \in \mathbb{Z}$. For each $k \in \mathbb{N}$, we have $v_k(x) = \sin(k\pi x/a)$ and $\mu_k = (k\pi/a)^2$. We solve for w for each μ_k. For each $k,l \in \mathbb{N}$, we have $w_{kl}(y) = \sin(l\pi y/b)$ and $\lambda_{kl} = (k\pi/a)^2 + (l\pi/b)^2$. For each $k,l \in \mathbb{N}$, we have

$$u_{kl}(x,y) = \sin(k\pi x/a)\sin(l\pi y/b)$$

and $\lambda_{kl} = (k\pi/a)^2 + (l\pi/b)^2$.

5.5.2 In Disk

Let the disk of radius a be $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$. We wish to solve the Dirichlet EVP in the disk Ω

$$\begin{cases} \frac{-1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \lambda u(r,\theta) & (r,\theta) \in \Omega \\ u(\theta) = u(\theta + 2\pi) & \theta \in \mathbb{R} \\ u(a,\theta) = 0 & \theta \in \mathbb{R}. \end{cases}$$

We look for solutions of the form $u(r,\theta) = v(r)w(\theta)$ (variable separated). Substituting u in separated form in the equation, we get

$$-\frac{w}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{v}{r^2}w''(\theta) = \lambda v(r)w(\theta).$$

Hence dividing by vw and multiplying by r^2, we get

$$-\frac{r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{1}{w}w''(\theta) = \lambda r^2.$$

$$\frac{r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \lambda r^2 = \frac{-1}{w}w''(\theta) = \mu.$$

Solving for non-trivial w, using the periodicity of w, we get for $\mu_0 = 0$, $w_0(\theta) = \frac{a_0}{2}$ and for each $k \in \mathbb{N}$, $\mu_k = k^2$ and

$$w_k(\theta) = a_k \cos k\theta + b_k \sin k\theta.$$

For each $k \in \mathbb{N} \cup \{0\}$, we have the equation,

$$r\frac{d}{dr}\left(r\frac{dv}{dr}\right) + (\lambda r^2 - k^2)v = 0.$$

Introduce change of variable $x = \sqrt{\lambda}r$ and $x^2 = \lambda r^2$. Then

$$r\frac{d}{dr} = x\frac{d}{dx}.$$

rewriting the equation in new variable $y(x)) = v(r)$

$$x\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right) + (x^2 - k^2)y(x) = 0.$$

Note that this none other than the *Bessel's equation*. We already know that for each $k \in \mathbb{N} \cup \{0\}$, we have the Bessel's function J_k as a solution to the Bessel's equation. Recall the boundary condition on v, $v(a) = 0$. Thus, $y(\sqrt{\lambda}a) = 0$. Hence $\sqrt{\lambda}a$ should be a zero of the Bessel's function.

For each $k \in \mathbb{N} \cup \{0\}$, let z_{kl} be the l-th zero of J_k, $l \in \mathbb{N}$. Hence $\sqrt{\lambda}a = z_{kl}$ and so $\lambda_{kl} = z_{kl}^2/a^2$ and $y(x) = J_k(x)$. Therefore, $v(r) = J_k(z_{kl}r/a)$. For each $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$, we have

$$u_{kl}(r,\theta) = J_k(z_{kl}r/a)\sin(k\theta) \text{ or } J_k(z_{kl}r/a)\cos(k\theta)$$

and $\lambda_{kl} = z_{kl}^2/a^2$.

Chapter 6
Heat Equation

In 1822, J. Fourier introduced in *Théorie analytique de la chaleur* the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t),$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. This heat flow model was based on Newton's law of cooling.

6.1 Derivation of Heat Equation

The heat propagation in a bar of length L is

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x)\sigma(x)} \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$$

where $\sigma(x)$ is the *specific heat* at x, $\rho(x)$ is *density* of bar at x and $\kappa(x)$ is the *thermal conductivity* of the bar at x. If the bar is *homogeneous*, i.e, its properties are same at every point, then

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma}\frac{\partial^2 u}{\partial x^2}$$

with ρ, σ, κ being constants.

Let a homogeneous material occupy a region represented by an open, bounded subset $\Omega \subset \mathbb{R}^n$ with C^1 boundary. Let κ denote the thermal conductivity (dimensionless quantity) and c be the heat capacity of the material. Let $u(x,t)$ denote the temperature of the material Ω at $x \in \Omega$ and time $t \in [0,\infty)$. The thermal energy stored at $x \in \Omega$ and time $t \in [0,\infty)$ is $cu(x,t)$. If $\mathbf{v}(x,t)$ denotes the velocity at (x,t) then, by Fourier law, the thermal energy changes according to the gradients of temperature, i.e., $cu(x,t)\mathbf{v}(x,t) = -\kappa\nabla u(x,t)$. The thermal energy is the quantity that is conserved (conservation law) and satisfies the continuity equation (1.4). Thus, we have

$$u_t(x,t) - \frac{\kappa}{c}\Delta u(x,t) = 0.$$

If the material occupying the region Ω is non-homogeneous, anisotropic, the temperature gradient may generate heat in preferred directions, which themselves may depend on $x \in \Omega$. Thus, the conductivity of such a material at $x \in \Omega$ is given by a $n \times n$ matrix $K(x) = (\kappa_{ij}(x))$. Thus, in this case, the heat equation becomes,

$$u_t(x,t) - \operatorname{div}_x\left(\frac{1}{c}K(x)\nabla_x u(x,t)\right) = 0.$$

The heat equation is an example of a second order equation in divergence form. The heat equation gives the temperature distribution $u(x,t)$ of the material with conductivity κ and capacity c. In general, we may choose $\kappa/c = 1$, since, for any κ and c, we may rescale our time scale $t \mapsto (\kappa/c)t$.

Remark 6.1 (Time Irreversiblity). The heat equation describes irreversible process, i.e., it is not possible to find the distribution of temperature at an earlier time $t < t_0$, if the temperature distribution is given at t_0. Equivalently, the transformation $t \mapsto -t$ changes the heat equation to a *backward equation*. If set $v(x,t) = u(x,-t)$ where $u_t - \Delta u = 0$, then v satisfies the backward equation $v_t + \Delta v = 0$.

Remark 6.2 (Invariance in space and time). For each fixed (y,τ), if u satisfies $u_t - \Delta u = 0$ then $v(x,t) := u(x-y, t-\tau)$ also satisfies $v_t - \Delta v = 0$.

Remark 6.3 (Parabolic Dilations). If u satisfies $u_t - \Delta u = 0$ then $v(x,t) = \beta u(\lambda x, \lambda^2 t)$ satisfies $v_t - \Delta v = 0$ for all $\lambda, \beta > 0$.

6.2 Duhamel's Principle

In this section we solve the inhomogeneous heat equation, using Duhamel's principle. The *Duhamel's principle* states that one can obtain a solution of the inhomogeneous IVP for heat from its homogeneous IVP.

For a given f, let $u(x,t)$ be the solution of the *inhomogeneous* heat equation,

$$\begin{cases} u_t(x,t) - c^2\Delta u(x,t) = f(x,t) & \text{in } \Omega \times (0,T) \\ u(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \\ u(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{6.1}$$

As a first step, for each $s \in (0,\infty)$, consider $w(x,t;s)$ as the solution of the *homogeneous* problem (auxiliary)

$$\begin{cases} w^s_t(x,t) - c^2\Delta w^s(x,t) = 0 & \text{in } \Omega \times (s,T) \\ w^s(x,t) = 0 & \text{in } \partial\Omega \times (s,T) \\ w^s(x,s) = f(x,s) & \text{on } \Omega \times \{s\}. \end{cases}$$

Since $t \in (s,T)$, introducing a change of variable $r = t - s$, we have $w^s(x,t) = w(x,t-s)$ which solves

$$\begin{cases} w_t(x,r) - c^2 \Delta w(x,r) = 0 & \text{in } \Omega \times (0, T-s) \\ w(x,r) = 0 & \text{in } \partial\Omega \times (0, T-s) \\ w(x,0) = f(x,s) & \text{on } \Omega. \end{cases}$$

Theorem 6.1 (Duhamel's Principle). *The function* $u(x,t)$ *defined as*

$$u(x,t) := \int_0^t w^s(x,t)\, ds = \int_0^t w(x,t-s)\, ds$$

solves (6.1).

Proof. Suppose w is $C^{2,1}(\mathbb{R}^n \times (0,T))$, we get

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t} \int_0^t w(x,t-s)\, ds \\ &= \int_0^t w_t(x,t-s)\, ds + w(x,t-t) \frac{d(t)}{dt} \\ &\quad - w(x,t-0) \frac{d(0)}{dt} \\ &= \int_0^t w_t(x,t-s)\, ds + w(x,0) \\ &= \int_0^t w_t(x,t-s)\, ds + f(x,t). \end{aligned}$$

Similarly,

$$\Delta u(x,t) = \int_0^t \Delta w(x,t-s)\, ds.$$

Thus,

$$\begin{aligned} u_t - c^2 \Delta u &= f(x,t) + \int_0^t \big(w_t(x,t-s) - c^2 \Delta w(x,t-s) \big)\, ds \\ &= f(x,t). \end{aligned}$$

6.3 Cauchy Problem for Bounded Domains

We have already elaborated on the way to choose boundary conditions (q.v. Section 2.3.3 and Remark 3.4). In view of those discussions, let Ω be an open subset of $\mathbb{R}^n$ with a non-empty boundary $\partial\Omega$. The Cauchy problem for heat equation

$$\begin{cases} u_t(x,t) - c^2 \Delta u(x,t) = 0 & \text{in } \Omega \times (0,T) \\ u(x,0) = g(x) & \text{on } \overline{\Omega} \times \{0\} \end{cases}$$

is well-posed if we choose to specify one of the following conditions on $\partial\Omega \times (0,T)$:

(i) (Dirichlet condition) $u(x,t) = h(x,t)$;
(ii) (Neumann condition) $\nabla_x u(x,t)\cdot\nu(x) = h(x,t)$, where $\nu(x)$ is the unit outward normal of $(x,t) \in \partial\Omega \times (0,T)$;
(iii) (Robin condition) $\nabla_x u(x,t)\cdot\nu + cu(x,t) = h(x,t)$ for any $c > 0$.
(iv) (Mixed condition) $u(x,t) = h(x,t)$ on Γ_1 and $\nabla_x u(x,t)\cdot\nu = h(x,t)$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega \times (0,T)$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Set $Q_T := \Omega \times (0,T)$ and

$$\partial_p Q_T := \overline{\Omega} \times \{t = 0\} \cup \partial\Omega \times (0,T].$$

Let $C^{2,1}(\overline{Q}_T)$ denote the class of all continuous functions which are twice continuously differentiable in the x-variable and once continuously differentiable in the t-variable.

Remark 6.4 (Steady State Equation). Consider the Cauchy problem with inhomogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t(x,t) - c^2\Delta u(x,t) = 0 & \text{in } \Omega \times (0,\infty) \\ u(x,t) = = h(x,t) & \text{in } \partial\Omega \times [0,\infty) \\ u(x,0) = g(x) & \text{on } \overline{\Omega} \end{cases}$$

such that, for all $x \in \partial\Omega$,

$$g(x) = h(x,0), g''(x) = h''(x,0).$$

The *steady-state* solution of the heat equation is defined as

$$v(x) = \lim_{t\to\infty} u(x,t).$$

Note that v satisfies the equation $\Delta v = 0$, since $v_t = 0$. Further, v satisfies the following condition on $\partial\Omega$:

$$v(x) = \lim_{t\to\infty} h(x,t).$$

The heat flows from higher to lower temperature regions. This is equivalent to saying that a solution of the homogeneous heat equation attains its maximum and minimum values on $\partial_p Q_T$. This is known as the maximum principle.

Theorem 6.2 (Weak Maximum Principle). *Let $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ be the solution of*

$$u_t(x,t) - \Delta u(x,t) = f(x,t) \text{ in } Q_T. \tag{6.2}$$

If $f(x,t) \le 0$ in Q_T then u attains its maximum on $\partial_p Q_T$, i.e.,

$$\max_{\overline{Q}_T} u(x,t) = \max_{\partial_p Q_T} u(x,t).$$

In particular, if u is negative on $\partial_p Q_T$ then it is negative in Q_T.

Proof. We already have

$$\max_{\overline{Q}_T} u(x,t) \geq \max_{\partial_p Q_T} u(x,t)$$

because $\partial_p Q_T \subset \overline{Q}_T$. It only remains to show the other inequality. We need to show the maximum of u is achieved in $\partial_p Q_T$. If not the maximum can be in the interior of Q_T or in the boundary $\Omega \times \{t = T\}$. To avoid the problem of differentiability at boundary, we consider $\varepsilon > 0$ such that $T - \varepsilon > 0$. We first claim that

$$\max_{\overline{Q}_{T-\varepsilon}} u \leq \max_{\partial_p Q_T} u + \varepsilon T. \tag{6.3}$$

Set $v = u - \varepsilon t$. Then

$$v_t(x,t) - \Delta v(x,t) = f(x,t) - \varepsilon < 0. \tag{6.4}$$

We claim that the maximum of v on $\overline{Q}_{T-\varepsilon}$ occurs on $\partial_p Q_{T-\varepsilon}$. Suppose not, then $(x_0,t_0) \in \Omega \times (0, T-\varepsilon]$ be a maximum point for v on $\overline{Q}_{T-\varepsilon}$. Thus, by the negative definiteness of Hessian matrix, $\Delta_x v(x_0,t_0) \leq 0$. Since $v(x_0,t_0) \geq v(x_0,t)$ for all $0 < t \leq T - \varepsilon$ being a maximum point,

$$v_t(x_0,t_0) := \lim_{h \to 0} \frac{v(x_0,t_0+h) - v(x_0,t_0)}{h} \leq 0.$$

For $t_0 < T - \varepsilon$, we obtain $v_t(x_0,t_0) = 0$. For the maximum attained at boundary $t_0 = T - \varepsilon$, $v_t(x_0,t_0) \geq 0$. In both cases,

$$v_t(x_0,t_0) - \Delta v(x_0,t_0) \geq 0$$

contradicting (6.4). Since $v \leq u$,

$$\max_{\overline{Q}_{T-\varepsilon}} v \leq \max_{\partial_p Q_{T-\varepsilon}} v \leq \max_{\partial_p Q_T} u. \tag{6.5}$$

On the other hand, $u = v + \varepsilon t \leq v + \varepsilon T$, and therefore, from (6.5) we get

$$\max_{\overline{Q}_{T-\varepsilon}} u \leq \max_{\overline{Q}_{T-\varepsilon}} v + \varepsilon T \leq \max_{\partial_p Q_T} u + \varepsilon T$$

which is 6.3. Since u is continuous in $\overline{Q}_T$, it is uniformly continuous and attains its maximum on $\overline{Q}_T$. Thus,

$$\max_{\overline{Q}_{T-\varepsilon}} u \to \max_{\overline{Q}_T} u \text{ as } \varepsilon \to 0.$$

Now, letting $\varepsilon \to 0$ in (6.3) we get $\max_{\overline{Q}_T} u \leq \max_{\partial_p Q_T} u$.

Corollary 6.1. *If $f(x,t) = 0$ in Q_T then u attains its maximum and its minimum on $\partial_p Q_T$. In particular,*

$$\min_{\partial_p Q_T} u \le u(x,t) \le \max_{\partial_p Q_T} u \quad \forall (x,t) \in Q_T.$$

Corollary 6.2 (Comparison and stability). *Let u and v satisfy*

$$u_t - \Delta u = f_1 \text{ and } v_t - \Delta v = f_2.$$

Then

(a) If $u \ge v$ on $\partial_p Q_T$ and $f_1 \ge f_2$ in Q_T then $u \ge v$ in all Q_T.
(b) The stability estimate holds:

$$\max_{\overline{Q}_T} |u-v| \le \max_{\partial_p Q_T} |u-v| + T \max_{\overline{Q}_T} |f_1 - f_2|. \tag{6.6}$$

In particular, the Dirichlet problem has at most one solution and depends continuously on the data.

Remark 6.5. The equation (6.6) is a uniform pointwise stability estimate, useful in several applications. In fact, if $u = g_1, v = g_2$ on $\partial_p Q_T$ and

$$\max_{\partial_p Q_T} |g_1 - g_2| \le \varepsilon \text{ and } \max_{\overline{Q}_T} |f_1 - f_2| \le \varepsilon$$

then

$$\max_{\overline{Q}_T} |u-v| \le \varepsilon(1+T).$$

Thus, in finite time, a small change in data implies a small change in the corresponding solutions.

Remark 6.6 (Strong Maximum Principle). The weak maximum principle gives no information about the solution achieving its maximum or minimum at an interior point too. The *strong maximum principle* states that if a solution of $u_t - \Delta u = 0$ achieves its maximum M (or minimum) at a point (x_1, t_1) with $x_1 \in \Omega$ and $0 < t_1 \le T$, then $u = M$ in $\overline{\Omega} \times [0, t_1]$.

Remark 6.7. Above Corollary gives uniqueness for the Dirichlet problem. One can show uniquenees without maximum principle but assuming continuity of derivatives of the solution up to $\partial_p Q_T$.

Theorem 6.3 (At most one solution). *Let Ω be a domain with C^1 boundary. The initial Dirichlet, Neumann, Robin and mixed problems have at most one solution in $C^{2,1}(\overline{Q}_T)$.*

Proof. Suppose u and v are solutions of one of the specified Cauchy problem. Set $w := u - v$. We claim that $w \equiv 0$. Observe that w satisfies the homogeneous equation

$$w_t(x,t) - \Delta w(x,t) = 0 \tag{6.7}$$

with initial condition $w(x,0) = 0.$ in $\overline{\Omega}$, and one of the homogeneous boundary conditions on $\partial\Omega \times (0,T]$. Multiply both sides of (6.7) by w and integrate on Ω, we get

$$\int_\Omega w w_t \, dx = \int_\Omega w \Delta w \, dx.$$

But LHS is, using Green's identity,

$$\int_\Omega w w_t \, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 \, dx.$$

Similarly, RHS is

$$\int_\Omega w \Delta w \, dx = \int_{\partial\Omega} w[\nabla w \cdot \nu] \, d\sigma - \int_\Omega |\nabla w|^2 \, dx.$$

By setting $F(t) := \int_\Omega w^2 \, dx$, and putting in LHS, we get

$$\frac{1}{2}F'(t) = \int_{\partial\Omega} w[\nabla w \cdot \nu] \, d\sigma - \int_\Omega |\nabla w|^2 \, dx.$$

If the Robin boundary condition is satisfied, then

$$\int_{\partial\Omega} w[\nabla w \cdot \nu] \, d\sigma = -c\int_\Omega w^2 \, dx \leq 0.$$

If one of the other boundary conditions are satisfied, then

$$\int_{\partial\Omega} w[\nabla w \cdot \nu] \, d\sigma = 0.$$

Thus, in all the cases it follows that $F'(t) \leq 0$ and, hence, F is a non-increasing function of t. Also,

$$F(0) = \int_\Omega w^2(x,0) \, dx = 0.$$

Thus, $F(t) \leq 0$ for all $t \geq 0$. But, by definition, $F(t) \geq 0$ for all $t \geq 0$. Hence, $F(t) = 0$ for all $t \geq 0$ which implies that $w(x,t) \equiv 0$ in Ω for all $t > 0$. Thus $u = v$ in Q_T.

Theorem 6.4 (Heat Flow on a Bar). *Let $\Omega = (0,L)$ be a homogeneous rod of length L insulated along sides and its ends are kept at zero temperature. The temperature zero at the end points of the rod is given by the Dirichlet boundary condition $u(0,t) = u(L,t) = 0$. The initial temperature of the rod, at time $t = 0$, is given by $u(x,0) = g(x)$, where $g : [0,L] \to \mathbb{R}$ be such that $g(0) = g(L) = 0$. Then there is a solution u of*

$$\begin{cases} u_t(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = 0 & \text{in } (0,\infty) \\ u(x,0) = g(x) & \text{on } [0,L] \end{cases}$$

where c is a constant.

Proof. We begin with the ansatz that $u(x,t) = v(x)w(t)$ (variable separated). Substituting u in separated form in the equation, we get

$$v(x)w'(t) = c^2 v''(x)w(t)$$

and, hence,

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since LHS, a function of t, and RHS, a function x, are equal they must be equal to some constant, say λ. Thus,

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)} = \lambda.$$

Therefore, we need to solve two ODE to obtain v and w,

$$w'(t) = \lambda c^2 w(t) \text{ and } v''(x) = \lambda v(x).$$

We first solve the eigenvalue problem involving v. For each $k \in \mathbb{N}$, there is a pair (λ_k, v_k) which solves the eigenvalue problem involving v, where $\lambda_k = -(k\pi)^2/L^2$ and $v_k(x) = \sin\left(\frac{k\pi x}{L}\right)$. For each $k \in \mathbb{N}$, we solve for w_k to get

$$\ln w_k(t) = \lambda_k c^2 t + \ln \alpha,$$

where α is integration constant. Thus, $w_k(t) = \alpha e^{-(kc\pi/L)^2 t}$. Hence,

$$u_k(x,t) = v_k(x)w_k(t) = \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t},$$

for some constants β_k, is a solution to the heat equation. By superposition principle, the general solution is

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t}.$$

We now use the initial temperature of the rod, given as $g : [0,L] \to \mathbb{R}$ to compute the constants. Since $u(x,0) = g(x)$,

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right).$$

Further, $g(0) = g(L) = 0$. Thus, g admits a Fourier Sine expansion and hence its coefficients β_k are given as

$$\beta_k = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right).$$

Theorem 6.5 (Circular Wire). *Let Ω be a circle (circular wire) of radius one insulated along its sides. Let the initial temperature of the wire, at time $t=0$, be given by a 2π-periodic function $g:\mathbb{R}\to\mathbb{R}$. Then there is a solution $u(r,\theta)$ of*

$$\begin{cases} u_t(\theta,t)-c^2u_{\theta\theta}(\theta,t)=0 & \text{in } \mathbb{R}\times(0,\infty)\\ u(\theta+2\pi,t)=u(\theta,t) & \text{in } \mathbb{R}\times(0,\infty)\\ u(\theta,0)=g(\theta) & \text{on } \mathbb{R}\times\{t=0\}\end{cases}$$

where c is a constant.

Proof. Note that $u(\theta,t)$ is 2π-periodic in θ-variable, i.e., $u(\theta+2\pi,t)=u(\theta,t)$ for all $\theta\in\mathbb{R}$ and $t\geq 0$. We begin with ansatz $u(\theta,t)=v(\theta)w(t)$ with variables separated. Substituting for u in the equation, we get

$$\frac{w'(t)}{c^2w(t)}=\frac{v''(\theta)}{v(\theta)}=\lambda.$$

For each $k\in\mathbb{N}\cup\{0\}$, the pair (λ_k,v_k) is a solution to the eigenvalue problem where $\lambda_k=-k^2$ and

$$v_k(\theta)=a_k\cos(k\theta)+b_k\sin(k\theta).$$

For each $k\in\mathbb{N}\cup\{0\}$, we get $w_k(t)=\alpha e^{-(kc)^2t}$. For $k=0$

$$u_0(\theta,t)=a_0/2 \quad \text{(To maintain consistency with Fourier series)}$$

and for each $k\in\mathbb{N}$, we have

$$u_k(\theta,t)=[a_k\cos(k\theta)+b_k\sin(k\theta)]\,e^{-k^2c^2t}.$$

Therefore, the general solution is

$$u(\theta,t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}[a_k\cos(k\theta)+b_k\sin(k\theta)]\,e^{-k^2c^2t}.$$

We now use the initial temperature on the circle to find the constants. Since $u(\theta,0)=g(\theta)$,

$$g(\theta)=u(\theta,0)=\frac{a_0}{2}+\sum_{k=1}^{\infty}[a_k\cos(k\theta)+b_k\sin(k\theta)]\,.$$

Further, g is 2π-periodic and, hence, admits a Fourier series expansion. Thus,

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}g(\theta)\cos(k\theta)\,d\theta$$

and

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}g(\theta)\sin(k\theta)\,d\theta.$$

Note that as $t\to\infty$ the temperature of the wire approaches a constant $a_0/2$.

Exercise 6.1. Solve the heat equation for 2D Rectangle and 2D Disk.

6.4 Method of Shifting the data

Without solving the problem, let us reduce the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin x \cos t & \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = t^2 & \text{in } (0,\infty) \\ u(\pi,t) = t & \text{in } (0,\infty) \\ u(x,0) = \cos x & \text{in } (0,\pi) \end{cases}$$

to a homogeneous problem with zero boundary conditions and an inhomogeneous problem with zero boundary and initial conditions. The idea is to subtract the boundary conditions from the solution. Define

$$v(x,t) := \left(1 - \frac{x}{\pi}\right)t^2 + \frac{x}{\pi}t.$$

Note that $v(0,t) = t^2$ and $v(\pi,t) = t$. Set $w(x,t) := u(x,t) - v(x,t)$. Then $w(x,t)$ satisfies

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) = \sin x \cos t - 2\left(1 - \frac{x}{\pi}\right)t - \frac{x}{\pi} & \text{in } (0,\pi)\times(0,\infty) \\ w(0,t) = 0 & \text{in } (0,\infty) \\ w(\pi,t) = 0 & \text{in } (0,\infty) \\ w(x,0) = \cos x & \text{in } (0,\pi). \end{cases}$$

Note that $w(x,t) = w^1(x,t) + w^2(x,t)$ where $w^1(x,t)$ and $w^2(x,t)$ solves, respectively,

$$\begin{cases} w^1_t(x,t) - w^1_{xx}(x,t) = 0 & \text{in } (0,\pi)\times(0,\infty) \\ w^1(0,t) = w^1(\pi,t) = 0 & \text{in } (0,\infty) \\ w^1(x,0) = \cos x & \text{in } (0,\pi) \end{cases}$$

and

$$\begin{cases} w^2_t(x,t) - w^2_{xx}(x,t) = \sin x \cos t - 2\left(1 - \frac{x}{\pi}\right)t - \frac{x}{\pi} & \text{in } (0,\pi)\times(0,\infty) \\ w^2(0,t) = w^2(\pi,t) = 0 & \text{in } (0,\infty) \\ w^2(x,0) = 0 & \text{in } (0,\pi). \end{cases}$$

6.5 Fundamental Solution of Heat Equation

The global Cauchy problem of the heat equation is

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{6.8}$$

where $g \in C(\mathbb{R}^n)$.

To solve the Cauchy problem (6.8) for unbounded domains, we shall derive the fundamental solution of (6.8). Taking Fourier transform w.r.t the x-variable both sides of the equation, we get

$$\begin{aligned}
\widehat{u_t - \Delta u} &= \hat{f} \\
\widehat{u_t}(\xi,t) - \widehat{\Delta u}(\xi,t) &= \hat{f}(\xi,t) \\
\hat{u}_t(\xi,t) - \sum_{j=1}^{n} \mathrm{i}^2 \xi_j^2 \hat{u}(\xi,t) &= \hat{f}(\xi,t) \\
\hat{u}_t(\xi,t) + |\xi|^2 \hat{u}(\xi,t) &= \hat{f}(\xi,t).
\end{aligned}$$

For each $\xi \in \mathbb{R}^n$, we have an ODE in t-variable with initial condition $\hat{u}(\xi,0) = \hat{g}(\xi)$ whose solution is given by (cf. Appendix **??**)

$$\hat{u}(\xi,t) = \hat{g}(\xi) e^{-|\xi|^2 t} + \int_0^t e^{-|\xi|^2 (t-s)} \hat{f}(\xi,s)\, ds.$$

Therefore, by inverse Fourier formula,

$$\begin{aligned}
u(x,t) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(\hat{g}(\xi) e^{-|\xi|^2 t} e^{i\xi\cdot x} + \int_0^t e^{i\xi\cdot x - |\xi|^2(t-s)} \hat{f}(\xi,s)\, ds \right) d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(y) e^{-i\xi\cdot y}\, dy \right) e^{-|\xi|^2 t} e^{i\xi\cdot x}\, d\xi \\
&\quad + (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^t \left(\int_{\mathbb{R}^n} f(y,s) e^{-i\xi\cdot y}\, dy \right) e^{i\xi\cdot x - |\xi|^2(t-s)}\, ds\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y) - |\xi|^2 t}\, d\xi \right) dy \\
&\quad + (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^t f(y,s) \left(\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y) - |\xi|^2(t-s)}\, d\xi \right) ds\, dy \\
&= \int_{\mathbb{R}^n} g(y) K(x,y,t)\, dy + \int_{\mathbb{R}^n} \int_0^t f(y,s) K(x,y,t-s)\, ds\, dy
\end{aligned}$$

where

$$K(x,y,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y) - |\xi|^2 t}\, d\xi.$$

Note that

$$i\xi\cdot(x-y) - |\xi|^2 t = -\left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}} \right) \cdot \left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}} \right) - \frac{|x-y|^2}{4t}$$

and, set $\eta = \left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}} \right)$. Therefore, $d\eta = \sqrt{t}\, d\xi$. Using this substituion in K and simplifying, we get

$$K(x,y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$$

called the *heat kernel* or the fundamental solution of heat equation. The function K can be motivated in another way. Recall that the solution of heat equation respectes parabolic dilation, i.e., if $u(x,t)$ is a solution of the heat equation, then $u(\lambda x, \lambda^2 t)$ is also a solution of the heat equation, for any $\lambda \neq 0$. Thus, we look for a solution $u(x,t) = v(t)w(r^2/t)$, where $r = |x|$. Substituting this separation of variable in the heat equation, we derive $v(t) = t^{-n/2}$ and $w(t) = e^{-r^2/4t}$. This motivates us to define the fundamental solution as

$$K(x,t) := \begin{cases} -\left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{r^2}{4t}} & x \in \mathbb{R}^n,\ t \in (0,T) \\ 0 & x \in \mathbb{R}^n,\ t < 0. \end{cases}$$

6.6 Fokas Method

Consider the initial value problem (IVP) for the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,T] \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

such that $u(x,t) \to 0$ sufficiently fast as $|x| \to \infty$, for all $t \geq 0$ and $T > 0$. From §6.5, we know that

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\xi) e^{i\xi x - \xi^2 t}\, d\xi$$

where

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(y) e^{-i\xi y}\, dy.$$

For each $z \in \mathbb{C}$, we define the one parameter family of PDE's, called the *local relation*,

$$\rho_t(x,t;z) + q_x(x,t;z) = 0$$

where $\rho(x,t;z) = e^{-izx+\omega(z)t}u$ and $\omega(z) = z^2$ is the dispersion relation for heat equation. The dispersion relation gives the relationship between the wave number and frequency. This family of PDE's are called the *local relation*. Note that

$$\begin{aligned} \rho_t(x,t;z) &= e^{-izx+\omega(z)t}u_t + \omega(z)e^{-izx+\omega(z)t}u \\ &= e^{-izx+\omega(z)t}u_{xx} + \omega(z)e^{-izx+\omega(z)t}u \\ &= (e^{-izx+\omega(z)t}u_x)_x + ize^{-izx+\omega(z)t}u_x + \omega(z)e^{-izx+\omega(z)t}u \\ &= (e^{-izx+\omega(z)t}u_x)_x + (ize^{-izx+\omega(z)t}u)_x - z^2e^{-izx+\omega(z)t}u + \omega(z)e^{-izx+\omega(z)t}u. \end{aligned}$$

Choosing $\omega(z) = z^2$, we get

$$\rho_t(x,t;z) = [e^{-izx+\omega(z)t}(u_x + izu)]_x$$

and hence $q(x,t;z) = -e^{-izx+\omega(z)t}(u_x + izu)$. Integrating both sides of local relation we get

$$\int_{\mathbb{R}\times(0,T)} [\rho_t + q_x]\,dt\,dx = 0$$

$$\int_{\mathbb{R}\times(0,T)} [(e^{-izx+\omega(z)t}u)_t - [e^{-izx+\omega(z)t}(u_x + izu)]_x\,dt\,dx = 0.$$

By Green's theorem in Ω and the fact that u vanishes at ∞, the LHS becomes

$$\int_{-\infty}^{\infty} e^{-izx}u(x,0)\,dx - \int_{-\infty}^{\infty} e^{-izx+\omega(z)T}u(x,T)\,dx = 0$$

$$\int_{-\infty}^{\infty} e^{-izx+\omega(z)T}u(x,T)\,dx = \int_{-\infty}^{\infty} e^{-izx}g(x)\,dx$$

$$\int_{-\infty}^{\infty} e^{-izx}u(x,T)\,dx = e^{-\omega(z)T}\int_{-\infty}^{\infty} e^{-izx}g(x)\,dx$$

$$\hat{u}(z,T) = e^{-\omega(z)T}\hat{g}(z).$$

Appendices

Appendix A
Leibniz Integral Rule

Theorem A.1. *Let $a, b \in C^1(x_0, x_1)$ and $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be such that both f and ∂_x are continuous in Ω, a region that includes $a(x) \leq y \leq b(x)$ and $x_0 \leq x \leq x_1$. Then*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)\,dy\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \partial_x f(x,y)\,dy.$$

Appendix B
Divergence Theorem

Definition B.1. For an open set $\Omega \subset \mathbb{R}^n$, its boundary $\partial\Omega$ is said to be C^k ($k \geq 1$) if, for every point $x \in \partial\Omega$, there is a $r > 0$ and a C^k diffeomorphism[1] $\phi : B_r(x) \to B_1(0)$ such that

1. $\phi(\partial\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ and
2. $\phi(\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n > 0\}$

The boundary $\partial\Omega$ is said to be C^∞ if $\partial\Omega$ is C^k, for all $k \in \mathbb{N}$, and $\partial\Omega$ is *analytic* if ϕ is analytic.

Equivalently, $\partial\Omega$ is C^k if, for every point $x \in \partial\Omega$, there exists a neighbourhood U_x of x and a C^k function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_x = \{x \in B_x \mid x_n > \phi(x_1, x_2, \ldots, x_{n-1})\}.$$

Theorem B.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega \frac{\partial v}{\partial x_i}\, dx = \int_{\partial\Omega} v \nu_i \, d\sigma$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit vector pointing outward and $d\sigma$ is the surface measure of $\partial\Omega$.

The hypothesis that Ω is bounded can be relaxed provided $|v|$ and $\left|\frac{\partial v}{\partial x_i}\right|$ decays as $|x| \to \infty$. Much weaker hypotheses on $\partial\Omega$ and v are considered in geometric measure theory.

Theorem B.2 (Integration by parts). *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $u, v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx + \int_\Omega v \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} u v \nu_i \, d\sigma.$$

[1] ϕ^{-1} exists and both ϕ and ϕ^{-1} are k-times continuously differentiable

Proof (Hint). Set $v := uv$ in the theorem above.

Theorem B.3 (Gauss). *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $V = (v_1, \ldots, v_n)$ on Ω is a vector field such that $v_i \in C^1(\overline{\Omega})$, for all $1 \le i \le n$, then*

$$\int_{\Omega} \nabla \cdot V \, dx = \int_{\partial\Omega} V \cdot \nu \, d\sigma. \tag{B.1}$$

The divergence of a vector field is the measure of the magnitude (outgoing nature) of all source (of the vector field) and absorption in the region. The divergence theorem was discovered by C. F. Gauss in 1813[2] which relates the outward flow (flux) of a vector field through a closed surface to the behaviour of the vector field inside the surface (sum of all its "source" and "sink"). The divergence theorem is the mathematical formulation of the conservation law.

Theorem B.4 (Green's Identities). *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $u, v \in C^2(\overline{\Omega})$ then*

(i)
$$\int_{\Omega} (v\Delta u + \nabla v \cdot \nabla u) \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\sigma,$$
where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$;

(ii)
$$\int_{\Omega} (v\Delta u - u\Delta v) \, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) d\sigma.$$

Proof (Hint). Apply divergence theorem to $V = v\nabla u$ to get the first formula. To get second formula apply divergence theorem for both $V = v\nabla u$ and $V = u\nabla v$ and subtract one from the other.

[2] J. L. Lagrange might have discovered this, before Gauss, in 1762

Appendix C
Surface Area and Volume of a Disk

Theorem C.1 (Polar coordinates). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and integrable. Then*

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \left(\int_{S_r(a)} f(y)\,d\sigma_y \right) dr$$

for each $a \in \mathbb{R}^n$. In particular, for each $r > 0$,

$$\frac{d}{dr}\left(\int_{B_r(a)} f(x)\,dx \right) = \int_{S_r(a)} f(y)\,d\sigma_y.$$

Theorem C.2. *Prove that*

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1.$$

Further, prove that the surface area ω_n of $S_1(0)$ in $\mathbb{R}^n$ is

$$\frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and the volume of the ball $B_1(0)$ in $\mathbb{R}^n$ is ω_n/n. Consequently, for any $x \in \mathbb{R}^n$ and the $r > 0$, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$ and the volume of $B_r(x)$ is $r^n\omega_n/n$.

Proof. We first observe that

$$e^{-\pi|x|^2} = e^{-\pi\left(\sum_{i=1}^n x_i^2\right)} = \prod_{i=1}^n e^{-\pi x_i^2}.$$

Therefore,

$$
\begin{aligned}
I_n := \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\pi x_i^2}\,dx \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\pi t^2}\,dt \\
&= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^n = (I_1)^n
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^{2(n/2)} = \left((I_1)^2\right)^{n/2} = (I_2)^{n/2} \\
&= \left(\int_{\mathbb{R}^2} e^{-\pi|y|^2}\,dy\right)^{n/2} \\
&= \left(\int_0^{2\pi}\int_0^{\infty} e^{-\pi|y|^2}\,dy\right)^{n/2} \\
&= \left(\int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2}\,r\,dr\,d\theta\right)^{n/2} \quad \text{(since jacobian is } r\text{)} \\
&= \left(2\pi\int_0^{\infty} e^{-\pi r^2}\,r\,dr\right)^{n/2} \\
&= \left(\pi\int_0^{\infty} e^{-\pi s}\,ds\right)^{n/2} \quad \text{(by setting } r^2 = s\text{)} \\
&= \left(\int_0^{\infty} e^{-q}\,dq\right)^{n/2} \quad \text{(by setting } \pi s = q\text{)} \\
&= (\Gamma(1))^{n/2} = 1.
\end{aligned}
$$

Let ω_n denote the surface area of the unit sphere $S_1(0)$ in $\mathbb{R}^n$, i.e.,

$$
\omega_n = \int_{S_1(0)} d\sigma,
$$

where $d\sigma$ is the $n-1$-dimensional surface measure. Now, consider

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx \\
&= \int_{S_1(0)}\int_0^{\infty} e^{-\pi r^2}\,r^{n-1}\,dr\,d\sigma \\
&= \omega_n \int_0^{\infty} e^{-\pi r^2}\,r^{n-1}\,dr \\
&= \frac{\omega_n}{2\pi^{n/2}}\int_0^{\infty} e^{-s}s^{(n/2)-1}\,ds \quad \text{(by setting } s = \pi r^2\text{)} \\
&= \frac{\omega_n \Gamma(n/2)}{2\pi^{n/2}}.
\end{aligned}
$$

Thus, $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. We shall now compute the volume of the disk $B_1(0)$. Consider,

$$\int_{B_1(0)} dx = \omega_n \int_0^1 r^{n-1}\, dr = \frac{\omega_n}{n}.$$

For any $x \in \mathbb{R}^n$ and $r > 0$, we observe by the shifting of origin that the surface area of $S_r(x)$ is same as the surface area of $S_r(0)$. Let $S_r(0) = \{s \in \mathbb{R}^n \mid |s| = r\}$. Now

$$\int_{S_r(0)} d\sigma_s = \int_{S_1(0)} r^{n-1}\, d\sigma_t = r^{n-1}\omega_n,$$

where $t = s/r$. Thus, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$. Similarly, volume of a disk $B_r(x)$ is $r^n\omega_n/n$.

References

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