First Week
- Motivation to Distribution Theory

Second Week
- Topology on the Space of Test Functions
- Regularization and Cut-off Technique

Third Week
- Space of Distributions

Fourth Week
- Topology on Distribution
- Principal Value Distribution

Fifth Week
- Distributional Derivative and Weak Derivative
- Product of Distributions

Sixth Week
- Support of Distribution
- Singular Support
- Shifting and Scaling
- Convolution
Sobolev Spaces and its Applications

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Fifteenth-Sixteenth Week

- Generalised Sobolev Imbedding
- Compact Imbedding
- Trace Theory
- Green’s Identity
Motivation

The need to introduce the notion of distribution is three fold:

(i) Generalising the notion of function.
(ii) Expanding the domain of differential calculus to ‘generalised function’.
(iii) Weakening the notion of solution to a differential equation.

Let us illustrate each of these motivation with specific examples!
Case for Generalising Function

- Consider a particle of mass $m$ moving along the $x$-axis with constant speed $v$
- and, say, at time $t = t_0$ an elastic collision with vertical wall bounces the particle with speed $-v$.
- Then, by Newton’s second law, if $F(t)$ is the force acting on the particle at time $t$ and; $v_1$ and $v_2$ are the speeds at time $t_1$ and $t_2$, respectively, then

$$m(v_2 - v_1) = \int_{t_1}^{t_2} F(t) \, dt.$$

- It is easy to observe that for $t_1 < t_2 < t_0$, $v_2 = v_1 = v$ and for $t_0 < t_1 < t_2$, $v_2 = v_1 = -v$; in both cases the LHS above is zero.
- However, for $t_2 < t_0 < t_1$, the LHS above is $2mv \neq 0$ and, hence, $F$ is such that the RHS is non-zero, which is a contradiction if $F$ is a ‘function’ in the classical sense.
- Thus, mechanical impulse cannot be modelled by a classical function.
Dirac Delta ‘function’

In late 1920’s, while studying the quantum theory of collision processes, P. A. M. Dirac introduced the δ-“function” defined on the real line \( \mathbb{R} \) satisfying the following properties:

1. \( \delta(x) = 0 \) for \( x \neq 0 \).
2. \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \).
3. For any continuous function defined on \( \mathbb{R} \),
   \[ f(a) = \int_{-\infty}^{\infty} f(x) \delta(a - x) \, dx \] for all \( a \in \mathbb{R} \).
4. \( \delta \) is infinitely differentiable and for any \( k \)-times continuously differentiable function \( f \) on \( \mathbb{R} \),
   \[ f^{(k)}(a) = \int_{-\infty}^{\infty} f(x) \delta^{(k)}(a - x) \, dx \] for all \( a \in \mathbb{R} \).
5. Given the Heaviside function
   \[ H(x) = \begin{cases} 
   1 & \text{if } x > 0 \\
   0 & \text{if } x < 0,
   \end{cases} \quad (1.1) \]
   then the delta function is the “derivative” of \( H \), i.e., \( \delta(x) = H'(x) \).
Case for Generalising Derivative

- Properties (4) and (5) listed above for Dirac Delta function is already a case for generalising derivative!

- Recall the linear transport equation $u_t(x, t) = cu_x(x, t)$ on $\mathbb{R} \times (0, \infty)$, has the general solution $u(x, t) = F(x + ct)$ where $F$ is chosen from the initial data $u(x, 0)$.

- Mathematical rigor imposes that one should exclude non-differentiable initial data in order to verify the PDE.

- For instance, consider $F$ to be the Heaviside function $H$ (cf. (1.1)). Now, how do we mathematically describe this kind of “solution”?
Case for Weakening Notion of Solution

- Consider the quasilinear Cauchy problem called the Burgers’ equation

\[
\begin{align*}
    u_t(x, t) + u(x, t)u_x(x, t) &= 0 & \text{in } \mathbb{R} \times (0, \infty) \\
    u(x, 0) &= u_0(x) & \text{on } \mathbb{R} \times \{0\}.
\end{align*}
\]

- Then \(u(x, t) = u_0(x - tu)\) is the solution in the implicit form which is constant along the projected characteristic curves \(x = u_0(r)t + r\) with slope \(\frac{1}{u_0(r)}\).

- If the Cauchy data \(u_0\) is such that, for \(r_1 < r_2, u_0(r_1) > u_0(r_2)\) then the characteristic curves passing through \(r_1\) and \(r_2\) will necessarily intersect.

- This situation leads to a multi-valued solution because

\[
u(x_0, t_0) = u(r_2, 0) = u_0(r_2) < u_0(r_1) = u(r_1, 0) = u(x_0, t_0).
\]

- Thus, even if the Cauchy data is a ‘smooth’ decreasing initial data one may not be able to find a solution for all time \(t\).
The space of test functions will be the argument for the ‘generalised function’.

If $f \in C^1(a, b)$ then, for all $\phi \in C^1(a, b)$ by classical integration by parts,

$$
\int_a^b \phi(x)f'(x) \, dx + \int_a^b f(x)\phi'(x) \, dx = f(b)\phi(b) - f(a)\phi(a).
$$

If $\phi \in C^1_c(a, b)$ then

$$
\int_a^b \phi(x)f'(x) \, dx = -\int_a^b f(x)\phi'(x) \, dx. \tag{1.2}
$$

Suppose $f$ is chosen from $C^k(a, b)$ and $\phi \in C^\infty_c(a, b)$, then above integration by parts could be repeated $k$ times to get

$$
\int_a^b \phi(x)f^{(k)}(x) \, dx = (-1)^k \int_a^b f(x)\phi^{(k)} \, dx.
$$
Space of Test Functions

- Observe that the maps \( \phi \mapsto \int_a^b f \phi \, dx \) and \( \phi \mapsto \int_a^b f \phi' \, dx \) are linear on \( C_c^\infty(a, b) \).

- Seek a suitable complete topology on \( C_c^\infty(a, b) \) such that

\[
\phi \mapsto \int_a^b f \phi \, dx \tag{1.3}
\]

is a continuous linear functional on \( C_c^\infty(a, b) \) identified with \( f \).

- Before we seek a suitable topology, let us understand the elements of \( C_c^\infty(\Omega) \) in all dimensions.
Smooth Non-analytic (Bump) Functions

- For any open set \( \Omega \subseteq \mathbb{R}^n \), \( C^\infty(\Omega) \) denotes the space all real valued functions whose partial derivatives of all order exist and are continuous in \( \Omega \). We know that polynomials, trigonometric functions, exponential are all \( C^\infty \) functions.

- Let \( C^\infty_c(\Omega) \) denote the subclass of \( C^\infty(\Omega) \) that are compactly supported in \( \Omega \). Any function in \( C^\infty_c(\Omega) \) falls to zero within a compact set.

**Exercise**

\( C^\infty_c(\Omega) \) is a vector space under usual addition and scalar multiplication of real-valued functions.

**Exercise**

Show that any non-zero function in \( C^\infty_c(\Omega) \) is non-analytic.
Gluing of Smooth Functions

- We seek smooth non-analytic functions in $C_c^\infty(\Omega)$.
- This can be viewed as gluing of the zero function, along the boundary of a compact set, with smooth function inside the compact set such that the smoothness is preserved in all of domain.
- The point of gluing can lead to non-differentiability! For instance, gluing of the smooth maps $x \mapsto -x$ on $(-\infty, 0]$ and $x \mapsto x$ on $[0, \infty)$ yields the non-differentiable map $|x|$ on $\mathbb{R}$.
- A tool to accomplish the gluing of smooth functions in a smooth way is the partition of unity.
- Recall that the positive function $e^{1/x}$ behaves badly at $x = 0$. From the right side it approaches $+\infty$ and from left side it approaches zero. However, for $x \neq 0$, $e^{1/x}$ is infinitely differentiable (smooth).
The function $f : \mathbb{R} \to \mathbb{R}$, referred to as the Cauchy’s exponential function, defined as

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is infinitely differentiable but not analytic.

The function

$$f(x) = \begin{cases} \exp \left( \frac{-b^2}{b^2 - |x|^2} \right) & \text{if } |x| < b \\ 0 & \text{if } |x| \geq b \end{cases}$$

is in $C^\infty_c(\mathbb{R}^n)$ with support in $\overline{B}(0; b)$, the disk with centre at origin and radius $b$. 

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For $\varepsilon > 0$, set $b = \varepsilon$, then

$$
\int_{\mathbb{R}^n} f_\varepsilon(x) \, dx = \int_{|x|<\varepsilon} \exp \left( \frac{-\varepsilon^2}{\varepsilon^2 - |x|^2} \right) \, dx
$$

$$
= \varepsilon^n \int_{|y|<1} \exp \left( \frac{-1}{1 - |y|^2} \right) \, dy \quad \text{(by setting } y = x/\varepsilon)\quad

= \varepsilon^n c^{-1},
$$

where

$$
c^{-1} = \int_{|y|\leq 1} \exp \left( \frac{-1}{1 - |y|^2} \right) \, dy.
$$

Now, set $\rho_\varepsilon(x) = c\varepsilon^{-n} f_\varepsilon(x)$, equivalently,

$$
\rho_\varepsilon(x) = \begin{cases} 
  c\varepsilon^{-n} \exp \left( \frac{-\varepsilon^2}{\varepsilon^2 - |x|^2} \right) & \text{if } |x| < \varepsilon \\
  0 & \text{if } |x| \geq \varepsilon.
\end{cases}
$$
Multi-Index Notations

- Note that a $k$-degree polynomial in one variable is written as $\sum_{1 \leq i \leq k} a_i x^i$.
- How does one denote a $k$-degree polynomial in $n$ variables (higher dimensions)?
- A $k$-degree polynomial in $n$ variables can be concisely written as $\sum_{|\alpha| \leq k} a_\alpha x^\alpha$ where
- the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a $n$-tuple where $\alpha_i$, for each $1 \leq i \leq n$, is a non-negative integer,
- $|\alpha| := \alpha_1 + \ldots + \alpha_n$,
- and, for any $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. 
Multi-Index Notations

- The partial differential operator of order $\alpha$ is denoted as

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$ 

- If $|\alpha| = 0$, then $\partial^\alpha f = f$.
- For each $k \in \mathbb{N}$, $D^k u(x) := \{\partial^\alpha u(x) \mid |\alpha| = k\}$.
- The case $k = 1$ is the gradient vector,

$$\nabla u(x) := D^1 u(x) = \left( \partial^{(1,0,\ldots,0)} u(x), \partial^{(0,1,0,\ldots,0)} u(x), \ldots, \partial^{(0,0,\ldots,0,1)} u(x) \right) = \left( \frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_n} \right).$$
Multi-Index Notations

- The case $k = 2$ is the Hessian matrix

$$D^2 u(x) = \begin{pmatrix}
\frac{\partial^2 u(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 u(x)}{\partial x_1 \partial x_n} \\
\frac{\partial^2 u(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u(x)}{\partial x_2 \partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 u(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u(x)}{\partial x_n^2}
\end{pmatrix}_{n \times n}.$$ 

- The Laplace operator, denoted as $\Delta$, is defined as the trace of the Hessian operator, i.e., $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Note that $\Delta = \nabla \cdot \nabla$. 

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Topology on $C(\Omega)$ for open $\Omega$

- If $\Omega$ is a compact subset, the norm on the space of continuous functions on $\Omega$, $C(\Omega)$, is defined as

$$
\|\phi\|_{\infty, \Omega} := \max_{x \in \Omega} |\phi(x)|.
$$

$C(\Omega)$ is a Banach space w.r.t the uniform norm and a sequence $\phi_k$ converges to $\phi$ in the uniform norm is said to converge *uniformly* in $\Omega$.

- For an open subset $\Omega$ of $\mathbb{R}^n$, the space $C(\Omega)$ is endowed with the topology of compact convergence (uniform convergence on compact sets) or the compact-open topology.

- $C(\Omega)$ is not normable but are metrizable. They form a locally convex complete metric space called *Fréchet space*.

We will describe this topology in the context of $C^\infty(\Omega)$ for completeness sake.
Topology on $C^\infty(\Omega)$

- For any open subset $\Omega$ of $\mathbb{R}^n$, there is a sequence $K_j$ of non-empty compact subsets of $\Omega$ such that $\Omega = \cup_{j=0}^{\infty} K_j$ and $K_j \subset \text{Int}(K_{j+1})$, for all $j$ (exhaustion of an open set by compact sets. This property is called the $\sigma$-compactness of $\Omega$.

- We define a countable family of semi-norms on $C^\infty(\Omega)$ as

$$p_j(\phi) = \sum_{|\alpha| = 0}^{j} \| D^\alpha \phi \|_{\infty, K_j}.$$ 

Note that $p_0 \leq p_1 \leq p_2 \leq \ldots$. The sets

$$\{ \phi \in C^\infty(\Omega) \mid p_j(\phi) < 1/j \}$$

form a local base for $C^\infty(\Omega)$. 
The metric induced by the family of semi-norms on $C^\infty(\Omega)$ is

$$d(\phi, \psi) = \max_{j \in \mathbb{N} \cup \{0\}} \frac{1}{2^j} \frac{p_j(\phi - \psi)}{1 + p_j(\phi - \psi)}.$$ 

$d$ is a complete metric because if $\{\phi_m\}$ is a Cauchy sequence w.r.t $d$ then $p_j(\phi_m - \phi_\ell) \to 0$, for all $j$ and as $m, \ell$ tends to infinity. Thus $\{\phi_m\}$ and all its derivatives converges uniformly on $K_j$ to some $\phi \in C^\infty(\Omega)$ and its derivatives, respectively. Then it is easy to see that $d(\phi, \phi_m) \to 0$.

**Exercise**

Show that the topology given in $C^\infty(\Omega)$ is independent of the choice the exhaustion compact sets $\{K_j\}$ of $\Omega$. 

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$C_c^\infty$ with Inherited Topology

- The space $C_c^\infty(\Omega)$ is a subset of $C^\infty(\Omega)$, the semi-norms defined in $C^\infty(\Omega)$ restricted to $C_c^\infty(\Omega)$ becomes a norm and, for any $\phi \in C_c^\infty(\Omega)$, the family of norms induces the same topology as the one inherited from $C^\infty(\Omega)$.

- However, this norm induced topology on $C_c^\infty(\Omega)$ is not complete and its completion is $C^\infty(\Omega)$.

**Exercise**

Let $\phi \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\phi) = [0, 1]$ and $\phi > 0$ in $(0, 1)$. Then the sequence

$$\psi_m(x) = \sum_{i=1}^{m} \frac{1}{i} \phi(x - i) = \phi(x - 1) + \frac{1}{2} \phi(x - 2) + \ldots + \frac{1}{m} \phi(x - m)$$

is Cauchy in the topology induced by the norms, but $\lim \psi_m \notin C_c^\infty(\mathbb{R})$. 
A closed subspace of $C^\infty(\Omega)$

- We shall construct a complete (non-metrizable) topology on $C_c^\infty(\Omega)$ different from the one inherited from $C^\infty(\Omega)$.
- For every compact subset $K \subset \Omega$, let $D_K(\Omega)$ denote the class of all functions in $C^\infty(\Omega)$ such that their support is in $K$.
- The space $D_K(\Omega)$ is given the topology inherited from $C^\infty(\Omega)$, the same induced by the family of norms

$$\|\phi\|_{j,K} = \sum_{|\alpha|=0}^{j} \|D_\alpha^j \phi\|_{\infty,K} \quad \forall j \geq 0.$$ 

**Theorem**

$D_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$ under the inherited topology of $C^\infty(\Omega)$. 
Proof

- For each $x \in \Omega$, define the functional $T_x : C^\infty(\Omega) \to \mathbb{R}$ as $T_x(\phi) = \phi(x)$.
- For each $x \in \Omega$, there is a $j_0$ such that $x \in K_j$ for all $j \geq j_0$. Then,
  \[ |T_x(\phi)| = |\phi(x)| \leq p_j(\phi) \quad \forall j \geq j_0. \]

- The functional $T_x$ is continuous because uniform convergence implies point-wise convergence.
- Therefore the kernel of $T_x$,
  \[ \ker(T_x) := \{ \phi \in C^\infty(\Omega) \mid T_x(\phi) = 0 \}, \]

  is a closed subspace of $C^\infty(\Omega)$. Note that $\ker(T_x)$ is precisely those $\phi \in C^\infty(\Omega)$ such that $\phi(x) = 0$. 

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We claim that

$$D_K(\Omega) = \bigcap_{x \in K^c} \ker(T_x).$$

If $\phi \in D_K(\Omega)$ then $\phi$ is in the intersection because $\phi(x) = 0$ for all $x \in K^c$.

Conversely, if $\phi(x) = 0$ for all $x \in K^c$, then $\text{supp}(\phi) \subseteq K$.

Thus, for any compact subset $K$ of $\Omega$, we have our claim that $D_K(\Omega)$ is an arbitrary intersection of closed sets.

Thus, $D_K(\Omega)$ is closed in $C^\infty(\Omega)$. 
Recall that $K_j$ is a sequence of non-empty compact subsets of $\Omega$ such that $\Omega = \bigcup_{j=1}^{\infty} K_j$ and $K_j \subset \text{Int}(K_{j+1})$, for all $j$.

Thus,

$$C_c^\infty(\Omega) = \bigcup_{j=1}^{\infty} D_{K_j}(\Omega)$$

and $D_{K_\ell}(\Omega) \subset D_{K_m}(\Omega)$ for all $\ell < m$.

With topology inherited from $C^\infty(\Omega)$, the inclusion map $I_{k \ell} : D_{K_\ell}(\Omega) \to D_{K_m}(\Omega)$ is continuous. This is because the local base in $D_{K_m}(\Omega)$ is

$$\left\{ \phi \in D_{K_m}(\Omega) \mid \|\phi\|_m < \frac{1}{m} \right\}.$$

For any such $\phi$ in the local base, we have $\|\phi\|_\ell < 1/\ell$ and is in the local base of $D_{K_\ell}(\Omega)$.

Thus, we endow $C_c^\infty$ with the finest topology that makes the inclusion maps $I_j : D_{K_j}(\Omega) \to C_c^\infty(\Omega)$ continuous, for all $j$. 
A set $U$ in $C_c^\infty(\Omega)$ is said to be open if and only if $I_j^{-1}(U)$ is open in $D_{K_j}(\Omega)$ for all $j \geq 1$.

This topology is called the *inductive limit topology* with respect to $D_{K_j}(\Omega)$ and the maps $I_{k\ell}$.

The space $C_c^\infty(\Omega)$ is complete with respect to the inductive limit topology because any Cauchy sequence is Cauchy in $D_{K_j}(\Omega)$, for some $j$. 
Non-metrizability of $C_c^\infty$

Though each $D_K(\Omega)$ is metrizable, the space $C_c^\infty(\Omega)$ is not metrizable.

**Exercise**

Every proper subspace of a topological vector space has empty interior.

**Theorem**

*The inductive limit topology on $C_c^\infty(\Omega)$ is not metrizable.*

**Proof.**

Recall that $C_c^\infty(\Omega) = \bigcup_{j=1}^{\infty} D_{K_j}(\Omega)$, where each closed set $D_{K_j}(\Omega)$ has empty interior (cf. Exercise 5). Therefore, the complete space $C_c^\infty(\Omega)$ is a countable union of no-where dense sets. If $C_c^\infty(\Omega)$ was metrizable then it would contradict the Baire’s category theorem.
Space of Test Functions

**Definition**

The space $\mathcal{C}_c^\infty(\Omega)$ endowed with the inductive limit topology, and denoted as $\mathcal{D}(\Omega)$, is called the space of *test functions*.

**Exercise**

Show that the topology defined on $\mathcal{D}(\Omega)$ is independent of the choice of (exhaustion sets) the sequence of compact sets $K_j$ of $\Omega$.

**Exercise**

A sequence of functions $\{\phi_m\} \subset \mathcal{D}(\Omega)$ converges to zero iff there exists a compact set $K \subset \Omega$ such that $\text{supp}(\phi_m) \subset K$, for all $m$, and $\phi_m$ and all its derivatives converge uniformly to zero on $K$. 
The technique of regularization by convolution was introduced by Leray and Friedrichs.

**Definition**

Let $f, g \in L^1(\mathbb{R}^n)$. The convolution $f \ast g$ is defined as,

$$(f \ast g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy \quad \forall x \in \mathbb{R}^n.$$ 

The integral on RHS is well-defined because since by Fubini’s Theorem and the translation invariance of the Lebesgue measure, we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x - y)g(y)| \, dx \, dy = \int_{\mathbb{R}^n} |g(y)| \, dy \int_{\mathbb{R}^n} |f(x)| \, dx = \|g\|_1\|f\|_1.$$ 

Thus, for a.e. $x \in \mathbb{R}^n$, the function $y \mapsto f(x - y)g(y)$ is in $L^1(\mathbb{R}^n)$. 

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Properties of Convolution

Exercise

The convolution operation on $L^1(\mathbb{R}^n)$ is both commutative and associative.

Exercise (Young's inequality)

Let $1 \leq p, q, r < \infty$ such that $(1/p) + (1/q) = 1 + (1/r)$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the convolution $f \ast g \in L^r(\mathbb{R}^n)$ and

$$\|f \ast g\|_r \leq \|f\|_p \|g\|_q.$$ 

In particular, for $1 \leq p < \infty$, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then the convolution $f \ast g \in L^p(\mathbb{R}^n)$ and

$$\|f \ast g\|_p \leq \|f\|_1 \|g\|_p.$$
Properties of Convolution

Exercise
Let \( f \in L^1(\mathbb{R}^n) \) and \( g \in L^p(\mathbb{R}^n) \), for \( 1 \leq p \leq \infty \). Then

\[
\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)
\]

If both \( f \) and \( g \) have compact support, then support of \( f * g \) is also compact.

The convolution operation preserves smoothness.

Exercise
Let \( f \in C^k_c(\mathbb{R}^n) \) (\( k \geq 1 \)) and let \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \). Then \( f * g \in C^k(\mathbb{R}^n) \) and for all \(|\alpha| \leq k\)

\[
D^\alpha(f * g) = D^\alpha f * g = f * D^\alpha g.
\]
Regularization

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let

$$\Omega_\varepsilon := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon \}.$$

If $f \in L^1_{\text{loc}}(\Omega)$ then $f_\varepsilon := \rho_\varepsilon \ast f$ is in $C^\infty(\Omega_\varepsilon)$.

Proof: Fix $x \in \Omega_\varepsilon$. Consider

$$\frac{f_\varepsilon(x + he_i) - f_\varepsilon(x)}{h} = \frac{1}{h} \int_{\Omega} \left[ \rho_\varepsilon(x + he_i - y) - \rho_\varepsilon(x - y) \right] f(y) \, dy$$

$$= \int_{B_\varepsilon(x)} \frac{1}{h} \left[ \rho_\varepsilon(x + he_i - y) - \rho_\varepsilon(x - y) \right] f(y) \, dy.$$
Now, taking \( \lim_{h \to 0} \) both sides, we get

\[
\frac{\partial f_\varepsilon(x)}{\partial x_i} = \lim_{h \to 0} \int_{B_\varepsilon(x)} \frac{1}{h} \left[ \rho_\varepsilon(x + he_i - y) - \rho_\varepsilon(x - y) \right] f(y) \, dy \\
= \int_{B_\varepsilon(x)} \frac{\partial \rho_\varepsilon(x - y)}{\partial x_i} f(y) \, dy \\
= \int_{\Omega} \frac{\partial \rho_\varepsilon(x - y)}{\partial x_i} f(y) \, dy = \frac{\partial \rho_\varepsilon}{\partial x_i} * f.
\]

(Interchange of limits is due to the uniform convergence)

Similarly, one can show that, for any tuple \( \alpha \), \( D^\alpha f_\varepsilon(x) = (D^\alpha \rho_\varepsilon * f)(x) \). Thus, \( f_\varepsilon \in C^\infty(\Omega_\varepsilon) \).
**Theorem (Regularization technique)**

\( C^\infty(\mathbb{R}^n) \) is dense in \( C(\mathbb{R}^n) \) under the uniform convergence on compact sets topology.

**Proof:** Let \( g \in C(\mathbb{R}^n) \) and \( K \subset \mathbb{R}^n \) be a compact subset. Note that \( g \) is uniformly continuous on \( K \). Hence, for every \( \eta > 0 \), there exist a \( \delta > 0 \) (independent of \( x \) and dependent on \( K \) and \( \eta \)) such that

\[
|g(x - y) - g(x)| < \eta \text{ whenever } |y| < \delta \text{ for all } x \in K.
\]

For each \( m \in \mathbb{N} \), set \( \rho_m := \rho_{1/m} \), the sequence of mollifiers. Define \( g_m := \rho_m * g \). Note that \( g_m \in C^\infty(\mathbb{R}^n) \) (\( D^\alpha g_m = D^\alpha \rho_m * g \)).
Now, for all $x \in \mathbb{R}^n$,

$$|g_m(x) - g(x)| = \left| \int_{|y| \leq \frac{1}{m}} g(x - y) \rho_m(y) \, dy - g(x) \int_{|y| \leq \frac{1}{m}} \rho_m(y) \, dy \right|$$

$$\leq \int_{|y| \leq 1/m} |g(x - y) - g(x)| \rho_m(y) \, dy$$

Hence, for all $x \in K$ and $m > 1/\delta$, we have

$$|g_m(x) - g(x)| \leq \int_{|y| < \delta} |g(x - y) - g(x)| \rho_m(y) \, dy$$

$$\leq \eta \int_{|y| < \delta} \rho_m(y) \, dy = \eta$$

Since the $\delta$ is independent of $x \in K$, we have $\|g_m - g\|_\infty < \eta$ for all $m > 1/\delta$. Hence, $g_m \rightarrow g$ uniformly on $K$. 

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Density of Smooth Bump Functions

**Theorem**

For any $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ under the uniform topology.

**Proof:** Let $g \in C_c(\Omega)$ and $K := \text{supp}(g)$. One can view $C_c(\Omega)$ as a subset of $C_c(\mathbb{R}^n)$ under the following identification: Each $g \in C_c(\Omega)$ is extended to $\mathbb{R}^n$ as $\tilde{g}$

$$
\tilde{g}(x) = \begin{cases} 
g(x) & x \in K \\
0 & x \in \mathbb{R}^n \setminus K. 
\end{cases}
$$

By Theorem 1, the sequence $g_m := \rho_m \ast \tilde{g}$ in $C^\infty(\mathbb{R}^n)$ converges to $\tilde{g}$ uniformly on every compact subsets of $\mathbb{R}^n$. Note that $\text{supp}(g_m) \subset K + B(0; 1/m)$ is compact because $K$ is compact. Since we want $g_m \in C_c^\infty(\Omega)$, we choose $m_0 \in \mathbb{N}$ such that $1/m_0 < \text{dist}(K, \Omega^c)$. Thus, $\text{supp}(g_m) \subset \Omega$ and $g_m \in C_c^\infty(\Omega)$, for all $m \geq m_0$. The proof of the uniform convergence of $g_m$ to $g$ on $\Omega$ is same as in Theorem 1.
Corollary

For any $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ is dense in $C(\Omega)$ under the uniform convergence on compact sets topology.
Density of Simple Functions

A simple function $\phi$ is a non-zero function on $\mathbb{R}^n$ having the (canonical) form

$$
\phi(x) = \sum_{i=1}^{k} a_i 1_{E_i}
$$

with disjoint measurable subsets $E_i \subset \mathbb{R}^n$ with $\mu(E_i) < +\infty$ and $a_i \neq 0$, for all $i$, and $a_i \neq a_j$ for $i \neq j$. By our definition, simple function is non-zero on a finite measure.

**Theorem**

Let $\Omega \subset \mathbb{R}^n$. The class of all simple functions are dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

**Proof:** Fix $1 \leq p < \infty$ and let $f \in L^p(\Omega)$ such that $f \geq 0$. Then, we have an increasing sequence of non-negative simple functions $\{\phi_k\}$ that converge point-wise a.e. to $f$ and $\phi_k \leq f$ for all $k$. 
Thus,
\[ |\phi_k(x) - f(x)|^p \leq 2^p |f(x)|^p \]
and, by Dominated Convergence Theorem, we have
\[ \lim_{k \to \infty} \|\phi_k - f\|_p^p = \lim_{k \to \infty} \int_\Omega |\phi_k - f|^p \to 0. \]

For an arbitrary \( f \in L^p(\Omega) \), we use the decomposition \( f = f^+ - f^- \) where \( f^+, f^- \geq 0 \). Thus we have sequences of simple functions \( \{\phi_k\} \) and \( \{\psi_k\} \) such that \( \phi_m - \psi_m \to f \) in \( L^p(\Omega) \) (using triangle inequality). Thus, the space of simple functions is dense in \( L^p(\Omega) \).
Density of Compactly Supported Functions

**Theorem**

The space of all compactly supported continuous functions on $\Omega$, denoted as $C_c(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

**Proof:** It is enough to prove the result for a characteristic function $1_F$, where $F \subset \Omega$ such that $F$ is bounded. By outer regularity, for a given $\varepsilon > 0$ there is an open (bounded) set $\omega$ such that $\omega \supset F$ and $\mu(\omega \setminus F) < \varepsilon/2$. Also, by inner regularity, there is a compact set $K \subset F$ such that $\mu(F \setminus K) < \varepsilon/2$. By Urysohn lemma, there is a continuous function $g : \Omega \to \mathbb{R}$ such that $g \equiv 0$ on $\Omega \setminus \omega$, $g \equiv 1$ on $K$ and $0 \leq g \leq 1$ on $\omega \setminus K$. Note that $g \in C_c(\Omega)$. Therefore,

$$\|1_F - g\|^p_p = \int_{\Omega} |1_F - g|^p = \int_{\Omega \setminus K} |1_F - g|^p \leq \mu(\Omega \setminus K) = \varepsilon.$$
Theorem (Regularization technique)

The space $C^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for $1 \leq p < \infty$, under the $p$-norm.

**Proof:** Let $f \in L^p(\mathbb{R}^n)$. For each $m \in \mathbb{N}$, set $\rho_m := \rho_{1/m}$, the sequence of mollifiers. Then the sequence $f_m := \rho_m \ast f$ is in $C^\infty(\mathbb{R}^n)$. Since $\rho_m \in L^1(\mathbb{R}^n)$, by Young’s inequality, $f_m \in L^p(\mathbb{R}^n)$. We shall prove that $f_m$ converges to $f$ in $p$-norm. For any given $\varepsilon > 0$, by Theorem 4, we choose a $g \in C_c(\mathbb{R}^n)$ such that $\|g - f\|_p < \varepsilon/3$. Therefore, by Theorem 2, there is a compact subset $K \subset \mathbb{R}^n$ such that $\|\rho_m \ast g - g\|_\infty < \varepsilon/3(\mu(K))^{1/p}$. Hence, $\|\rho_m \ast g - g\|_p < \varepsilon/3$. Thus, for sufficiently large $m$, we have

$$
\|f - f\|_p \leq \|\rho_m \ast f - \rho_m \ast g\|_p + \|\rho_m \ast g - g\|_p + \|g - f\|_p < \|\rho_m \ast (f - g)\|_p + \frac{2\varepsilon}{3} \leq \|f - g\|_p \|\rho_m\|_1 + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
$$

The first term has been handled using Young’s inequality.
**Theorem (Cut-Off Technique)**

For $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

**Proof:** Any $f \in L^p(\Omega)$ can be viewed as an element in $L^p(\mathbb{R}^n)$ under the extension

$$\tilde{f}(x) = \begin{cases} 
  f(x) & x \in \Omega \\
  0 & x \in \Omega^c.
\end{cases}$$

By Theorem 5, the sequence $f_m := \rho_m \ast \tilde{f}$ converges to $\tilde{f}$ in $p$-norm. The sequence $\{f_m\}$ may fail to have compact support in $\Omega$ because support of $\tilde{f}$ is not necessarily compact in $\Omega$. To fix this issue, we shall multiply the sequence with suitable choice of test functions in $C_c^\infty(\Omega)$. Choose the sequence of exhaustion compact sets $\{K_m\}$ in $\Omega$. In particular, for $\Omega = \mathbb{R}^n$, we can choose $K_m = B(0; m)$. Note that $\Omega = \bigcup_m K_m$. Consider (The type of functions, $\phi_k$, are called cut-off functions) $\{\phi_m\} \subset C_c^\infty(\Omega)$ such that $\phi_m \equiv 1$ on $K_m$ and $0 \leq \phi_m \leq 1$, for all $m$. We extend $\phi_m$ by zero on $\Omega^c$. 

[Author and date information]
Define $F_m := \phi_m f_m$ and, hence, $F_m \in C_c^\infty(\Omega)$. Also, $F_m = f_m$ on $K_m$ and $|F_m| \leq |f_m|$ in $\mathbb{R}^n$.

Thus,

$$\|F_m - f\|_{p,\Omega} = \|F_m - \tilde{f}\|_{p,\mathbb{R}^n} \leq \|\phi_m f_m - \phi_m \tilde{f}\|_{p,\mathbb{R}^n} + \|\phi_m \tilde{f} - \tilde{f}\|_{p,\mathbb{R}^n} \leq \|f_m - \tilde{f}\|_{p,\mathbb{R}^n} + \|\phi_m \tilde{f} - \tilde{f}\|_{p,\mathbb{R}^n}.$$ 

The first term converges to zero by Theorem 5 and the second term converges to zero by Dominated convergence theorem.

**Remark**

The case $p = \infty$ is ignored in the above results, because the $L^\infty$-limit of $\rho_m * f$ is continuous and we do have discontinuous functions in $L^\infty(\Omega)$. 
Definition

A linear functional $T$ on $\mathcal{D}(\Omega)$ is said to be continuous if inverse image of open sets of $\mathbb{R}$ are open in $\mathcal{D}(\Omega)$. A linear functional $T$ on $\mathcal{D}(\Omega)$ is said to be sequential continuous if $T\phi_m \to 0$ in $\mathbb{R}$ whenever $\phi_m \to 0$ in $\mathcal{D}(\Omega)$.

- It is enough to define for zero convergent sequences because addition operation is continuous.
- For a first countable space the notion of continuity and sequential continuity are equivalent.
- A Hausdorff topological vector space is metrizable iff it is first countable.
- We know that $\mathcal{D}(\Omega)$ is not metrizable (cf. Exercise 2) and hence cannot be first countable.
Exercise

For any compact set $K \subset \Omega$, the restriction to $D_K(\Omega)$ of any continuous map on $C_c^\infty(\Omega)$ is also continuous on $D_K(\Omega)$. Similarly, for any compact set $K \subset \Omega$, the restriction to $D_K(\Omega)$ of any sequentially continuous map on $C_c^\infty(\Omega)$ is also sequentially continuous on $D_K(\Omega)$.

Theorem

Let $T : D(\Omega) \to \mathbb{R}$ be a linear map. Then the following are equivalent:

1. $T$ is continuous, i.e., inverse image of open set in $\mathbb{R}$, under $T$, is open in $D(\Omega)$.

2. For every compact subset $K \subset \Omega$, there exists a constant $C_K > 0$ and an integer $N_K \geq 0$ (both depending on $K$) such that

$$|T(\phi)| \leq C_K \|\phi\|_{N_K}, \quad \forall \phi \in D_K(\Omega).$$

3. $T$ is sequentially continuous.
(i) $\implies$ (ii)

- Let $T$ be continuous on $D(\Omega)$.
- Then, for any compact subset $K \subset \Omega$, the restriction of $T$ to $D_K(\Omega)$ is continuous (cf. Exercise 12).
- The inverse image of $(-c, c)$ under $T$ is an open set in $D_K(\Omega)$ containing origin.
- Since $D_K(\Omega)$ is first countable (normed space), there is a local base at 0.
- Thus, there is a $N_K$ and for all $\phi \in D_K(\Omega)$ such that $\|\phi\|_{N_K} \leq 1/N_K$, we have $|T(\phi)| < c$.
- Thus, for $\phi \in D_K(\Omega)$,

$$
\left| T \left( \frac{\phi}{N_K \|\phi\|_{N_K}} \right) \right| < c
$$

and hence $|T(\phi)| < N_K c \|\phi\|_{N_K}$. 

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Let \( \phi_m \to 0 \) in \( \mathcal{D}(\Omega) \).

By the inductive limit topology, there is a compact set \( K \) such that \( \| \phi_m \|_j \to 0 \) for all \( j \).

Using the fact that \( |T(\phi)| \leq C_K \| \phi \|_{N_K} \), we get \( |T(\phi_m)| \to 0 \).

Thus \( T(\phi_m) \to 0 \).

(iii) \( \implies \) (i): Let \( T \) be sequentially continuous.

Then, by Exercise 12, the restriction of \( T \) on \( D_K(\Omega) \), for every compact subset \( K \subset \Omega \), is also sequentially continuous.

But \( D_K(\Omega) \) is metrizable and hence \( T \) on \( D_K(\Omega) \) is continuous, for all compact \( K \) of \( \Omega \).

Thus, \( T \) is continuous on \( \mathcal{D}(\Omega) \).
Distribution and its Order

Definition

A linear functional $T$ on $\mathcal{D}(\Omega)$ is said to be a distribution on $\Omega$, if for every compact subset $K \subset \Omega$, there exists a constant $C_K > 0$ and an integer $N_K \geq 0$ (both depending on $K$) such that

$$|T(\phi)| \leq C_K \|\phi\|_{N_K}, \quad \forall \phi \in \mathcal{D}_K(\Omega).$$

- By Theorem 7, any continuous linear functional on $\mathcal{D}(\Omega)$ is a distribution.
- The space of all distributions in $\Omega$ is denoted by $\mathcal{D}'(\Omega)$.

Definition

If the $N_K$ is independent of $K$, i.e., the same $N$ is enough for all compact sets $K$, then the smallest such $N$ is called the order of $T$. If there exist no such $N$, we say $T$ is of infinite order.
Examples

Exercise

Which of the following are distributions? If your answer is affirmation, give the order of the distribution. If your answer is in negation, give reasons.

For $\phi \in \mathcal{D}(\mathbb{R})$, $T(\phi)$ is defined as:

(i) $\phi'(1) - \phi''(-2)$.
(ii) $\sum_{k=0}^{\infty} \phi^{(k)}(\pi)$.
(iii) $\sum_{k=0}^{\infty} \phi^{(k)}(k)$.
(iv) $\sum_{k=1}^{\infty} \frac{1}{k} \phi^{(k)}(k)$.
(v) $\int_{\mathbb{R}} \phi^2(x) \, dx$.

Exercise

Show that if $T \in \mathcal{D}'(\Omega)$ and $\omega$ is an open subset of $\Omega$, then $T \in \mathcal{D}'(\omega)$. 
Locally Integrable Functions

**Definition**

For $1 \leq p < \infty$, we say a function $f$ is locally $p$-integrable in $\Omega$, denoted as $f \in L^p_{\text{loc}}(\Omega)$, if $f$ is measurable and $\int_K |f(x)|^p \, dx < +\infty$, for every compact set $K \subset \Omega$.

- Any locally $p$-integrable function is locally integrable. i.e., if $f \in L^p_{\text{loc}}(\Omega)$ for all $1 < p < \infty$, then $f \in L^1_{\text{loc}}(\Omega)$. Because the Hölder’s inequality implies that for any compact subset $K$ of $\Omega$,

  $$
  \int_K |f(x)| \, dx \leq \left( \int_K |f|^p \, dx \right)^{1/p} (\mu(K))^{1/q} < +\infty.
  $$

- Any $L^p(\Omega)$, for $1 \leq p < \infty$, is in $L^p_{\text{loc}}(\Omega)$ and, hence, are in $L^1_{\text{loc}}(\Omega)$.

- In particular, $L^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$. This inclusion is strict because, if $\Omega$ is not of finite measure, then constant functions do not belong to $L^1(\Omega)$ but they are in $L^1_{\text{loc}}(\Omega)$.

- If $f$ is continuous on $\Omega$, then $f \in L^1_{\text{loc}}(\Omega)$. 

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Examples

- The function $\ln |x| \in L^1_{\text{loc}}(\mathbb{R})$ because

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} |\ln x| \, dx = \lim_{\varepsilon \to 0^+} - (x \ln x - x) \bigg|_{\varepsilon}^{1} = 1.$$ 

Because $\varepsilon \ln(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (use L’Hospital’s rule after proper substitution).

- The function $1/x \not\in L^1_{\text{loc}}(\mathbb{R})$ (after assigning a real value at $x = 0$). Similarly, $e^{1/x} \not\in L^1_{\text{loc}}(\mathbb{R})$. In both these cases the integral is not finite on a compact set containing origin.

- For any integer $n \geq 1$ and $\alpha > 0$, the function $|x|^{-\alpha} \in L^1_{\text{loc}}(\mathbb{R}^n)$, for all $0 < \alpha < n$, because

$$\int_{B_{\rho}(0)} |x|^{-\alpha} \, dx = \int_{S_{\rho}(0)} \int_{0}^{\rho} r^{-\alpha + n - 1} \, dr \, d\sigma.$$ 

Thus, for $-\alpha + n - 1 > -1$ or $\alpha < n$, the integral is finite and is equal to $\rho^{2n-\alpha-1} \frac{\omega_n}{n-\alpha}$, where $\omega_n$ is the surface measure of the unit ball.
Functions as Distributions

- We shall now observe that to every locally integrable function one can associate a distribution.
- For a $f \in L^1_{\text{loc}}(\Omega)$, we define the functional $T_f$ on $\mathcal{D}(\Omega)$ defined as,

$$T_f(\phi) = \int_{\Omega} f(x)\phi(x) \, dx.$$ 

- The functional $T_f$ is continuous on $\mathcal{D}(\Omega)$ (and, hence, is in $\mathcal{D}'(\Omega)$) because for every compact set $K$ in $\Omega$,

$$|T_f(\phi)| \leq \left( \int_K |f| \, dx \right) \|\phi\|_0, \quad \forall \phi \in \mathcal{D}_K(\Omega).$$

- The distribution $T_f$ is of zero order.
- We usually identify the distribution $T_f$ with the function $f$ that induces it.
Well Definedness

The constant function 0 on Ω induces the zero distribution (the zero functional on $\mathcal{D}(Ω)$).

**Exercise**

If $f \in L^1_{\text{loc}}(Ω)$, such that $T_f \equiv 0$, i.e., for all $\phi \in \mathcal{D}(Ω)$,

$$\int_{Ω} f \phi \, dx = 0$$

then $f = 0$ a.e. in $Ω$.

**Proof:** Note that it is enough to prove that $f = 0$ a.e. on all compact subsets $K$ of $Ω$. For any compact subset $K$ of $Ω$, choose $\phi_K \in \mathcal{D}(Ω)$ such that $\phi_K \equiv 1$. Define, for each $x \in K$,

$$f_ε(x) := f \phi_K * ρ_ε(x) = \int_{K} f(y)\phi_K(y)ρ_ε(x - y) \, dy.$$ 

Since $y \mapsto \phi_K(y)ρ_ε(x - y)$ is in $\mathcal{D}(Ω)$, we have $f_ε \equiv 0$ on $K$. 
Regular and Singular Distribution

Since $f_\varepsilon \to f$ in $L^1(K)$,

$$0 = \lim_{\varepsilon \to 0} \| f_\varepsilon - f \|_{1,K} = \lim_{\varepsilon \to 0} \int_K |f_\varepsilon - f| \, dx = \int_K |f| \, dx = \| f \|_{1,K}$$

and hence $f = 0$ a.e. in $K$.

- The association $f \mapsto T_f$ is well-defined because if $f = g$ a.e. in $\Omega$, then the distributions $T_f = T_g$.
- Thus, we $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$. In particular, $\mathcal{D}(\Omega) \subset \mathcal{D}'(\Omega)$.
- Any distribution that is induced by a locally integrable function is called a regular distribution. Otherwise, the distribution is called singular.
Radon Measures

- Let $\mathcal{M}_b(\Omega)$ denote the space of all signed finite Radon measures on the Borel $\sigma$-algebra of $\Omega$. It is a Banach space in the total variation norm.
- The Lebesgue measure on $\Omega$ restricted to the Borel $\sigma$-algebra of $\Omega$ is a Radon measure.
- The Dirac measure is a Radon measure that assigns a total mass of 1 at a point $a \in \Omega$. Thus, on the Borel measurable subsets of $\Omega$,

$$
\delta_a(E) = \begin{cases} 
1 & \text{if } a \in E \\
0 & \text{if } a \notin E.
\end{cases}
$$
Measures as Distributions

- To each Radon measure $\mu \in \mathcal{M}_b(\Omega)$, we associate a linear functional on $\mathcal{D}(\Omega)$ as follows:

$$T_\mu(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in \mathcal{D}(\Omega).$$

The functional $T_\mu$ is continuous on $\mathcal{D}(\Omega)$ (and is in $\mathcal{D}'(\Omega)$) because for every compact set $K$ in $\Omega$,

$$|T_\mu(\phi)| \leq \left( \int_K d\mu \right) \|\phi\|_{\infty,K} = |\mu|(K)\|\phi\|_{\infty,\Omega}, \quad \forall \phi \in D_K(\Omega),$$

where $|\mu|$ denotes the total variation of the measure.

- The distribution $T_\mu$ is of zero order.
Examples

- The distribution induced by the Lebesgue measure is same as that induced by the locally integrable constant function 1 on $\Omega$. Thus, the distribution induced by Lebesgue measure is a \textit{regular} distribution.

- The distribution induced by Dirac measure $\delta_a$, called \textit{Dirac distribution}, is:

  $$
  \delta_a(\phi) = \int_{\Omega} \phi(x) \, d\delta_a.
  $$

Exercise

Show that, for any function $g$ on $\Omega$,

$$
\int_{\Omega} g(x) \, d\delta_a = g(a).
$$

With the above observations, we note that the Dirac distribution is just

$$
\delta_a(\phi) = \phi(a) \quad \forall \phi \in \mathcal{D}(\Omega). \quad (3.1)
$$
A measure \( \mu \) is absolutely continuous w.r.t another measure \( \nu \) if for every element \( E \) of the \( \sigma \)-algebra \( \mu(E) = 0 \) whenever \( \nu(E) = 0 \), denoted as \( \mu \ll \nu \).

The Lebesgue measure is absolutely continuous w.r.t counting measure but the converse is not true.

The absolutely continuous measures w.r.t the Lebesgue measure are precisely the one that induce regular distribution.

For each \( f \in L^1_{\text{loc}}(\Omega) \), one can define a signed measure

\[
\mu_f(E) := \int_E f \, dx
\]  

(3.2)

for all measurable subsets \( E \) of \( \Omega \).

Conversely, the Radon-Nikodym theorem states that, for any absolutely continuous measure \( \mu \) w.r.t the Lebesgue measure, there is a \( f \in L^1_{\text{loc}}(\Omega) \) such that, for all \( \mu \)-measurable subsets \( E \) of \( \Omega \) (3.2) is valid.
Proposition

The Dirac distribution is a singular distribution, i.e., there is no $f \in L^1_{loc}(\Omega)$ such that $T_f = \delta_a$.

Proof:

- Note that for any $a \in \Omega$, since $\Omega$ is open, there is a $\varepsilon_0 > 0$ such that $B(a, \varepsilon_0) \subset \Omega$.
- For each $0 < \varepsilon < \varepsilon_0$, we choose $\phi_\varepsilon \in D(\Omega)$ with support in $B(a; \varepsilon)$, $0 \leq \phi_\varepsilon \leq 1$ and $\phi_\varepsilon = 1$ in $B(a; \frac{\varepsilon}{2})$.
- Thus,

$$\delta_a(\phi_\varepsilon) = \phi_\varepsilon(a) = 1 \quad \text{for all } 0 < \varepsilon < \varepsilon_0.$$
Suppose \( f \) is a locally integrable function such that \( T_f = \delta_a \), then
\[
\delta_a(\phi_\varepsilon) = T_f(\phi_\varepsilon) = \int_\Omega f\phi_\varepsilon \, dx = \int_{B(a;\varepsilon)} f\phi_\varepsilon \, dx \leq \int_{B(a;\varepsilon)} |f| \, dx.
\]

Therefore, \( 1 \leq \int_{B(a;\varepsilon)} |f| \, dx \).

Since \( f \in L^1_{\text{loc}}(\Omega) \), the quantity on RHS is finite.

Hence, as \( \varepsilon \to 0 \), we get a contradiction \( 1 \leq 0 \).
Theorem

A distribution $T \in D'(\Omega)$ is of zero order iff $T$ is a distribution induced by a Radon measure.

Proof:

- The implication that a Radon measure induced distribution is of zero order is already shown.

- Conversely, let $T$ be a distribution of zero order.

- Then, for every compact set $K \subset \Omega$, there is a $C_K > 0$ such that

$$|T(\phi)| \leq C_K \|\phi\|_0 \quad \forall \phi \in D_K(\Omega).$$
The idea is to continuously extend $T$ to $C_c(\Omega)$, in a unique way, and then consider the Radon measure obtained from

**Theorem (Riesz-Markov)**

Consider $C_c(\Omega)$ endowed with the inductive limit topology. Then there is an isometric isomorphism between the dual of $C_c(\Omega)$ and $\mathcal{M}_b(\Omega)$, i.e. for any continuous linear functional $T : C_c(\Omega) \to \mathbb{R}$ there is a unique Radon measure $\mu \in \mathcal{M}_b(\Omega)$ such that

$$T(\phi) = \int_{\Omega} \phi(x) \, d\mu \quad \forall \phi \in C_c(\Omega).$$

This association $T \mapsto \mu$ defines an isometry, i.e., $\| T \| = |\mu|$.

- Let $\phi \in C_c(\Omega)$ and define $\phi_\varepsilon := \phi \ast \rho_\varepsilon$.
- By Theorem 1, $\phi_\varepsilon$ converges uniformly to $\phi$. 
The sequence \( \{ T(\phi_\varepsilon) \} \) is Cauchy in \( \mathbb{R} \) because

\[
|T(\phi_\varepsilon) - T(\phi_\delta)| = |T(\phi_\varepsilon - \phi_\delta)| \leq C_K \|\phi_\varepsilon - \phi_\delta\|_0.
\]

Hence the sequence \( \{ T(\phi_\varepsilon) \} \) converges, thus we extend \( T \) uniquely on \( C_c(\Omega) \) as

\[ T(\phi) = \lim_{\varepsilon \to 0} T(\phi_\varepsilon). \]

Thus, Theorem 9, there is a Radon measure associated to the extended \( T \).
Multipole Distributions

- The formulation of Dirac distribution as given in the equation (3.1) motivates, for a fixed $n$-tuple $\alpha$ of non-negative integers and $a \in \Omega$ of $\mathbb{R}^n$, the linear functional on $\mathcal{D}(\Omega)$

$$\delta^\alpha_a(\phi) = D^\alpha \phi(a), \quad \forall \phi \in \mathcal{D}(\Omega).$$

- This functional is continuous on $\mathcal{D}(\Omega)$ because, for all $\phi \in D_K(\Omega)$,

$$|\delta^\alpha_a(\phi)| = |D^\alpha \phi(a)| \leq \|\phi\| |\alpha|.$$

- The order of this distribution is, at most, $|\alpha|$. The situation $|\alpha| = 0$ corresponds to the Dirac distribution. The situation $|\alpha| = 1$ is called the dipole or doublet distribution.
Proposition

Any $k$-th multipole distribution, $\delta_0^{(k)}$ cannot have order less than $k$ and hence is of order $k$.

Proof:

- Suppose that for all compact set $K \subset \mathbb{R}$ and $\phi \in D_K(\Omega)$ we have
  \[ |\delta_0^{(k)}(\phi)| \leq C \|\phi\|_j \quad \text{for some } 0 \leq j < k. \]

- Now, choose $\phi \in D_K(\Omega)$ such that $0 \in \text{Int}(K)$ and $\phi^{(k)}(0) \neq 0$.
- For each integer $m \geq 1$, set $\phi_m(x) := m^{-k}\phi(mx)$.
- Note that $\phi_m^{(j)}(x) = m^{j-k}\phi^{(j)}(mx)$. 

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Thus,

\[ 0 \neq |\phi^{(k)}(0)| = |\phi_m^{(k)}(0)| = |\delta_0^{(k)}(\phi_m)| \leq C\|\phi_m\|_j = C\sum_{i=0}^{j} \|\phi_m^{(i)}\|_0 \]

\[ \neq C\sum_{i=0}^{j} m^{i-k} \|\phi^{(i)}\|_0 \leq Cm^{j-k} \sum_{i=0}^{j} \|\phi^{(i)}\|_0 = Cm^{j-k} \|\phi\|_j. \]

This is a contradiction because \( j < k \) and RHS converges to zero, as \( m \) increases whereas LHS is strictly positive.
The dipole distribution is singular. In fact, the situation is much worse. The dipole distribution is not induced by any Radon measure.

**Proposition**

The dipole distribution $\delta_a^{(1)} \in \mathcal{D}(\mathbb{R})$ is not induced by any Radon measure, i.e., there is no Radon measure $\mu$ such that $T_{\mu} = \delta_a^{(1)}$.

**Proof:**

- Let $\Omega = \mathbb{R}$ and $a = 0$. Choose $\phi \in \mathcal{D}(\mathbb{R})$ with support in $[-1, 1]$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $[-1/2, 1/2]$.
- Set $\phi_m(x) = \sin(mx)\phi(x)$.
- Thus, $\text{supp}(\phi_m) \subset [-1, 1]$ and $|\phi_m| \leq 1$.
- Note that the derivative of $\phi_m$ is $\phi'_m(x) = \sin(mx)\phi'(x) + m\cos(mx)\phi(x)$ and, hence, $\phi'_m(0) = m$ for all positive integer $m > 0$. 
Therefore, the dipole distribution takes the value,

$$\delta_0^{(1)}(\phi_m) = \phi'_m(0) = m, \quad \forall m \in \mathbb{N}.$$ 

Suppose there exists a Radon measure $\mu$ inducing the dipole distribution $\delta_0^{(1)}$, then

$$m = |\delta_0^{(1)}(\phi_m)| = |T_\mu(\phi_m)| = \left| \int_{-1}^{1} \phi(x) \sin(mx) \, d\mu \right| \leq |\mu|(B(0; 1)).$$

The inequality $|\mu|(B(0; 1)) \geq m$, for all $m \in \mathbb{N}$, implies that $|\mu|(B(0; 1))$ is infinite which contradicts the fact that the Radon measure $\mu$ is finite on compact subsets of $\mathbb{R}$. 

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Example

Define the functional $T$ on $\mathcal{D}(\mathbb{R})$ as $T(\phi) := \sum_{k=0}^{\infty} \phi^{(k)}(k)$. Without loss of generality, consider the compact set $[0, m]$. If $\phi \in C^\infty[0, m]$, then

$$|T(\phi)| \leq \sum_{k=0}^{m} |\phi^{(k)}(k)| = \sum_{k=0}^{m} \|\phi^{(k)}\|_0 = \|\phi\|_m.$$ 

The larger the compact set becomes, the higher the derivatives of $\phi$ needs to be taken in. Thus, there is no fixed $m$ for all compact subsets of $\mathbb{R}$. 
Example

Define the functional $T \in \mathcal{D}'(0, \infty)$ as

$$T(\phi) = \sum_{k=1}^{\infty} \phi^{(k)}(1/k).$$

Let $\phi \in \mathcal{D}(0, \infty)$ be such that $\text{supp}(\phi) \subset [1/m, m]$ (exhaustion sets), then

$$T(\phi) = \sum_{k=1}^{m} \phi^{(k)}(1/k) \leq \sum_{k=1}^{m} \|\phi^{(k)}\|_0 \leq \sum_{k=0}^{m} \|\phi^{(k)}\|_0 = \|\phi\|_m.$$

Thus, $T \in \mathcal{D}'(0, \infty)$ is a distribution of infinite order.
For every $\phi \in \mathcal{D}(\Omega)$, one can define the linear functional $\Lambda_\phi : \mathcal{D}'(\Omega) \to \mathbb{R}$ as follows, $\Lambda_\phi(T) = T(\phi)$. Thus, the linear functionals $\Lambda_\phi$ are included in the second (algebraic) dual of $\mathcal{D}(\Omega)$.

For every open subset of $V \subset \mathbb{R}$, consider the collection of subsets $\{\Lambda_\phi^{-1}(V)\}$ of $\mathcal{D}'(\Omega)$, for all $\phi \in \mathcal{D}(\Omega)$. The topology given is the topology generated by this collection of subsets in $\mathcal{D}'(\Omega)$.

The endowed topology is the weak-* topology; the coarsest (weakest or smallest topology, one with fewer open sets) topology on $\mathcal{D}'(\Omega)$ such that all the linear maps $\Lambda_\phi : \mathcal{D}'(\Omega) \to \mathbb{R}$, corresponding to each $\phi \in \mathcal{D}(\Omega)$, are continuous.
Cauchy Sequence of $\mathcal{D}'(\Omega)$

Since $\mathcal{D}(\Omega)$ is not metrizable, the space $\mathcal{D}'(\Omega)$ is sequentially weak-∗ complete but may not be weak-∗ complete.

**Definition**

We say a sequence of distributions $\{T_m\}$ is Cauchy in $\mathcal{D}'(\Omega)$, if $T_m(\phi)$ is Cauchy in $\mathbb{R}$, for all $\phi \in \mathcal{D}(\Omega)$. A sequence $\{T_m\} \subset \mathcal{D}'(\Omega)$ converges to $T$ in weak-∗ topology of $\mathcal{D}'(\Omega)$ or in the distribution sense if $T_m(\phi) \rightarrow T(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$, denoted as $T_m \rightharpoonup T$.

**Exercise**

If $S_m$ and $T_m$ are sequences of distributions converging to $S$ and $T$, respectively, then show that $\lambda S_m + \mu T_m$ converges to $\lambda S + \mu T$, for all $\lambda, \mu \in \mathbb{R}$. 
Proposition

Let \( \{T_m\} \) be a Cauchy sequence in \( \mathcal{D}'(\Omega) \) and let \( T(\phi) := \lim_{m \to \infty} T_m(\phi) \) (\( T \) is well-defined because \( \mathbb{R} \) is complete). Then \( T \in \mathcal{D}'(\Omega) \).

Proof:

- To show \( T \in \mathcal{D}'(\Omega) \), it is enough to show that \( T : D_K(\Omega) \to \mathbb{R} \) is continuous for all compact sets \( K \subset \Omega \).
- Fix a compact set \( K \subset \Omega \) and \( \phi \in D_K(\Omega) \).
- Since \( \{T_m(\phi)\} \) is convergent in \( \mathbb{R} \) it is bounded and hence there is a real constant \( C_{\phi} > 0 \) (may depend on \( \phi \)) such that \( \sup_m |T_m(\phi)| \leq C_{\phi} \).
- The family \( \{T_m\} \) is point-wise bounded on \( D_K(\Omega) \). Then, by uniform boundeness principle on \( D_K(\Omega) \), \( T \) is uniformly bounded, i.e., there is a constant \( C > 0 \) (independent of \( \phi \)) such that \( \sup_m |T_m(\phi)| \leq C \) for all \( \phi \in D_K(\Omega) \).
Therefore,

\[ |T(\phi)| \leq \sup_m |T_m(\phi)| \leq C \quad \forall \phi \in D_K(\Omega) \]

and, hence, \( T \) is continuous on \( C^\infty(K) \) for all compact subsets \( K \subset \Omega \).

Thus, \( T \) is continuous on \( \mathcal{D}(\Omega) \) and hence \( T \in \mathcal{D}'(\Omega) \).
Comparing Topologies

- The weak-* convergence in $\mathcal{D}'(\Omega)$ is weaker than any other topology that preserves the continuity of the functionals $\Lambda_\phi$.
- For instance, this topology is weaker than $L^p$-topology or uniform norm topology etc., i.e. convergence of classical functions in $p$-norm or uniform norm implies convergence in distribution sense.

Example

Consider the sequence $f_m(x) = e^{imx}$ on $\mathbb{R}$ which converges point-wise to 1 iff $x \in 2\pi \mathbb{Z}$. The sequence of distributions corresponding to $f_m$ converges to zero in the distributional sense because

$$\int e^{imx} \phi(x) \, dx = -\frac{1}{im} \int e^{imx} \phi'(x) \, dx.$$ 

We have used integration by parts to get the second integral and since the integrand is bounded in the second integral, it converges to zero as $m \to \infty$. 

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Let $\rho \in C_c(\mathbb{R}^n)$ be such that $\text{supp}(\rho) \subset B(0; 1)$ and $\int_{\mathbb{R}^n} \rho(y) \, dy = 1$. Now, set $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$. This sequence is called the Dirac Sequence.

In particular, the sequence of mollifiers (cf. (1.4)) is one example.

Another trivial example is

$$
\rho(x) = \begin{cases} 
2 & |x| < 1 \\
0 & |x| \geq 1.
\end{cases}
$$
We claim that the sequence $\rho_\varepsilon$ converges to the Dirac distribution $\delta_0$, in the distribution sense.

Let $T_\varepsilon$ denote the distribution corresponding to $\rho_\varepsilon$. For any $\phi \in \mathcal{D}(\mathbb{R}^n)$, consider

$$T_\varepsilon(\phi) = \int_{\mathbb{R}^n} \varepsilon^{-n} \rho \left( \frac{x}{\varepsilon} \right) \phi(x) \, dx = \int_{\mathbb{R}^n} \rho(y) \phi(\varepsilon y) \, dy.$$ 

Taking limit $\varepsilon \to 0$ both sides, we get

$$\lim_{\varepsilon \to 0} T_\varepsilon(\phi) = \int_{\mathbb{R}^n} \rho(y) \phi(0) \, dy = \phi(0) = \delta_0(\phi).$$

The interchange of limit is possible due to uniform continuity of $\phi$ which induces the uniform convergence of $\phi(\varepsilon y) \to \phi(0)$.

Thus, $T_\varepsilon \to \delta_0$ in the distribution sense, whereas the point-wise limit at $x = 0$ did not exist for $\rho_\varepsilon$. 
Example

- Consider the sequence of functions

\[
  f_m(x) = \begin{cases} 
    m^2 x & 0 \leq x < \frac{1}{m} \\
    m^2 \left( \frac{2}{m} - x \right) & \frac{1}{m} < x \leq \frac{2}{m} \\
    0 & \text{otherwise.}
  \end{cases}
\]

- Note that they converge point-wise to zero for all \( x \in \mathbb{R} \).

- Let \( T_m \) denote the distribution corresponding to \( f_m \) and hence

\[
  T_m(\phi) = m^2 \int_0^{1/m} x \phi(x) \, dx + m^2 \int_{1/m}^{2/m} \left( \frac{2}{m} - x \right) \phi(x) \, dx.
\]

- Both the integral above converges to \( \phi(0)/2 \) and hence the sequence of distributions converges to \( \delta_0 \).

Exercise

Give a proof that both the integral converges to the limit as claimed above.
Weak Dominated Convergence Theorem

The following theorem gives the condition that is violated by the above example for the point-wise limit to coincide with the distributional limit.

**Theorem**

Let \( \{f_m\} \subset L^1_{loc}(\Omega) \) such that \( f_m(x) \to f(x) \) point-wise for a.e. \( x \in \Omega \) and there is a \( g \in L^1_{loc}(\Omega) \) such that for every compact set \( K \subset \Omega \) \( |f_m| \leq g \) for all \( m \). Then \( f \in L^1_{loc}(\Omega) \) and \( f_m \rightharpoonup f \) in \( \mathcal{D}'(\Omega) \).

**Proof:**

- Let \( T_m \) denote the distribution corresponding to \( f_m \) and fix \( \phi \in \mathcal{D}(\Omega) \).
- Set \( h_m(x) = f_m(x)\phi(x) \) and \( h(x) = f(x)\phi(x) \). Note that \( h_m(x) \to h(x) \) point-wise a.e. in \( \Omega \) (in fact in the \( \text{supp}(\phi) \)).
- Also, \( |h_m| \leq g|\phi| \) and \( g|\phi| \in L^1(\Omega) \). By classical Lebesgue's dominated convergence theorem, \( h \in L^1(\Omega) \) and

\[
\int_{\Omega} h(x) \, dx = \lim_{m \to \infty} \int_{\Omega} h_m(x) \, dx = \lim_{m \to \infty} \int_{\Omega} f_m(x)\phi(x) \, dx.
\]
Proof Continued...

- In particular, choosing $\phi \equiv 1$ on a given compact set $K$ of $\Omega$ and since for this $\phi$, $h \in L^1(\Omega)$, we have $f \in L_{\text{loc}}^1(\Omega)$ because

$$\infty > \int_\Omega |h| \, dx = \int_\Omega |f\phi| \, dx = \int_K |f| \, dx.$$ 

- Let $T_f$ be the distribution corresponding to $f$. Moreover,

$$\lim_{m \to \infty} T_m(\phi) = \lim_{m \to \infty} \int_\Omega f_m \phi \, dx = \int_\Omega f \phi \, dx = T_f(\phi).$$

**Corollary**

Let $\{f_m\} \subset C(\Omega)$ be a sequence of continuous functions that converges uniformly on compact subsets of $\Omega$ to $f$. Then $f_m \rightharpoonup f$ in $\mathcal{D}'(\Omega)$. 

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Series of Distributions

- By the vector space structure of $\mathcal{D}'(\Omega)$, we already have the finite sum of distributions, i.e. if $\{ T_i \}_{1}^{k} \subset \mathcal{D}'(\Omega)$ then
  $$T := \sum_{i=1}^{k} T_i \in \mathcal{D}'(\Omega)$$
defined as $T(\phi) = \sum_{i=1}^{k} T_i(\phi)$.

- The topology on $\mathcal{D}'(\Omega)$ can be used to give the notion of series of distributions.

**Definition**

*For any countable collection of distributions $\{ T_i \}_{1}^{\infty} \subset \mathcal{D}'(\Omega)$ the series $\sum_{i=1}^{\infty} T_i$ is said to converge to $S \in \mathcal{D}'(\Omega)$ if the sequence of partial sums $T_m := \sum_{i=1}^{m} T_i$ converges to $S$ in $\mathcal{D}'(\Omega)$.***
Singularity at Infinity

In practice, we encounter functions whose integral diverges. This motivates other ways of studying divergent integrals, like principal part, finite part of divergent integral.

**Definition**

We say that the integral of a function $f$ (with singularity at $\infty$) exists in the generalised sense if the limit

$$\lim_{a, b \to +\infty} \int_{-a}^{b} f(x) \, dx$$

exists. We say that the principal value of the integral exists if the limit (with $b = a$)

$$\lim_{a \to +\infty} \int_{-a}^{b} f(x) \, dx$$

exists.
Singularity at a Point

Definition

We say that the integral of a compact supported function $f$ (with singularity at a point 0) exists in the generalised sense if the limit

$$\lim_{a,b \to 0^+} \int_{-\infty}^{-a} f(x) \, dx + \int_{b}^{\infty} f(x) \, dx$$

exists. We say that the principal value of the integral exists if the limit (with $b = a$)

$$\lim_{a \to 0^+} \int_{|x| > a} f(x) \, dx$$

exists.
Principal Value Distribution

We noted that $1/x \notin L^1_{\text{loc}}(\mathbb{R})$, however, we do know that $1/x \in L^1_{\text{loc}}(\mathbb{R} \setminus \{0\})$ and, hence, induces a distribution in $\mathcal{D}(\mathbb{R} \setminus \{0\})$. One can extend this distribution to yield a distribution corresponding $1/x$ on $\mathbb{R}$.

**Definition**

We define the linear functional $PV\left(\frac{1}{x}\right)$ on $\mathcal{D}(\mathbb{R})$ as

$$PV\left(\frac{1}{x}\right)(\phi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \phi(x) \, dx.$$  \hspace{1cm} (4.1)

The limit on RHS is called the Cauchy’s Principal Value.

Equivalently, $PV(1/x)$ is defined as the distributional limit, $f_\varepsilon \rightharpoonup PV\left(\frac{1}{x}\right)$ in $\mathcal{D}'(\Omega)$, where

$$f_\varepsilon(x) := \begin{cases} \frac{1}{x} & |x| > \varepsilon, \\ 0 & |x| \leq \varepsilon. \end{cases}$$
The limit on RHS exists

**Lemma**

The principal value of $1/x$ (limit on the RHS) exists.

**Proof:**

- Let $\phi \in \mathcal{D}(\mathbb{R})$ and, WLOG, let $\text{supp}(\phi) \subset [-a, a]$ for some real $a > 0$.
- For $\varepsilon > 0$ small enough, consider

\[
\int_{|x| \geq \varepsilon} \frac{\phi(x)}{x} \, dx = \int_{-a}^{-\varepsilon} \frac{\phi(x)}{x} \, dx + \int_{\varepsilon}^{a} \frac{\phi(x)}{x} \, dx
\]

\[
= \int_{\varepsilon}^{a} \frac{\phi(x)}{x} \, dx - \int_{\varepsilon}^{a} \frac{\phi(-x)}{x} \, dx
\]

\[
= \int_{\varepsilon}^{a} \frac{\phi(x) - \phi(-x)}{x} \, dx.
\]
Therefore, taking limit both sides as $\varepsilon \to 0$,

\[
\text{PV} \left( \frac{1}{x} \right) (\phi) = \int_0^a \frac{\phi(x) - \phi(-x)}{x} \, dx
\]

\[
= \int_0^a \frac{1}{x} \left( \int_{-x}^x \phi'(s) \, ds \right) \, dx \quad \text{(Using FTC)}
\]

\[
= \int_0^a \frac{1}{x} \left( \int_{-1}^1 x\phi'(xt) \, dt \right) \, dx.
\]

Set $\psi(x) = \int_{-1}^1 \phi'(xt) \, dt$, then

\[
\left| \text{PV} \left( \frac{1}{x} \right) (\phi) \right| = \left| \int_0^a \frac{1}{x} x\psi(x) \, dx \right| = \left| \int_0^a \psi(x) \, dx \right| \leq \int_0^a |\psi(x)| \, dx.
\]

But $|\psi(x)| \leq 2\|\phi'\|_0 \leq 2\|\phi\|_1$. Hence,

\[
\left| \text{PV} \left( \frac{1}{x} \right) (\phi) \right| \leq 2a\|\phi\|_1.
\]

Since RHS is finite, the limit is finite.
Order of PV(1/x)

- The proof above highlights that the PV(1/x) distribution can have order, at most, one.
- Infact, it cannot be of zero order and hence its order is one.
- Observe the following characterisation of principal value distribution in the above argument when supp(φ) ⊂ [−a, a],
  \[ \text{PV} \left( \frac{1}{x} \right)(φ) = \int_0^a \frac{\phi(x) - \phi(-x)}{x} \, dx. \]
- Choose φ such that supp(φ) ⊂ [0, a], 0 ≤ φ ≤ 1 and φ ≡ 1 on [c, d] ⊂ [0, a] with c ≠ 0 and d ≠ a.
Order of $\text{PV}(1/x)$

Then,

$$\text{PV}(1/x)(\phi) = \int_0^c \frac{\phi(x)}{x} \, dx + \int_c^d \frac{\phi(x)}{x} \, dx + \int_d^a \frac{\phi(x)}{x} \, dx$$

$$> \int_c^d \frac{1}{x} \, dx$$ \hspace{1cm} \text{I and III integral being non-negative}

$$> \frac{1}{a} \int_c^d \, dx$$ \hspace{1cm} \text{since } x < a

$$= \frac{1}{a} (d - c) \geq \frac{d - c}{a} \sup_{x \in [0,a]} |\phi| = \frac{d - c}{a} \|\phi\|_0.$$ 

Thus, the distribution cannot be of zero order.
Alternate Ways for PV(1/x)

- The principal value distribution is not the only choice for the function $1/x$.
- Consider the linear functional on $\mathcal{D}(\mathbb{R})$ corresponding to $1/x$,

$$
T(\phi) = \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} \, dx + \int_{2\varepsilon}^{\infty} \frac{\phi(x)}{x} \, dx.
$$

**Exercise**
Show that the $T$ defined above is a distribution. Also, compare $T$ and PV(1/x).
Alternate Ways for PV($1/x$)

- Consider the function $f_{\varepsilon} : \mathbb{R} \to \mathbb{C}$, for each $\varepsilon > 0$, defined as
  
  $$f_{\varepsilon}(x) = \frac{1}{x + i\varepsilon}.$$

- The limit of $f_{\varepsilon}$, as $\varepsilon \to 0$, when $x \neq 0$ is $1/x$ but is, classically, not defined for $x = 0$.
- Also, note that the integral $\int_{-1}^{1} (1/x) \, dx$ does not converge.
- However,
  
  $$\lim_{\varepsilon \to 0} \int_{-1}^{1} f_{\varepsilon}(x) \, dx = \lim_{\varepsilon \to 0} \ln(x + i\varepsilon) \big|_{-1}^{1} = \ln(1) - \ln(-1) = -i\pi.$$

  Thus, we expect $\int_{-1}^{1} (1/x) \, dx = -i\pi$.
- This cannot be made sense classically, but using the theory of distributions one can give a meaning to this observation.
Alternate Ways for PV(1/x)

Yet another way to defining $PV(1/x)$ is: consider the distributions $T_\varepsilon$ corresponding to

$$f_\varepsilon(x) := \frac{1}{x \pm i\varepsilon}.$$

**Exercise**

Show that $T_\varepsilon \rightharpoonup PV(1/x) \mp i\pi\delta_0$. 
Functions, but not Distributions

- It is not always possible to extend the notion of distributions to non-locally integrable functions as done above.
- The function $e^{1/x} \notin L^1_{loc}(\mathbb{R})$ but is in $L^1_{loc}(\mathbb{R} \setminus \{0\})$.
- Let $T \in \mathcal{D}'(0, \infty)$ be the distribution corresponding to $e^{1/x}$ on $(0, \infty)$.

Exercise

Show that there is no distribution $S \in \mathcal{D}'(\mathbb{R})$ corresponding to $e^{1/x}$ on $\mathbb{R}$ whose restriction coincides with the distribution $T$ on $(0, \infty)$.

Exercise

Consider the distribution $T \in \mathcal{D}'(0, \infty)$ defined in Example 2, i.e., $T(\phi) = \sum_{k=1}^{\infty} \phi^{(k)}(1/k)$. Show that there is no distribution $S \in \mathcal{D}'(\mathbb{R})$ whose restriction to $(0, \infty)$ coincides with $T$. 
Differentiation

- The space of distributions $\mathcal{D}'(\Omega)$ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) with the usual operation of addition and scalar multiplication. Hence, for any two $S, T \in \mathcal{D}'(\Omega)$, $(S + T)(\phi) = S(\phi) + T(\phi)$. Also, for any $\lambda \in \mathbb{R}$ and $T \in \mathcal{D}'(\Omega)$, the distribution $\lambda T$ is defined as $(\lambda T)(\phi) = \lambda T(\phi)$.

- Recall the discussion leading to (1.2). For any $f \in C^\infty(\mathbb{R})$, $\phi \in \mathcal{D}(\mathbb{R})$ and all $k \in \mathbb{N}$, using integration by parts we have
  \[
  \int f^{(k)} \phi \, dx = (-1)^k \int f \phi^{(k)} \, dx.
  \]
  This motivates the following definition of derivative of a distribution.

**Definition**

For any $T \in \mathcal{D}'(\Omega)$ and $n$-tuple $\alpha$, the derivative $D^\alpha T$ of $T$ is defined as

\[
(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \quad \phi \in \mathcal{D}(\Omega).
\]
Order of Distributional Derivative

- The derivative $D^\alpha T$ is also a distribution because for all compact subsets $K$ of $\Omega$ there are $C_K > 0$ and non-negative integer $N_K$ such that

$$|(D^\alpha T)(\phi)| = |T(D^\alpha \phi)| \leq C_K \|D^\alpha \phi\|_{N_K} \leq C_K \|\phi\|_{N_K+|\alpha|}. $$

Thus, $D^\alpha T \in \mathcal{D}'(\Omega)$.

- For a finite order distribution $T$, the derivative $D^\alpha T$ has $|\alpha|$ order more than $T$, i.e., differentiation operation increases the order of the distribution.

- Observe that every distribution is infinitely differentiable and the mixed derivatives are equal because the same holds for test functions.

Exercise

Let $S, T \in \mathcal{D}'(\Omega)$. Show that

$(i)$ $D^\beta (D^\alpha T) = D^\alpha (D^\beta T) = D^{\alpha+\beta} T$ for all multi-indices $\alpha, \beta$.

$(ii)$ $D^\alpha (\lambda S + \mu T) = \lambda D^\alpha S + \mu D^\alpha T$, for each $\lambda, \mu \in \mathbb{R}$. 
Exercise

Let $n = 1$, $\Omega = (0, 2)$ and

$$u(x) = \begin{cases} 
  x & \text{if } 0 < x \leq 1 \\
  1 & \text{if } 1 \leq x < 2.
\end{cases}$$

Compute the first distributional derivative of $u$.

Exercise

Let $n = 1$, $\Omega = (0, 2)$ and

$$u(x) = \begin{cases} 
  x & \text{if } 0 < x \leq 1 \\
  2 & \text{if } 1 < x < 2.
\end{cases}$$

Compute the first distributional derivative of $u$. 
Derivatives of Dirac Measure

Example

The $k$-th distributional derivative of the Dirac distribution is the $k$-th multipole distribution. For $\phi \in \mathcal{D}(\Omega)$,

$$D^\beta \delta^\alpha_a(\phi) = (-1)^{|\beta|} \delta^\alpha_a (D^\beta \phi) = (-1)^{|\beta|} D^\alpha (D^\beta \phi(a)) = (-1)^{|\beta|} D^{\alpha+\beta} \phi(a) = (-1)^{|\beta|} \delta^\alpha_a + \beta (\phi).$$

In particular, the distributional derivative of the Dirac distribution is the dipole distribution, up to a factor of sign.
Weak Derivatives

**Definition**

A function \( f \in L^1_{\text{loc}}(\Omega) \) is said to be \( \alpha \)-weakly differentiable if \( D^\alpha T_f \) is a regular distribution. In other words, for any given multi-index \( \alpha \), a function \( f \in L^1_{\text{loc}}(\Omega) \) is said to be \( \alpha \)-weakly differentiable if there exists a \( g_\alpha \in L^1_{\text{loc}}(\Omega) \) such that

\[
\int_{\Omega} g_\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} f D^\alpha \phi \, dx \quad \forall \phi \in \mathcal{D}(\Omega).
\]

Consider the continuous function \( f : \mathbb{R} \to \mathbb{R} \) defined as \( f(x) = |x| \). Classically, its derivative exists a.e. and is the function

\[
f'(x) = \begin{cases} 
-1 & -\infty < x < 0 \\
1 & 0 < x < \infty.
\end{cases}
\]
Weak Derivative of $|x|$

- The distribution corresponding to $f'$ is

$$T_{f'}(\phi) = - \int_{-\infty}^{0} \phi(x) \, dx + \int_{0}^{\infty} \phi(x) \, dx$$

- However, for $\phi \in \mathcal{D}(\mathbb{R})$, the distributional derivative of $T_f$ is

$$DT_f(\phi) = - T_f(\phi') = - \int_{-\infty}^{0} (-x)\phi'(x) \, dx - \int_{0}^{\infty} x\phi'(x) \, dx$$

$$= x\phi(x) \bigg|_{-\infty}^{0} - \int_{-\infty}^{0} \phi(x) \, dx + \int_{0}^{\infty} \phi(x) \, dx - x\phi(x) \bigg|_{0}^{\infty}$$

$$= \int_{0}^{\infty} \phi(x) \, dx - \int_{-\infty}^{0} \phi(x) \, dx$$

$$= T_{f'}(\phi).$$

- Therefore, $DT_f = T_{f'}$ and $|x|$ is weakly differentiable.
Example

Consider the everywhere discontinuous function

\[ f(x) = \begin{cases} 
1 & x \in \mathbb{Q}^c \\
0 & x \in \mathbb{Q}.
\end{cases} \]

The function \( f \) is not differentiable at any point. The Lebesgue measure of \( \mathbb{Q} \) is zero, hence,

\[ DT_f(\phi) = -T_f(\phi') = -\int_{\mathbb{R}} \phi'(x) \, dx = 0. \]

Thus, \( DT_f = 0 \) and \( f \) is weakly differentiable because \( DT_f = T_0 \).
Example

Consider the everywhere discontinuous function

\[ f(x) = \begin{cases} 
\sin x & x \in \mathbb{Q}^c \\
0 & x \in \mathbb{Q}.
\end{cases} \]

The function \( f \) is not differentiable at any point. The Lebesgue measure of \( \mathbb{Q} \) is zero, hence,

\[ DT_f(\phi) = -T_f(\phi') = -\int_{\mathbb{R}} \sin x \phi'(x) \, dx = \int_{\mathbb{R}} \cos x \phi(x) \, dx. \]

Thus, \( DT_f = T_{\cos x} \) and hence \( f \) is weakly differentiable.
Classical and Distributional Derivative can Differ

- Consider the locally integrable function $H_a$, for every $a \in \mathbb{R}$,

  \[ H_a(x) = \begin{cases} 
  1 & \text{if } x > a \\
  0 & \text{if } x \leq a.
  \end{cases} \]

- Classically, the function $H_a$ is differentiable a.e. and $H'_a(x) = 0$ a.e. Therefore, $T_{H'_a} = 0$.

- The distributional derivative of the regular distribution $T_{H_a}$, induced by $H_a$, is for $\phi \in \mathbb{R}$,

  \[ DT_{H_a}(\phi) = -T_{H_a}(\phi') = - \int_a^\infty \phi'(x) \, dx = \phi(a) = \delta_a(\phi). \]

- The distributional derivative of the function $H_a$ is the Dirac distribution, a singular distribution.

- Therefore, $DT_{H_a} = \delta_a(\phi) \neq 0 = T_{H'_a}$.

- We have also observed that $H_a$ is not weakly differentiable.
When do they Coincide

- We have already seen in the examples above that the two notions of differentiability need not coincide.

- They do coincide for smooth functions. Let $f \in C^\infty(\Omega)$ and $T_f$ be its induced distribution. For any $n$-tuple $\alpha$, $g := D^\alpha f \in C^\infty(\Omega)$ and let $T_g$ be its induced distribution. Then

$$D^\alpha T_f(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi) = (-1)^{|\alpha|} \int f D^\alpha \phi \, dx$$

$$= \int D^\alpha f \phi \, dx = \int g \phi = T_g(\phi).$$

- Thus, the two notions of derivatives coincide for all functions which respect integration by parts. In the one dimension case, it is precisely the class of absolutely continuous functions.
Observe that the function $H_a$ had a jump discontinuity and Dirac measure appeared as its derivative at the point of “jump”. This is the feature of Dirac measure. It appears as a derivative at points of jump.

However, note that in Example 5 and 6, the jump at every point did not give rise to a Dirac measure because outside the set of jump points (in that case $\mathbb{Q}$) the function is continuous.

Exercise

Is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} |x| & |x| < a \\ 0 & |x| > a \end{cases}$$

weakly differentiable?
Exercise

Is the function

\[ f(x) = \begin{cases} 
    x^2 + x & x < 1 \\
    e^{-5x} & x > 1.
\end{cases} \]

weakly differentiable?

Exercise

Is the Cantor function \( f_C : [0, 1] \to [0, 1] \) extended continuously to \( \mathbb{R} \) by setting

\[ f_C(x) = \begin{cases} 
    0 & x \leq 0 \\
    1 & x \geq 1.
\end{cases} \]

weakly differentiable?
Exercise
Compute the distributional derivative of $\ln |x|$.

Exercise
For any $\alpha \in \mathbb{R}$, we have already noted that the function $f(x) = |x|^{-\alpha}$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$ for $\alpha < n$. Show that the function is weakly differentiable for $\alpha + 1 < n$ and its weak partial derivative is $f_{x_i}(x) = -\frac{\alpha}{|x|^{|\alpha+1|}} x_i |x|$. 

Exercise
Compute and compare the classical and distributional derivative of

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Theorem (Exercise)

The differential operator $D^\alpha$, for each $\alpha$, is a continuous map from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)$.

Example

The above theorem is very special to the space of distributions $\mathcal{D}'(\Omega)$. For instance, the result is not true in $C^\infty(\Omega)$. Consider the sequence $f_m = (1/\sqrt{m}) \sin mx$ that converges uniformly to 0 and hence converges to 0 in the distribution sense too. However, it’s derivative $f'_m(x) = \sqrt{m} \cos mx$ does not converge pointwise but converges to 0 in the distribution sense.

Corollary (Term-by term differentiation of series)

If $S := \sum_{i=1}^\infty T_i$, then $D^\alpha S = \sum_{i=1}^\infty D^\alpha T_i$. 
Now that we have addition of distributions, a natural question is whether one can define product of any two distributions.

The answer is in negation. The space of distribution cannot be made an algebra which extends the classical notion of point-wise multiplication. This is called the Schwartz impossibility result.

L. Schwartz himself showed that it is not possible to define product of Dirac distributions, i.e., $\delta_0 \cdot \delta_0$.

However, one may define product of a distribution with a $C^\infty$ function which is a generalisation of the point-wise multiplication of two functions.
To motivate this definition, we note that if \( f \in L_{\text{loc}}^1(\Omega) \) and \( \psi \in C^\infty(\Omega) \) then
\[
\int_\Omega \left[ \psi(x) f(x) \right] \phi(x) \, dx = \int_\Omega f(x) \left[ \psi(x) \phi(x) \right] \, dx.
\]

If \( T_f \) is the distribution induced by \( f \), then above equality is same as saying \( T_{\psi f}(\phi) = T_f(\psi \phi) \).

But \( T_f(\psi \phi) \) makes sense only when \( \psi \phi \in \mathcal{D}(\Omega) \). This is the reason we demand \( \psi \in C^\infty(\Omega) \) because then \( \text{supp}(\psi \phi) \subseteq \text{supp}(\phi) \) and \( \psi \phi \) has compact support. Also, by Leibniz’ rule,
\[
D^\alpha(\psi \phi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta \psi D^{\alpha - \beta} \phi
\]
and hence \( \psi \phi \in \mathcal{D}(\Omega) \).

**Definition**

Let \( \psi \in C^\infty(\Omega) \) and \( T \in \mathcal{D}'(\Omega) \), then we define the functional
\[
\psi T : \mathcal{D}(\Omega) \to \mathbb{R} \text{ as } \psi T(\phi) = T(\psi \phi).
\]
**Proposition**

For every $\psi \in C^\infty(\Omega)$, $\psi T \in \mathcal{D}'(\Omega)$.

**Proof.**

We need to show that the functional $\psi T$ is continuous on $\mathcal{D}(\Omega)$. For any $\phi \in C^\infty(K)$,

$$
|\psi T(\phi)| = |T(\psi\phi)| \leq C_K \|\psi\phi\|_{N_K} = C_K \sum_{|\alpha|=0}^{N_K} \|D^\alpha(\psi\phi)\|_0
$$

$$
\leq C_K \sum_{|\alpha|=0}^{N_K} \sum_{\beta \leq \alpha} \left| \frac{\alpha!}{\beta!(\alpha - \beta)!} \right| \|D^\beta \psi\|_0 \|D^{\alpha - \beta} \phi\|_0
$$

$$
\leq C_K C_0 \|\phi\|_{N_K}.
$$

If $T$ is a distribution of order $k$, then it is enough to demand $\psi \in C^k(\Omega)$. 

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Example
Let \( \psi(x) = x \) on \( \mathbb{R} \) and \( T = \delta_a \in \mathcal{D}'(\mathbb{R}) \), then \( x\delta_a \in \mathcal{D}'(\mathbb{R}) \) and \( x\delta_a(\phi) = \delta_a(x\phi) = a\phi(a) \).

Exercise (Leibniz’ formula)
If \( \psi \in C^\infty(\Omega) \), then

\[
D^\alpha (\psi T) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta \psi \ D^{\alpha - \beta} T.
\]

Exercise
1. Find all real valued functions \( f \) on \( \mathbb{R} \) which is a solution to the equation \( xf(x) = 0 \) for all \( x \in \mathbb{R} \).
2. Find all distributions \( T \in \mathcal{D}'(\mathbb{R}) \) that solve the equation \( xT = 0 \) in the distribution sense.
3. Find all distributions \( T \in \mathcal{D}'(\mathbb{R}) \) that solve the equation \( x^{2022} T = 0 \) in the distribution sense.
Example

Let $\psi(x) = x$ on $\mathbb{R}$ and $T = \text{PV}(1/x) \in \mathcal{D}'(\mathbb{R})$, then $x\text{PV}(1/x) \in \mathcal{D}'(\mathbb{R})$ and

\[
x\text{PV} \left(\frac{1}{x}\right) (\phi) = \text{PV} \left(\frac{1}{x}\right) (x\phi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{1}{x} x \phi(x) \, dx
\]

\[
= \int_{\mathbb{R}} \phi(x) \, dx = T_1(\phi)
\]

where $T_1$ is the distribution corresponding to the constant function 1.

Exercise

Find all distributions $T \in \mathcal{D}'(\mathbb{R})$ that solve the equation $xT = 1$ in the distribution sense.
● In light of the above example, we come back to the issue of not being able to have a product in the space of distributions.

● Suppose we have a notion generalising product to distributions, the product cannot be associative because

\[ PV\left(\frac{1}{x}\right) \cdot (x \cdot \delta_0) = 0 \neq \delta_0 = (PV\left(\frac{1}{x}\right) \cdot x) \cdot \delta_0. \]

● The failure to define a suitable notion of product is what makes the theory of distributions unsuitable for nonlinear differential equations.

**Exercise**

For any given \( \psi \in C^\infty(\mathbb{R}) \), find the solutions of the equation

\[ DT + \psi T = 0 \text{ in } \mathcal{D}'(\mathbb{R}). \]

**Exercise**

For any given \( \psi \in C^\infty(\mathbb{R}) \) and \( S \in \mathcal{D}'(\mathbb{R}) \), find the solutions of the equation

\[ DT + \psi T = S \text{ in } \mathcal{D}'(\mathbb{R}). \]
Lemma

If $\psi_m \to 0$ in $C^\infty(\Omega)$ then $\psi_m \phi \to 0$ in $\mathcal{D}(\Omega)$.

Proof.

Since $D^\alpha \psi_m \to 0$ uniformly on all compact subsets of $\Omega$, in particular on $K = \text{supp}(\phi)$. Thus, $D^\alpha \psi_m \phi \to 0$ uniformly on $K$, for all $\alpha$, and $\text{supp}(\psi_m \phi) \subseteq K$, for all $m$. \hfill \square

Theorem

If $T_m \to T$ in $\mathcal{D}'(\Omega)$ and $\psi_m \to \psi$ in $C^\infty(\Omega)$ then $\psi_m T_m \to \psi T$ in $\mathcal{D}'(\Omega)$. 
Support of Distribution

- The fact that we cannot have a product on $\mathcal{D}'(\Omega)$, generalising point-wise multiplication, that makes $\mathcal{D}'(\Omega)$ an algebra leads us to look for other ways to make $\mathcal{D}'(\Omega)$ an algebra.
- One such choice is the convolution operation and whether this can be extended to distributions.
- We shall see later than one can define a notion of convolution for distributions provided one of them has compact support!
- This motivates us to understand the support of a distribution which coincides for classical functions.
- Classically, support of a function is complement of the largest open set on which the function vanishes.
### Definition (Localisation)

The restriction of \( T \in \mathcal{D}'(\Omega) \) to an open subset \( \omega \subset \Omega \), denoted as \( T|_\omega \), is defined as

\[
T|_\omega (\phi) = T(\phi) \quad \forall \phi \in \mathcal{D}(\omega) \subset \mathcal{D}(\Omega).
\]

We say a distribution \( T \) vanishes on an open set \( \omega \subset \Omega \) if \( T|_\omega = 0 \). More generally, two distributions \( S \) and \( T \) coincide on a open set \( \omega \) if \( S|_\omega = T|_\omega \).

The inclusion \( \mathcal{D}(\omega) \subset \mathcal{D}(\Omega) \) is continuous because any \( \phi \in \mathcal{D}(\omega) \) can be extended by zero outside \( \omega \) which belongs to \( \mathcal{D}(\Omega) \). Since \( T|_\omega \in \mathcal{D}'(\omega) \), \( T \) vanishing on \( \omega \) is same as the zero distribution in \( \mathcal{D}'(\omega) \).

### Example

The restriction of the Dirac distribution \( \delta_a \) to any open set \( \omega \subset \Omega \setminus \{a\} \) is zero.
Exercise

If $T \in \mathcal{D}'(\Omega)$ vanishes on the open subsets $\omega_i$ of $\Omega$, then $T$ vanishes on the union $\bigcup_{i \in I} \omega_i$.

Definition

The support of a distribution $T$ is the complement of the largest open set $\omega$ such that $T|_{\omega} = 0$. Equivalently, the support of a distribution is the relative complement in $\Omega$ of the union of all open sets $\omega$ of $\Omega$ such that $T|_{\omega} = 0$.

If $E$ is the support of a distribution $T$, then $T(\phi) = 0$ for all $\phi \in \mathcal{D}(\Omega)$ such that $E \cap \text{supp}(\phi) = \emptyset$ because all such $\phi$ is in $\mathcal{D}(E^c)$. Obviously, the support of the zero distribution is the empty set.

Exercise

Show that for any continuous function $f$, $\text{supp}(T_f) = \text{supp}(f)$. More generally, for any $f \in L^1_{\text{loc}}(\Omega)$, there exists a measurable function $g$ such that $g = f$ a.e. and $\text{supp}(T_f) = \text{supp}(g)$. 
Example

The support of the Dirac distribution $\delta_a$ is the singleton set $\{a\}$. Similarly, the support of the derivatives of Dirac distribution $\delta^\alpha_a$ is also the singleton set $\{a\}$. The open set $\Omega \setminus \{a\} = \bigcup_i B_i$, is the union of punctured open balls of rational radius $r_i$ with centre $\{a\}$ removed. $\delta^\alpha_a$ vanishes on $B_i$ for each $i$ and hence vanishes on $\Omega \setminus \{a\}$.

Exercise

Show that the support of $DT \in D(\mathbb{R})$ is contained in the support of $T \in D(\mathbb{R})$.

- A distribution is said to have *compact support*, if its support is a compact subset of $\Omega$.
- A nice characterisation of distributions with compact support, which is a subclass of $D'(\Omega)$, is that it can be identified as a dual of $C^\infty_c(\Omega)$ endowed with the topology of uniform convergence on compact sets.
Recall the motivation for the choice of $C^\infty_c(\Omega)$ as a test function. The compact support of test function was necessary to kill the boundary evaluation while applying integration by parts. If the function $f$ itself is compact, to begin with, then we can expand the space of test functions to $C^\infty(\Omega)$.

For convenience sake (for ease of notation), we set $C^\infty(\Omega) = E(\Omega)$, the space of infinitely differentiable functions on $\Omega$ with the topology of uniform convergence on compact subsets.

The inclusion $\mathcal{D}(\Omega) \subset E(\Omega)$ is continuous w.r.t the respective topology. This is because if $\phi_m \to 0$ in $\mathcal{D}(\Omega)$ then $\phi_m \to 0$ in $E(\Omega)$, as well.
Theorem

The dual space $\mathcal{E}'(\Omega)$ is the collection of all distributions with compact support.

Proof:

- Let $T \in \mathcal{D}'(\Omega)$ such that support of $T$ is compact. Set $K = \text{supp}(T)$.
- Thus, $T(\phi) = 0$ for all $\phi \in \mathcal{D}(\Omega)$ such that $\text{supp}(\phi) \subset K^c$.
- Choose $\chi \in \mathcal{D}(\Omega)$ such that $\chi \equiv 1$ on $K$. For all $\phi \in \mathcal{D}(\Omega)$, consider
  \[ T(\phi) - \chi T(\phi) = T(\phi - \chi \phi) = T((1 - \chi)\phi). \]
- Since $1 - \chi = 0$ on $K$, $\text{supp}(1 - \chi)\phi \subset K^c$ and hence $T = \chi T$ for any $\chi$ such that $\chi \equiv 1$ on $K$.
- This fact helps us to extend $T$ as a functional on $\mathcal{E}(\Omega)$, because for all $\psi \in \mathcal{E}(\Omega)$, $\psi \chi \in \mathcal{D}(\Omega)$.
- We extend $T$ to $\mathcal{E}(\Omega)$ by setting $T(\psi) = T(\chi \psi)$.
- The assignment is well-defined (it is independent of choice of $\chi$). Suppose $\chi_1 \in \mathcal{D}(\Omega)$ is such that $\chi_1 \neq \chi$ and $\chi_1 \equiv 1$ on $K$, then $(\chi - \chi_1)\psi = 0$ on $K$. Therefore, $\text{supp}((\chi - \chi_1)\psi) \subset K^c$ and $T((\chi - \chi_1)\psi) = 0$, By linearity of $T$, $T(\chi \psi) = T(\chi_1 \psi)$. 

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Proof Continued...

- The continuity of $T$ on $\mathcal{E}(\Omega)$ is a consequence of Lemma 6. Thus, $T \in \mathcal{E}'(\Omega)$.
- Conversely, let $T \in \mathcal{E}'(\Omega)$. We need to show that $T \in \mathcal{D}'(\Omega)$ and has a compact support.
- The fact that $T \in \mathcal{D}'(\Omega)$ is obvious due to the continuous inclusion of $\mathcal{D}(\Omega)$ in $\mathcal{E}(\Omega)$.
- Let $\{K_i\}$ be the collection of compact subsets of $\Omega$ such that $K_i \subset K_{i+1}$ and $\Omega = \bigcup_{i=1}^{\infty} K_i$ (exhaustion subsets).
- Suppose that the support of $T$ is not compact in $\Omega$, then the support of $T$ intersects $\Omega \setminus K_i$, for all $i$. Thus, for each $i$, there is a $\phi_i \in \mathcal{D}(\Omega)$ such that $\text{supp}(\phi_i) \subset \Omega \setminus K_i$ and $T(\phi_i) \neq 0$.
- Say $T(\phi_i) = \alpha_i \neq 0$. Then $T(\alpha_i^{-1}\phi_i) = 1$. Set $\psi_i = \alpha_i^{-1}\phi_i$. Thus, $\text{supp}(\psi_i) \subset \Omega \setminus K_i$ and $T(\psi_i) = 1$. Hence, the sequence $\psi_i \to 0$ in $\mathcal{E}(\Omega)$ because their support tend to empty set. Now, by continuity of $T$ on $\mathcal{E}(\Omega)$, $T(\psi_i) \to 0$ which contradicts the fact that $T(\psi_i) \to 1$. Hence, $T$ has a compact support.
Proposition

If $T \in \mathcal{E}'(\Omega)$ then order of $T$ is finite.

Proof.

Since $T \in \mathcal{E}'(\Omega)$, $T$ has compact support, say $K$. Let $\chi \in \mathcal{D}(\Omega)$ be such that $\chi \equiv 1$ on $K$, then we know that (cf. proof of last theorem) $T = \chi T$. Let $K'$ be the support of $\chi$ and hence $K \subset K'$. Now, for any compact subset $K \subset \Omega$ and $\phi \in C^\infty(K)$, $\text{supp}(\phi \chi) \subset K'$. Consider,

$$|T(\phi)| = |\chi T(\phi)| = |T(\chi \phi)| \leq C_{K'} \|\chi \phi\|_{N_{K'}} \leq C_0 \|\phi\|_{N_{K'}}.$$

The $N_{K'}$ is independent of all compact subsets $K$. Thus, $T$ has finite order.
Definition

A distribution \( T \in \mathcal{D}'(\Omega) \) is said to be \( C^\infty \) on \( \omega \subset \Omega \) if there is a \( f \in C^\infty(\omega) \) such that \( T = T_f \) on \( \mathcal{D}(\omega) \).

Example

The Dirac distribution, \( \delta_a \), is \( C^\infty \) in \( \Omega \setminus \{a\} \) because \( \delta_a \) restricted to the open set \( \Omega \setminus \{a\} \) is the zero function.

Local behaviour of “compatible” distributions can be patched up to get a global description of the distribution. This concept will facilitate the definition of singular support.

Theorem

Let \( T_i \in \mathcal{D}'(\omega_i) \), an arbitrary collection of distributions, such that \( T_i \mid_{\omega_i \cap \omega_j} = T_j \mid_{\omega_i \cap \omega_j} \) then there is a unique distribution \( T \in \mathcal{D}'(\Omega) \), where \( \Omega = \bigcup_i \omega_i \), such that \( T \mid_{\omega_i} = T_i \) for all \( i \).
Proof:

Let \( \{\phi_i\} \subset \mathcal{D}(\Omega) \) be a \( C^\infty \) locally finite partition of unity subordinate to \( \{\omega_i\} \). Thus, \( \text{supp}(\phi_i) \subset \omega_i \) and \( 1 = \sum_i \phi_i \). We define the functional \( T : \mathcal{D}(\Omega) \to \mathbb{R} \) as \( T(\phi) = \sum_i T_i(\phi \phi_i) \). We first check the continuity of \( T \). One way is to show the sequential continuity. Alternately, For any \( \phi \in C^\infty(K) \) (compact subset of \( \Omega \)), there exist finitely many \( i_1, i_2, \ldots, i_k \) such that \( K \cap \text{supp}(\phi_{i_m}) \neq \emptyset \). Then

\[
|T(\phi)| \leq \sum_{m=1}^k |T_{i_m}(\phi \phi_{i_m})| \leq \sum_m C_k \|\phi \phi_{i_m}\|_{N_k} \leq C_0 \|\phi\|_{N_0}.
\]

It only remains to show that the restriction of \( T \) to \( \omega_k \) is \( T_k \). Let \( \phi \in \mathcal{D}(\omega_k) \), then \( \phi \phi_i \in \mathcal{D}(\omega_i \cap \omega_k) \) for all \( i \). Thus,

\[
T(\phi) = \sum_i T_i(\phi \phi_i) = \sum_i T_k(\phi \phi_i) = T_k(\sum_i \phi \phi_i) = T_k(\phi).
\]

The uniqueness of \( T \) follows from Proposition 37.
Corollary

If $T \in \mathcal{D}'(\Omega)$ is $C^\infty$ on an arbitrary collection of open subsets $\omega_i$, then $T$ is $C^\infty$ on the union $\bigcup_{i \in I} \omega_i$.

A consequence of the above corollary is the following definition.

Definition

The singular support of $T \in \mathcal{D}'(\Omega)$ is the complement of the largest open set on which $T$ is $C^\infty$, denoted as $\text{sing.supp}(T)$.

A simple observation is that $\text{sing.supp}(T) \subset \text{supp}(T)$. 
Shifting

- Given any function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and a fixed \( a \in \mathbb{R}^n \), we introduce the shift operator \( \tau_a f(x) := f(x - a) \).
- In particular, \( \tau_0 f = f \) for all \( f \).
- Observe that
  \[
  \tau_a \tau_b f(x) = \tau_a f(x - b) = f(x - b - a) = f(x - (b + a)) = \tau_{a+b} f(x).
  \]
- Note that if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) then \( \tau_a f \in L^1_{\text{loc}}(\mathbb{R}^n) \).
We have the following relation between $T_f$ and $T_{\tau a} f$. For any $\phi \in \mathcal{D}(\mathbb{R}^n)$, consider

\[
T_{\tau a} f(\phi) = \int_{\mathbb{R}^n} \tau_a f(x) \phi(x) \, dx = \int_{\mathbb{R}^n} f(x-a) \phi(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} f(y) \phi(y+a) \, dy = \int_{\mathbb{R}^n} f(y) \tau_- a \phi(y) \, dy
\]

\[
= T_f(\tau_- a \phi).
\]

The last equality is valid because $\tau_- a \phi \in \mathcal{D}(\mathbb{R}^n)$ whenever $\phi \in \mathcal{D}(\mathbb{R}^n)$.

We wish to extend the notion of shift to distributions such that for $\tau_a T_f = T_{\tau a} f$.

**Definition**

*For any $T \in \mathcal{D}'(\mathbb{R}^n)$, we define its shift by $a \in \mathbb{R}^n$ as $\tau_a T(\phi) = T(\tau_- a \phi)$.*
Scaling

- Given any function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and a fixed scalar \( \lambda \in (0, \infty) \), we introduce the scaling function \( f_{\lambda}(x) := f(\lambda x) \).

- If \( f = g + h \) then \( f_{\lambda}(x) = f(\lambda x) = g(\lambda x) + h(\lambda x) = g_\lambda(x) + h_\lambda(x) \).
  Also, \( \tau_a(f + g) = \tau_a f + \tau_a g \).

- Note that if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) then \( f_\lambda \in L^1_{\text{loc}}(\mathbb{R}^n) \).

- We have the following relation between \( T_f \) and \( T_{f_\lambda} \). For any \( \phi \in D(\mathbb{R}^n) \), consider

\[
T_{f_{\lambda}}(\phi) = \int_{\mathbb{R}^n} f(\lambda x)\phi(x) \, dx = \lambda^{-n} \int_{\mathbb{R}^n} f(y)\phi\left(\frac{y}{\lambda}\right) \, dy
\]
\[
= \lambda^{-n} \int_{\mathbb{R}^n} f(y)\phi_{1/\lambda}(y) \, dy = \lambda^{-n} T_f(\phi_{1/\lambda}).
\]

The last equality is valid because \( \phi_{1/\lambda} \in D(\mathbb{R}^n) \) whenever \( \phi \in D(\mathbb{R}^n) \).
Similarly, one can argue for $\lambda \in (-\infty, 0)$ and deduce that

$$T_{f, \lambda}(\phi) = (-1)^n \lambda^{-n} T_f(\phi_{1/\lambda}).$$

Thus, for any $\lambda \in \mathbb{R} \setminus \{0\}$, we have $T_{f, \lambda}(\phi) = |\lambda|^{-n} T_f(\phi_{1/\lambda})$.

We wish to extend the notion of scaling to distributions such that for $(T_f)_\lambda = T_{f, \lambda}$.

**Definition**

For any $T \in \mathcal{D}'(\mathbb{R}^n)$, we define its scaling by $\lambda \in \mathbb{R} \setminus \{0\}$ as $T_{\lambda}(\phi) = |\lambda|^{-n} T(\phi_{1/\lambda})$. In particular, when $\lambda = -1$, we denote $T_{-1}$ as $\tilde{T}$ and hence $\tilde{T}(\phi) = T(\tilde{\phi})$. We say $T \in \mathcal{D}'(\mathbb{R}^n)$ is even if $\tilde{T} = T$.

**Exercise**

Show that $\tilde{T} = T$. 

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Recall the definition of convolution of functions given in Definition 1. Thus, for any $x \in \mathbb{R}^n$,

$$(f \ast g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \, dy.$$ 

In this section, we wish to extend the convolution operation to distributions such that for $L^1$ distributions the notion coincide. An obvious extension is visible when the convolution integral is rewritten using the shift and scaling operator. Note that, for a fixed $x \in \mathbb{R}^n$, $f(x - y) = f(-(y - x)) = \check{f}(y - x) = \tau_x \check{f}(y)$. With this notation the convolution is rewritten as

$$(f \ast g)(x) = \int_{\mathbb{R}^n} \tau_x \check{f}(y)g(y) \, dy = \int_{\mathbb{R}^n} f(y)\tau_x \check{g}(y) \, dy.$$ 

The second equality is due to the commutativity of convolution operation. The reformulation above motivates the following definition of convolution between a distribution and a test function.
Definition

Let \( T \in \mathcal{D}'(\mathbb{R}^n) \) and \( \phi \in \mathcal{D}(\mathbb{R}^n) \). The convolution \( T \ast \phi : \mathbb{R}^n \to \mathbb{R} \) is defined as

\[
(T \ast \phi)(x) = T(\tau_x \check{\phi}).
\]

The definition is meaningful because \( \tau_x \check{\phi} \in \mathcal{D}(\mathbb{R}^n) \), for all \( x \in \mathbb{R}^n \), whenever \( \phi \in \mathcal{D}(\mathbb{R}^n) \). In particular, \( (T \ast \check{\phi})(x) = T(\tau_x \phi) \) since \( \check{\phi} = \phi \).

Thus, \( T(\phi) = (T \ast \check{\phi})(0) \). As a consequence, if \( T \ast \phi = 0 \) for all \( \phi \in \mathcal{D}(\mathbb{R}^n) \) then \( T = 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \).

Exercise

(i) Show that \( T \ast (\phi_1 + \phi_2) = T \ast \phi_1 + T \ast \phi_2 \).

(ii) Show that \( (S + T) \ast \phi = S \ast \phi + T \ast \phi \).

(iii) For any \( a \in \mathbb{R}^n \), show that \( \tau_a(T \ast \phi) = \tau_a T \ast \phi = T \ast \tau_a \phi \).

(iv) Show that, for any multi-index \( \alpha \), \( D^\alpha(T \ast \phi) = D^\alpha T \ast \phi = T \ast D^\alpha \phi \).

In particular, \( T \ast \phi \in \mathcal{E}(\mathbb{R}^n) \).

(v) For \( \chi \in \mathcal{D}(\mathbb{R}^n) \), \( T \ast (\phi \ast \chi) = (T \ast \phi) \ast \chi \).
For any $\psi \in \mathcal{E}(\mathbb{R}^n)$, $\tau_x \psi \in \mathcal{E}(\mathbb{R}^n)$. Thus, the convolution notion can be extended to any $\psi \in \mathcal{E}(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$.

**Definition**

Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{E}(\mathbb{R}^n)$. The convolution $T \ast \psi : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$(T \ast \psi)(x) = T(\tau_x \psi).$$

**Exercise**

Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{E}(\mathbb{R}^n)$, then

(i) for any $a \in \mathbb{R}^n$, show that $\tau_a(T \ast \psi) = \tau_a T \ast \psi = T \ast \tau_a \psi$.

(ii) show that, for any multi-index $\alpha$, $D^\alpha(T \ast \psi) = D^\alpha T \ast \psi = T \ast D^\alpha \psi$.

In particular, $T \ast \phi \in \mathcal{E}(\mathbb{R}^n)$. Further, if $\phi \in \mathcal{D}(\mathbb{R}^n)$ then $T \ast \phi \in \mathcal{D}(\mathbb{R}^n)$ and

$$T \ast (\psi \ast \phi) = (T \ast \psi) \ast \phi = (T \ast \phi) \ast \psi.$$
Definition

Let \( S, T \in \mathcal{D}'(\mathbb{R}^n) \) such that one of them has compact support, i.e. either \( T \) or \( S \) is in \( \mathcal{E}'(\mathbb{R}^n) \). We define the convolution \( S \ast T \) as,

\[
(S \ast T)(\phi) = (S \ast (T \ast \check{\phi}))(0), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).
\]

Equivalently, one can also define the convolution \( S \ast T \) to be

\[
(S \ast T) \ast \phi = S \ast (T \ast \phi) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).
\]

Exercise

Let \( S, T \in \mathcal{D}'(\mathbb{R}^n) \) such that one of them is in \( \mathcal{E}'(\mathbb{R}^n) \).

1. Show that \( T \ast S = S \ast T \).
2. \( \text{supp}(S \ast T) \subset \text{supp}(S) + \text{supp}(T) \).
3. For any multi-index \( \alpha \), \( D^\alpha(T \ast S) = D^\alpha T \ast S = T \ast D^\alpha S \).
**Theorem**

Let \( T \in \mathcal{D}'(\mathbb{R}^n) \), then \( T = T \ast \delta_0 = \delta_0 \ast T \). Also, for any multi-index \( \alpha \), \( D^\alpha T = D^\alpha \delta_0 \ast T \).

**Proof.**

Since \( \delta_0 \) has compact support \( \{0\} \), the convolution makes sense. By commutativity, \( T \ast \delta_0 = \delta_0 \ast T \). For \( \phi \in \mathcal{D}(\mathbb{R}^n) \), consider \( (T \ast \delta_0)(\phi) = (T \ast (\delta_0 \ast \check{\phi}))(0) \). Set \( \psi := \delta_0 \ast \check{\phi} \). Then \( (T \ast \delta_0)(\phi) = T(\check{\psi}) \). Note that

\[
\check{\psi}(x) = \psi(-x) = (\delta_0 \ast \check{\phi})(-x) = \delta_0(\tau_{-x}\phi) = \tau_{-x}\phi(0) = \phi(x).
\]

Hence, \( (T \ast \delta_0)(\phi) = T(\phi) \). \( \square \)

**Theorem**

\( \mathcal{D}(\Omega) \) is dense in \( \mathcal{D}'(\Omega) \).
Exercise

Let $T \in \mathcal{D}'(\mathbb{R})$ be such that $DT = 0$, then show that $T$ is a regular distribution generated by a constant function.

Proof.

For a given sequence of mollifiers, let $T_\varepsilon(x) = (T \ast \rho_\varepsilon)(x)$. Thus, $T_\varepsilon \in \mathcal{E}(\mathbb{R})$ and $DT_\varepsilon = DT \ast \rho_\varepsilon = 0$. Hence, $T_\varepsilon = \lambda_\varepsilon$, where $\lambda_\varepsilon$ is a constant function for each $\varepsilon$. Also, $\lambda_\varepsilon \rightharpoonup T$ in $\mathcal{D}'(\mathbb{R})$. In particular, choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $\int \phi = 1$, then the sequence of real numbers $\lambda_\varepsilon$ converges to some $\lambda$. Thus, $T = T_\lambda$. \qed
Exercise

For any three distributions $T_1, T_2, T_3 \in \mathcal{D}'(\mathbb{R}^n)$ such that at least two of them have compact support then

$$T_1 \ast (T_2 \ast T_3) = (T_1 \ast T_2) \ast T_3.$$ 

Exercise

Let $T$ denote the distribution corresponding to the constant function $1$, $\delta_0^{(1)}$ be the dipole distribution at $0$ and $H$ be the Heaviside function

$$H(x) = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0.
\end{cases}$$

Show that $(T \ast \delta_0^{(1)}) \ast H \neq T \ast (\delta_0^{(1)} \ast H)$. 

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Proof

Consider

\[(1 * \delta_0^{(1)})(\phi) = 1 * (\delta_0^{(1)} * \tilde{\phi})(0) = (1 * \psi)(0) = T_1(\tilde{\psi}) = \int_\mathbb{R} \tilde{\psi}(x) \, dx.\]

But \(\tilde{\psi}(x) = \delta_0^{(1)} * \tilde{\phi}(-x) = \delta_0^{(1)}(\tau_x \phi) = -\tau_x \phi'(0) = -\phi'(x)\). Hence

\[(1 * \delta_0^{(1)})(\phi) = -\int_\mathbb{R} \phi'(x) \, dx = 0.\]

Further, \(0 * H = 0\). Now, consider

\[(\delta_0^{(1)} * H)(\phi) = \delta_0^{(1)} * (H * \tilde{\phi})(0) = (\delta_0^{(1)} * \psi)(0) = \delta_0^{(1)}(\tilde{\psi}) = (\tilde{\psi})'(0).\]

But \((\tilde{\psi})'(0) = (H' * \tilde{\phi})(0) = (\delta_0 * \tilde{\phi})(0) = \delta_0(\phi) = \phi(0)\). Since \(\delta_0\) is the identity distribution under convolution operation, \(1 * \delta_0 = 1\).

Alternately, notice that by derivative of convolution

\[(1 * \delta_0^{(1)}) * H = (D1 * \delta_0) * H = D1 * H = 0 * H = 0.\]

On the other hand,

\[1 * (\delta_0^{(1)} * H) = 1 * (\delta_0 * DH) = 1 * (\delta_0 * \delta_0) = 1 * \delta_0 = 1.\]
The *Schwartz Space* is the class of all smooth function on $\mathbb{R}^n$ which decay at infinity faster than polynomial of any degree, i.e,

$$S(\mathbb{R}^n) = \left\{ \phi \in C^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha \phi(x)| < \infty \; \forall \alpha, \beta \right\}.$$

We say a sequence $\{\phi_k\} \subset S(\mathbb{R}^n)$ converges to $\phi$ if

$$\sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha (\phi_n - \phi)| \to 0 \; \forall \alpha, \beta.$$ 

**Example:** Note that $C^\infty_0(\mathbb{R}^n) \subset S(\mathbb{R}^n)$. There are elements in $S(\mathbb{R}^n)$ which do not have compact support. For instance, $e^{-|x|^2} \in S(\mathbb{R}^n)$ and has no compact support. There are also elements in $S(\mathbb{R}^n)$ which do not have exponential decay as $|x| \to \infty$. 

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Proposition (Properties)

The space $S(\mathbb{R}^n)$ satisfies the following properties:

1. If $\phi \in S(\mathbb{R}^n)$ then $D^\alpha \phi \in S(\mathbb{R}^n)$.
2. $S(\mathbb{R}^n)$ is an algebra, i.e., if $\phi, \psi \in S(\mathbb{R}^n)$ then $\phi \psi \in S(\mathbb{R}^n)$.
3. If $\phi \in S(\mathbb{R}^n)$ and $P(x)$ a polynomial then $P\phi \in S(\mathbb{R}^n)$.

Definition

Let $f \in L^1(\mathbb{R}^n)$. The *Fourier transform* of $f$, denoted as $\hat{f} : \mathbb{R}^n \to \mathbb{C}$, is defined as

$$
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx.
$$

In particular, for each $f \in S(\mathbb{R}^n)$, its Fourier transform, $\hat{f}$, is defined because $S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. 
Proposition

The map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ defined as $\mathcal{F}(f) = \hat{f}$. Further, the following formulae hold: $D^\beta \hat{f} = (-i x)^\beta f(x)$ and $D^\beta \hat{f}(\xi) = (i \xi)^\beta \hat{f}(\xi)$.

Proof: Computing the $j$-th partial derivative of

$$
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i x \cdot \xi} \, dx
$$

we get, differentiating under the integral sign (Exercise: justify!),

$$
\frac{\partial \hat{f}}{\partial \xi_j}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)(-i x_j) e^{-i x \cdot \xi} \, dx = (-i x_j) \hat{f}(x)(\xi).
$$
Proof Continued...

On the other hand

$$\hat{\frac{\partial f}{\partial x_j}}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)e^{-ix\cdot\xi} \, dx.$$ 

Integration by parts gives

$$\hat{\frac{\partial f}{\partial x_j}}(\xi) = (\imath \xi_j)\hat{f}(\xi).$$

Iteration of these formulae gives the result. Use $\hat{f}$ is continuous if $f \in L^1(\mathbb{R}^n)$. 

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Proposition

Let \( f \in S(\mathbb{R}^n) \). Then there exist a constant \( C_0 > 0 \) (depending on \( n, k \) and \( f \)) such that

\[
|\hat{f}(\xi)| \leq \frac{C_0}{(1 + |\xi|)^k}.
\]

Proof: Let \( \xi = (\xi_1, \ldots, \xi_n) \) be fixed and \( j \) denote the index such that

\[
|\xi_j| = \max_{1 \leq i \leq n} |\xi_i|.
\]

Then

\[
|\xi| = \left(\sum_{i=1}^{n} \xi_i^2\right)^{1/2} \leq \sqrt{n}|\xi_j|.
\]

Therefore,

\[
(1 + |\xi|)^k = \sum_{i=0}^{k} \binom{k}{i} |\xi|^{i} \leq 2^k \sum_{i=0}^{k} n^{i/2} |\xi_j|^{i} \leq 2^k n^{k/2} \sum_{|\alpha| \leq k} |\xi^\alpha|.
\]

Consequently,

\[
(1 + |\xi|)^k \hat{f}(\xi) \leq 2^k n^{k/2} \sum_{|\alpha| \leq k} |(i\xi)^\alpha \hat{f}(\xi)| \leq 2^k n^{k/2} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha f(x)| \, dx
\]

where the last RHS term is set as \( C_0 \).
Corollary

The mapping $f \mapsto \hat{f}$ defines a continuous linear mapping of $S(\mathbb{R}^n)$ into itself.

Proof.

Consider for any $\alpha, \beta$ multi-index

$$
\xi^\alpha (D^\beta \hat{f})(\xi) = \xi^\alpha \hat{g}(\xi) = (-i)^{\alpha |} D^\alpha \hat{g}(\xi)
$$

where $g(x) = (-ix)^\beta f(x)$. Since $D^\alpha g \in S(\mathbb{R}^N) \subset L^1(\mathbb{R}^n)$, we get that

$$
\widehat{D^\alpha g}(\xi) \in L^\infty(\mathbb{R}^n).
$$

Thus, we have shown $\xi^\alpha (D^\beta \hat{f}) \in S(\mathbb{R}^n)$ for any $\hat{f}$. \qed
Corollary (Riemann-Lebesgue Lemma)

If \( f \in L^1(\mathbb{R}^n) \), then its Fourier transform \( \hat{f}(\xi) \) has the following properties:

1. \( \hat{f} \in C(\mathbb{R}^n) \) and is bounded;
2. \( \| \hat{f} \|_{\infty} \leq C_0 \| f \|_1 \);
3. \( \lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \).

Proof: If \( f \in L^1(\mathbb{R}^n) \) then, by definition of \( \hat{f} \),

\[
|\hat{f}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)||e^{-ix \cdot \xi}| \, dx.
\]

It is obvious that \( \| \hat{f} \|_{\infty} \leq C_0 \| f \|_1 \). \( \hat{f} \) is continuous by dominated convergence theorem. Since \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( L^1(\mathbb{R}^n) \) and the space of all continuous function which vanish at \( \infty \) is Banach space, it is enough to prove

\[
\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \quad \forall f \in \mathcal{S}(\mathbb{R}^n).
\]

This is obviously true because if \( f \in \mathcal{S}(\mathbb{R}^n) \) then \( \hat{f} \in \mathcal{S}(\mathbb{R}^n) \).
The Fourier transform map $\mathcal{F} : L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$ is defined as $\mathcal{F}(f) = \hat{f}$. Note that $\mathcal{F}$ is a bounded linear with $\|\mathcal{F}\| \leq 1$, since $\|\hat{f}\|_\infty \leq \|f\|_1$.

**Lemma**

Let $f(x) = e^{-\frac{|x|^2}{2}}$ for $x \in \mathbb{R}^n$. Then $\hat{f} \equiv f$.

**Proof:** Now

$$
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ix \cdot \xi} \, dx
$$

$$
= \prod_{j=1}^n \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} e^{-\frac{1}{2}x_j^2} e^{-ix_j \xi_j} \, dx_j.
$$

Thus, it suffices to look at the case $n = 1$ and in one dimension $e^{-\frac{x^2}{2}}$ satisfies the following differential equation:

$$
f'(x) + xf(x) = 0.
$$
Taking Fourier transform, we get $(i\xi)\hat{f}(\xi) + i\hat{f}'(\xi) = 0$ and, hence, 
$\hat{f}'(\xi) + \xi \hat{f}(\xi) = 0$. This is a first order ordinary differential equation (ODE) of $\hat{f}$ which can be easily integrated to get

$$\hat{f}(\xi) = C_0 e^{-\frac{\xi^2}{2}}.$$ 

Evaluating for $\xi = 0$, we see that

$$C_0 = \hat{f}(0) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \, dx = 1.$$ 

Hence the result.
**Definition**

The *inverse Fourier transform* of $f$ is defined as

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi \cdot x} \, d\xi.$$ 

**Theorem**

$\hat{\hat{f}} = f$ and $\check{\check{f}} = f$.

**Theorem**

*The Fourier transform is an isomorphism from $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$. In fact, we have*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi \cdot x} \, d\xi \quad \text{for } f \in S(\mathbb{R}^n).$$

**Proof:** Step-1: For $f, g \in L^1(\mathbb{R}^n)$, by Fubini’s theorem, we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi) \, d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(\xi)e^{-i\xi \cdot x} \, dx \, d\xi = \int_{\mathbb{R}^n} f(x)\hat{g}(x) \, dx.$$
Proof Continued...

Step-2: For a $\lambda > 0$, choose $g(x) := \phi\left(\frac{x}{\lambda}\right)$ where $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi\left(\frac{x}{\lambda}\right) e^{ix \cdot \xi} \, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lambda^n \phi(y) e^{iy \cdot \lambda \xi} \, dy$$

$$= \lambda^n \hat{\phi}(\lambda \xi).$$

Therefore, from Step 1, we obtain

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \phi\left(\frac{\xi}{\lambda}\right) \, d\xi = \int_{\mathbb{R}^n} f(\xi) \lambda^n \hat{\phi}(\lambda \xi) \, d\xi.$$ 

Equivalently,

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \phi\left(\frac{\xi}{\lambda}\right) \, d\xi = \int_{\mathbb{R}^n} f\left(\frac{y}{\lambda}\right) \hat{\phi}(y) \, dy.$$
Proof Continued...

Letting $\lambda$ tend to $\infty$ and, by dominated convergence theorem,

$$
\phi(0) \int_{\mathbb{R}^n} \hat{f}(\xi) \, d\xi = f(0) \int_{\mathbb{R}^n} \hat{\phi}(y) \, dy.
$$

Step-3: Choose $\phi(x) = e^{-\frac{|x|^2}{2}}$, then $\hat{\phi}(y) = \phi(y)$. Therefore,

$$
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{\phi}(y) \, dy = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(y) \, dy = \hat{\phi}(0) = \phi(0) = 1.
$$

Thus, we get

$$
f(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) \, d\xi.
$$

Step-4: Apply Step 3 to the function $y \mapsto f(y + x)$ to get the result.
Corollary (Plancherel Formula)

The Fourier transform defines an unitary isomorphism between $L^2(\mathbb{R}^n)$ spaces.

**Proof:** We shall show that

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}.$$ 

Since $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it is enough to show the above relation on $S(\mathbb{R}^n)$. In fact, by inversion formula

$$\int_{\mathbb{R}^n} f \bar{g} \, dx = \int_{\mathbb{R}^n} \bar{g}(x) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi \, dx$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) \left[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \bar{g}(x) e^{ix \cdot \xi} \, dx \right] \, d\xi$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x) e^{-ix \cdot \xi} \, dx \, d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi) \, d\xi.$$ 

This above identity is **Parseval Formula**. In particular, putting $g = f$ we obtain the Plancherel formula.
Remark (Decay of Fourier modes implies regularity)

If $f$ is such that $\xi^\alpha \hat{f}(\xi) \in L^1(\mathbb{R}^n)$, for all $|\alpha| \leq k$, then $f \in C^k(\mathbb{R}^n)$. Consider

$$D^\alpha f(x) = \int_{\mathbb{R}^n} \widehat{D^\alpha f}(\xi) e^{i x \cdot \xi} \, d\xi$$

$$= \int_{\mathbb{R}^n} \xi^\alpha \hat{f}(\xi) e^{i x \cdot \xi} \, d\xi.$$

Therefore, by Riemann-Lebesgue Lemma, $D^\alpha f \in C(\mathbb{R}^n)$ and so $f \in C^k(\mathbb{R}^n)$. 
Lemma

The convolution product is transformed into ordinary product after Fourier transformation, i.e., \( \hat{\phi} \ast \hat{\psi}(\xi) = (2\pi)^{\frac{n}{2}} \hat{\phi}(\xi)\hat{\psi}(\xi) \).

Proof.

Consider

\[
\begin{align*}
\hat{\phi} \ast \hat{\psi}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{\phi} \ast \hat{\psi}(x) e^{-i x \cdot \xi} \, dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{\psi}(y) \int_{\mathbb{R}^n} \phi(x - y) e^{-i x \cdot \xi} \, dx \, dy \\
&= \int_{\mathbb{R}^n} \hat{\psi}(y) \hat{\phi}(\xi) e^{-i y \cdot \xi} \, dy \\
&= (2\pi)^{\frac{n}{2}} \hat{\phi}(\xi)\hat{\psi}(\xi).
\end{align*}
\]
Corollary
\[ \hat{\phi} \ast \hat{\psi} = \hat{\phi \psi}. \]

Proof.
By Fourier Inversion
\[ \hat{\phi} \ast \hat{\psi} = \hat{\phi \psi} = (\hat{\phi \psi}) = \hat{\phi \psi}. \]
Tempered Distributions

Note that $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The inclusion is also dense because, for any $f \in \mathcal{S}(\mathbb{R}^n)$, $\phi_m f \to f$ in $\mathcal{S}$ where $\phi_m(x) = \phi(x/m)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ is such that $\phi \equiv 1$ on the unit ball in $\mathbb{R}^n$. Hence the dual of $\mathcal{S}$, denoted $\mathcal{S}'$, can be identified with a subspace of $\mathcal{D}'(\mathbb{R}^n)$.

**Definition**

The space of distributions $\mathcal{S}'$ is called the space of tempered distributions.

**Example**

Any distribution with compact support is tempered, i.e., $\mathcal{E}' \subset \mathcal{S}'$ because $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$ and all inclusions are dense.
Any slowly increasing measure is a tempered distribution

A measure \( \mu \) on \( \mathbb{R}^n \) is said to be *slowly increasing* if, for some integer \( k \geq 0 \),

\[
\int_{\mathbb{R}^n} \frac{d\mu(x)}{(1 + |x|^2)^k} < +\infty.
\]

For instance, any bounded measure is slowly increasing. Set

\[
T_\mu(f) = \int_{\mathbb{R}^n} f(x) \, d\mu(x), \quad f \in S(\mathbb{R}^n).
\]

Then \( T_\mu \) is a linear functional on \( S(\mathbb{R}^n) \) and

\[
|T_\mu(f)| \leq \left( \sup_{x \in \mathbb{R}^n} |f(x)(1 + |x|^2)^k| \right) \int_{\mathbb{R}^n} \frac{d\mu(x)}{(1 + |x|^2)^k}.
\]

Further, it follows that if \( f_m \to 0 \) in \( S(\mathbb{R}^n) \) then \( T_\mu(f_m) \to 0 \). Thus, \( T_\mu \in S'(\mathbb{R}^n) \).
Integrable functions are Tempered Distributions

If $1 \leq p \leq \infty$ then $L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$, define

$$T_f(\phi) = \int_{\mathbb{R}^n} f \phi \, dx \quad \forall \phi \in S(\mathbb{R}^n).$$

If $1 < p < \infty$, let $q$ be the conjugate exponent of $p$ given as $(1/p) + (1/q) = 1$. Then for $k > \frac{n}{2q}$, $g(x) = (1 + |x|^2)^{-k} \in L^q(\mathbb{R}^n)$. Hence, by Hölder’s inequality,

$$|T_f(\phi)| \leq \left( \sup_{x \in \mathbb{R}^n} |(1 + |x|^2)^k \phi(x)| \right) \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Thus, $T_f$ is tempered, and $f \mapsto T_f$ is continuous on $L^p(\mathbb{R}^n)$. If $p = 1$,

$$|T_f(\phi)| \leq \|\phi\|_{\infty, \mathbb{R}^n} \|f\|_{1, \mathbb{R}^n}$$

and if $p = \infty$,

$$|T_f(\phi)| < \sup_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k \phi(x) \right| \|f\|_{\infty, \mathbb{R}^n} \|g\|_{1, (\mathbb{R}^n)}$$

where $g$ is as above with $k > \frac{n}{2}$. 
Definition

Let \( T \in S'(\mathbb{R}^n) \). The Fourier transform of \( T \), denoted as \( \hat{T} \), is defined as \( \hat{T}(\hat{f}) = T(\hat{f}) \) for all \( f \in S(\mathbb{R}^n) \).

Since \( f \mapsto \hat{f} \) is continuous on \( S \), it follows that \( \hat{T} \in S' \).

Remark

Since \( S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \). It seems that, a priori, there are two definitions of the Fourier transform on \( S(\mathbb{R}^n) \), each inherited from \( L^1(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \), respectively. But they are one and the same because if \( f \in S(\mathbb{R}^n) \) then, for any \( g \in S(\mathbb{R}^n) \), by Parseval relation,

\[
\hat{T}_f(g) = \int_{\mathbb{R}^n} f \hat{g} = \int_{\mathbb{R}^n} \hat{f} g = T_{\hat{f}}(g).
\]

Hence \( \hat{T}_f = T_{\hat{f}} \) and, hence, both definitions of the Fourier transform coincide on \( S(\mathbb{R}^n) \).
Recall that \( \mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n) \), defined as \( \mathcal{F}(f) = \hat{f} \), is an isomorphism on \( S(\mathbb{R}^n) \), it extends as a bijection to \( S'(\mathbb{R}^n) \), as well.

**Remark**

Since \( S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \), then there are two definitions of Fourier transform on \( L^2(\mathbb{R}^n) \) inherited from \( S'(\mathbb{R}^n) \) and the other by extending from \( S(\mathbb{R}^n) \). It turns out that both are same. Consider the weak-* convergence of tempered distributions, i.e., \( T_m \to T \) weak-* sense if, for every \( \phi \in S(\mathbb{R}^n) \), \( T_m(\phi) \to T(\phi) \). If \( T_m \to T \) in \( S'(\mathbb{R}^n) \) then \( \hat{T}_m \to \hat{T} \) in \( S' \) because

\[
\hat{T}_m(\phi) = T_m(\hat{\phi}) \to T(\hat{\phi}) = \hat{T}(\phi).
\]

Let \( f \in L^2(\mathbb{R}^n) \) and let \( f_k \in S(\mathbb{R}^n) \) such that \( f_k \to f \) in \( L^2(\mathbb{R}^n) \). Since the inclusion of \( L^2(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \) is continuous, \( f_k \to f \) in \( S'(\mathbb{R}^n) \), as well. Hence \( \hat{f}_k \to \hat{f} \) in \( S'(\mathbb{R}^n) \). On the other hand, \( \mathcal{F}(f_k) \to \mathcal{F}(f) \) in \( L^2(\mathbb{R}^n) \) and hence in \( S'(\mathbb{R}^n) \). But as \( f_k \in S(\mathbb{R}^n) \), we know that \( \mathcal{F}(f_k) = \hat{f}_k \). Now, by uniqueness of the weak-* limit \( \hat{f} = \mathcal{F}(f) \).
By Proposition 7, for any polynomial \( P \) and multi-index \( \alpha \), if \( T \in S'(\mathbb{R}^n) \) then \( PT \in S'(\mathbb{R}^n) \) and \( D^\alpha T \in S'(\mathbb{R}^n) \).

We now extend the result of Theorem 8 to tempered distribution.

**Theorem**

Let \( T \in S'(\mathbb{R}^n) \) and let \( \alpha \) be a multi-index. Then \( D^\alpha \hat{T} = (-i)^{|\alpha|} \hat{(x^\alpha T)} \) and \( \hat{(D^\alpha T)} = (i)^{|\alpha|} \xi^\alpha \hat{T} \).

**Proof:** Let \( f \in S \). Then

\[
\hat{(x^\alpha T)}(f) = (x^\alpha T)(\hat{f}) = T(x^\alpha \hat{f}(x))
\]

\[
= \frac{1}{(i)^{|\alpha|}} T(\hat{D^\alpha f}) = \frac{1}{(i)^{|\alpha|}} \hat{T}(D^\alpha f)
\]

\[
= \frac{1}{(-i)^{|\alpha|}} D^\alpha \hat{T}(f).
\]

Thus, the first relation is proved. The second relation can be proved similarly.
Example

\[ \delta_0 = 1, \quad \left( \frac{\partial \delta_0}{\partial x_k} \right)(\xi) = \nu \xi_k \text{ and } \hat{1} = \delta_0. \] Let \( \phi \in \mathcal{S}(\mathbb{R}^n). \) Then

\[ \hat{\delta}_0(\phi) = \delta_0(\hat{\phi}) = \hat{\phi}(0) = \int_{\mathbb{R}^n} \phi(x) \, dx. \]

Thus, \( \hat{\delta}_0 = 1, \) i.e., the distribution induced by the constant function 1. Further,

\[ \left( \frac{\partial \delta_0}{\partial x_k} \right)(\xi) = \nu \xi_k \hat{\delta}_0 = \nu \xi_k. \]

The \( \hat{1} = \delta_0 \) follows by the Fourier inversion formula.
Definition

Let \( L = \sum_{|\alpha| \leq k} c_\alpha D^\alpha \) be a \( k \)-th order linear differential operator with constant coefficients. A distribution \( K \) is said to be a fundamental solution of the operator \( L \) if \( L(K) = \delta_0 \), where \( \delta_0 \) is the Dirac distribution at \( \{0\} \).

The existence of fundamental solution of a linear differential operator for constant coefficients is the famous result of Malgrange and Ehrenpreis. Suppose we wish to solve \( LT = S \) for a given distribution \( S \), we first find the fundamental solution \( LK = \delta_0 \). Then \( K \ast S \) (as long as the convolution makes sense) is a solution of the equation \( L(\cdot) = S \), because \( L(K \ast S) = LK \ast S = \delta_0 \ast S = S \). The fundamental solution is not unique because any solution \( U \) of the homogeneous equation \( L(U) = 0 \) can be added to the fundamental solution \( K \) to obtain other fundamental solutions, since \( L(U + K) = L(U) + L(K) = \delta_0 \).
Hölder Continuity at a Point

We introduce some class of function spaces in $C^k(\Omega)$, for all integers $k \geq 0$, which can be viewed in some sense as spaces of “fractional” derivatives.

**Definition**

Let $\gamma \in (0, 1]$. We say a function $u : \Omega \to \mathbb{R}$ is Hölder continuous of exponent $\gamma$ at $x_0 \in \Omega$, if

$$p_\gamma(u)(x_0) := \sup_{\begin{subarray}{c} x \in \Omega \\ x \neq x_0 \end{subarray}} \left\{ \frac{|u(x) - u(x_0)|}{|x - x_0|^{\gamma}} \right\} < +\infty.$$  

Note that the modulus in the numerator and denominator are in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.
Note that any Hölder continuous function at \( x_0 \in \Omega \) satisfies the estimate

\[
|u(x) - u(x_0)| \leq p_{\gamma}(u)(x_0)|x - x_0|^\gamma, \quad \text{for } x \in \Omega. \tag{8.1}
\]

The constant \( p_{\gamma}(u)(x_0) \) may depend on \( u, \Omega, \gamma \) and \( x_0 \).

It follows from (8.1) that any Hölder continuous function at \( x_0 \), for any exponent \( \gamma \), is also continuous at \( x_0 \). Because, for every \( \varepsilon > 0 \), we can choose \( \delta = \left[ \frac{\varepsilon}{p_{\gamma}(u)(x_0)} \right]^{1/\gamma} \).

The case when \( \gamma = 1 \) corresponds to the Lipshitz continuity of \( u \) at \( x_0 \).
Hölder Continuous

**Definition**

Let $\gamma \in (0, 1]$. We say a function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous of exponent $\gamma$, if

$$p_\gamma(u) := \sup_{x, y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right\} < +\infty \quad (8.2)$$

and is denoted by $C^{0, \gamma}(\Omega)$. The quantity $p_\gamma(u)$ is called the $\gamma$-th Hölder coefficient of $u$. If the Hölder coefficient is finite on every compact subsets of $\Omega$, then $u$ is said to be locally Hölder continuous with exponent $\gamma$, denoted as $C^{0, \gamma}_{loc}(\Omega)$.

Note that $p_\gamma(u) = \sup_{x_0 \in \Omega} p_\gamma(u)(x_0)$. Thus, the class of uniformly Hölder continuous functions do not include necessarily include all functions that are Hölder continuous at all points of $\Omega$. However, bounded locally Hölder continuous functions are Hölder continuous at all points of $\Omega$. 

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Example

Note that, for all $\delta < 0$, $u(x) = |x|^\delta$ on the open ball $B_1(0) \subset \mathbb{R}^n$ do not belong to $C(B_1(0))$.

For $\delta \in [0, \infty)$, $|x|^{\delta} \in C(B_1(0))$ and, for $\delta \in [2, \infty) \cup \{0\}$, $|x|^{\delta} \in C^1(B_1(0))$.

For $\delta \in (0, 1]$, $|x|^{\delta}$ is Hölder continuous, with exponent $\gamma$, for all $0 < \gamma \leq \delta$, but is not Hölder continuous for $\gamma > \delta$. In particular, $|x|^{\delta} \in C^{0,\delta}(B_1(0))$ for each $\delta \in [0, 1]$. Thus, we have one example from each of the space $C^{0,\delta}(B_1(0))$.

We first check the Hölder continuity at $x_0 = 0$. The $\gamma$-th Hölder coefficient is

$$\sup_{x \in B_1(0)} \frac{|x|^{\delta}}{|x|^\gamma} = |x|^{\delta-\gamma} \leq 1 \quad \text{for } \gamma \leq \delta.$$

For $\gamma > \delta$ the $\gamma$-th Hölder coefficient blows up.
More generally, for \( x \neq 0 \) and \(|x| > |y|\) (wlog),

\[
\frac{|x|^{\delta} - |y|^{\delta}}{|x - y|^{\gamma}} = \frac{|x|^{\delta-\gamma} \left| 1 - \left( \frac{|y|}{|x|} \right)^{\delta} \right|}{\left| \frac{x}{|x|} - \frac{y}{|x|} \right|^\gamma}.
\]

Since \( 0 < 1 - \frac{|y|}{|x|} < 1 \) and \( 0 < \delta \leq 1 \), \( \left| 1 - \left( \frac{|y|}{|x|} \right)^{\delta} \right| \leq \left| 1 - \frac{|y|}{|x|} \right| \). Thus,

\[
|x|^{\delta-\gamma} \left| 1 - \left( \frac{|y|}{|x|} \right)^{\delta} \right| \leq \frac{|x|^{\delta-\gamma} \left| 1 - \frac{|y|}{|x|} \right|}{\left| \frac{x}{|x|} - \frac{y}{|x|} \right|^\gamma} \leq \frac{||x| - |y||}{|x|^{1-\gamma}|x - y|^{\gamma}}.
\]

The last inequality is true for \( \gamma \leq \delta \) and we have used \( |x|^{\delta-\gamma} \leq 1 \). For one dimension, the last quantity is equal to 1 because

\[
\frac{||x| - |y||}{|x|^{1-\gamma}|x - y|^{\gamma}} = \left| x \right| \left| 1 - \frac{|y|}{|x|} \right| \leq \frac{1 - \frac{y}{x}}{1 - \frac{y}{x}} = 1.
\]

Complete the argument for higher dimensions!
Example

The Cantor function \( f_C \in C^{0,\gamma}([0,1]) \), i.e., is Hölder continuous with exponent \( 0 < \gamma \leq \log_3 2 \). Geometrically, the graph of Hölder continuous function have fractal appearance which increases with smaller \( \gamma \).

\[ \log_3 2 \approx 0.6309 \ldots \]

The case \( \gamma = 0 \) corresponds to bounded functions on \( \Omega \) and hence is ignored as an possible exponent.

In fact, continuous functions in \( C^{0,0}(\Omega) \) can be identified with the space \( C_b(\Omega) \) of bounded continuous functions on \( \Omega \).

The space \( C(\Omega) \) can be identified with continuous functions in \( C^{0,0}_{loc}(\Omega) \), the space of all locally Hölder continuous with exponent \( \gamma = 0 \).

The case \( \gamma = 1 \) corresponds to \( u \) being Lipschitz continuous. The space of all Lipschitz functions is denoted as \( \text{Lip}(\Omega) = C^{0,1}(\Omega) \).
The case $\gamma > 1$

**Exercise**

If $u \in C^{0,\gamma}(\Omega)$ with exponent $\gamma > 1$, then $u$ is constant in each of the connected component of $\Omega$.

**Proof.**

Let $u \in C^{0,\gamma}(\Omega)$. For any $x \in \Omega$,

$$
|u_{x_i}(x)| = \lim_{t \to 0} \frac{1}{|t|} |u(x + te_i) - u(x)| \leq \lim_{t \to 0} \frac{p_\gamma(u)(x)}{|t|} |t|^\gamma
$$

$$
= \lim_{t \to 0} p_\gamma(u)(x)|t|^{\gamma - 1} = 0.
$$

Therefore, the gradient $Du(x) = 0$ for all $x \in \Omega$ and hence $u$ is constant in each connected components of $\Omega$. $\square$
Exercise

If $\Omega$ has finite diameter, then for any $0 < \gamma < \delta \leq 1$ we have $C^{0,1}(\Omega) \subseteq C^{0,\delta}(\Omega) \subsetneq C^{0,\gamma}(\Omega) \subsetneq C(\Omega)$.

Proof: The last inclusion is trivial because every $u \in C^{0,\gamma}(\Omega)$ is uniformly continuous by choosing $\delta = \left[\frac{\varepsilon}{p_{\gamma}(u)}\right]^{1/\gamma}$, for any given $\varepsilon > 0$. Thus, $C^{0,\gamma}(\Omega) \subsetneq C(\Omega)$ for all $0 < \gamma \leq 1$. On the other hand, let $u \in C^{0,\delta}(\Omega)$. Consider

$$p_{\gamma}(u) = \sup_{x, y \in \Omega \atop x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}$$

$$= \sup_{x, y \in \Omega \atop x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\delta} |x - y|^\delta - \gamma \right\}$$

$$\leq \text{diam}(\Omega)^{\delta - \gamma} p_\delta(u).$$
Thus, $p_\gamma(u) < +\infty$ and $u \in C^{0,\gamma}(\Omega)$. The inclusions are strict by Example 162. If $\gamma < \delta$, then $u(x) = |x|^\gamma$ is in $C^{0,\gamma}(-1,1)$ but not in $C^{0,\delta}(-1,1)$. In particular, $\sqrt{x} \in C^{0,1/2}(-1,1)$ but is not Lipschitz, i.e. do not belong to $C^{0,1}(-1,1)$. 

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A natural question at this juncture is: what is the relation between the spaces $C^1(\Omega)$ and $C^{0,1}(\Omega)$. For instance, $|x| \in C^{0,1}(-1,1)$ and not in $C^1(-1,1)$. Given the inclusion relation in the above exercise, it might be tempting to believe that $C^1(\Omega)$ is contained in $C^{0,1}(\Omega)$, but this is not true. In fact, only a subclass of $C^1(\Omega)$ belongs to Lipschitz class.

**Theorem**

Let $\Omega$ be a convex domain. If $u \in C^1(\Omega)$ and $D^\alpha u$ is bounded on $\Omega$, for each $|\alpha| = 1$, then $u \in C^{0,1}(\Omega)$.

**Proof:** Let $u \in C^1(\Omega)$ such that there is a $C_0 > 0$ such that $|\nabla u(x)| \leq C_0$ for all $x \in \Omega$. For any two given points $x, y \in \Omega$, we define the function $F : [0, 1] \to \mathbb{R}$ as $F(t) = u((1 - t)x + ty)$. $F$ is well defined due to the convexity of $\Omega$. Since $u$ is differentiable, $F$ is differentiable in $(0, 1)$. Thus, for all $t \in (0, 1)$, $F'(t) = \nabla u[((1 - t)x + ty) \cdot (y - x)$. Therefore, $|F'(t)| \leq C_0|y - x|$. By mean value theorem, there is a $\xi \in (0, 1)$ such that $F'(\xi) = F(1) - F(0)$.
Thus,

\[ |u(y) - u(x)| = |F(1) - F(0)| = |F'(\xi)| \leq C_0|y - x|. \]

Hence, \( u \) is Lipschitz continuous.

What fails in the converse of the above theorem is the fact that a Lipschitz function can fail to be differentiable. If we assume, in addition to Lipschitz continuity, that \( u \) is differentiable then \( u \in C^1 \) and derivative is bounded.

**Exercise**

Prove the above theorem when \( \Omega \) is path connected.
Theorem

For any $\gamma \in (0, 1]$, the space $C^{0, \gamma}(\Omega)$ can be identified with the space $C^{0, \gamma}(\overline{\Omega})$.

Proof: The space $C^{0, \gamma}(\overline{\Omega})$ can be identified with a subset of $C^{0, \gamma}(\Omega)$, by identifying any $u \in C^{0, \gamma}(\overline{\Omega})$ with $u|_{\Omega} \in C^{0, \gamma}(\Omega)$. On the other hand, for any $u \in C^{0, \gamma}(\Omega)$ we wish to extend it uniquely to a function $\tilde{u} \in C^{0, \gamma}(\overline{\Omega})$ such that $\tilde{u}|_{\Omega} = u$. Let $u \in C^{0, \gamma}(\Omega)$. Set $\tilde{u}(x) = u(x)$ for all $x \in \Omega$. Now, for any $x_0 \in \partial \Omega$, choose a sequence $\{x_m\} \subset \Omega$ such that $\lim_m x_m = x_0$. Then, we have

$$|u(x_k) - u(x_l)| \leq C|x_k - x_l|^\gamma \to 0 \text{ as } k, l \to \infty.$$ 

Thus, $\{u(x_m)\}$ is a Cauchy sequence and converges in $\mathbb{R}$. Set $\tilde{u}(x_0) = \lim_m u(x_m)$. We now show that the definition of $\tilde{u}(x_0)$ is independent of the choice of the sequence.
If \( \{y_m\} \subset \Omega \) is any other sequence converging to \( x_0 \), then
\[
|u(x_m) - u(y_m)| \leq C|x_m - y_m|^{\gamma} \to 0 \text{ as } m \to \infty.
\]
Thus, \( \tilde{u}(x_0) \) is well-defined for all \( x_0 \in \partial \Omega \). Moreover, for all \( x, y \in \overline{\Omega} \), we have
\[
|\tilde{u}(x) - \tilde{u}(y)| = \lim_{m \to \infty} |u(x_m) - u(y_m)| \leq C \lim_{m \to \infty} |x_m - y_m|^{\gamma} = C|x - y|^{\gamma}.
\]
Hence, \( \tilde{u} \) satisfies the Hölder estimate with exponent \( \gamma \) and is in \( C^{0,\gamma}(\overline{\Omega}) \).

The above result is very special to uniformly continuous spaces and, hence, Hölder spaces. An interesting point in the proof of above result is that the Hölder coefficient are same for the extension, i.e., \( p_\gamma(u) = p_\gamma(\tilde{u}) \). An advantage of above theorem is that, without loss of generality, we can use the space \( C^{0,\gamma}(\Omega) \) and \( C^{0,\gamma}(\overline{\Omega}) \) interchangeably.
Exercise

Show that the H"{o}lder coefficient, $p_\gamma$, defines a semi-norm on $C^{0,\gamma}(\Omega)$.

Proof: For any constant function $u$, $p_\gamma(u) = 0$. Also,

$$p_\gamma(u + v) = \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) + v(x) - u(y) - v(y)|}{|x - y|^\gamma} \right\}$$

$$= \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y) + v(x) - v(y)|}{|x - y|^\gamma} \right\}$$

$$\leq \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y)| + |v(x) - v(y)|}{|x - y|^\gamma} \right\}$$

$$\leq p_\gamma(u) + p_\gamma(v).$$

Further $p_\gamma(\lambda u) = |\lambda| p_\gamma(u)$. 
Note that for bounded open subsets $\Omega$, the space $C^{0,\gamma}(\Omega)$ inherits the uniform topology from $C(\Omega)$.

**Exercise**

For any bounded open set $\Omega \subset \mathbb{R}^n$, show that $C^{0,\gamma}(\Omega)$ is dense in $C(\Omega)$.

To make the space $C^{0,\gamma}(\Omega)$ complete on bounded $\Omega$, we define the $\gamma$-Hölder norm on $C^{0,\gamma}(\Omega)$ as

$$
\| u \|_{C^{0,\gamma}(\Omega)} := \| u \|_{\infty} + p_\gamma(u),
$$

where $\| u \|_{\infty} := \sup_{x \in \Omega} |u(x)|$.

**Exercise**

For any bounded open set $\Omega \subset \mathbb{R}^n$, show that $\| \cdot \|_{C^{0,\gamma}(\Omega)}$ is a norm on $C^{0,\gamma}(\Omega)$. 
Theorem

For any bounded open set $\Omega \subset \mathbb{R}^n$, the space $C^{0,\gamma}(\Omega)$ is a Banach space with norm $\| \cdot \|_{C^{0,\gamma}(\Omega)}$.

Proof: We need to prove the completeness of the space $C^{0,\gamma}(\Omega)$ w.r.t the norm $\| \cdot \|_{C^{0,\gamma}(\Omega)}$. Let $\{u_m\}$ be a Cauchy sequence in $C^{0,\gamma}(\Omega)$, then $\{u_m\} \subset C(\Omega)$ is Cauchy w.r.t the supremum norm. Thus, there is a $u \in C(\Omega)$ such that $\|u_m - u\|_\infty \to 0$, as $m \to \infty$. We first show that $u \in C^{0,\gamma}(\Omega)$. For $x, y \in \Omega$ with $x \neq y$, consider

$$\frac{|u(x) - u(y)|}{|x - y|^{\gamma}} = \lim_m \frac{|u_m(x) - u_m(y)|}{|x - y|^{\gamma}} \leq \limsup_m p_\gamma(u_m) \leq \lim_m \|u_m\|_{C^{0,\gamma}(\Omega)}.$$

Since $\{u_m\}$ is Cauchy, $\lim_m \|u_m\|_{C^{0,\gamma}(\Omega)} < \infty$. Hence $u \in C^{0,\gamma}(\Omega)$.

Finally, we show that the sequence $\{u_m\}$ converges to $u$ in $C^{0,\gamma}(\Omega)$ w.r.t the $\gamma$-Hölder norm.
Consider,
\[
\frac{|u_m(x) - u(x) - u_m(y) + u(y)|}{|x - y|^\gamma} = \lim_{k} \frac{|u_m(x) - u_k(x) - u_m(y) + u_k|}{|x - y|^\gamma} \leq \limsup_{k} p_\gamma(u_m - u_k).
\]

Therefore, \( \lim_m p_\gamma(u_m - u) \leq \lim_m \limsup_k p_\gamma(u_m - u_k) \) and the RHS converges to 0 since the sequence is Cauchy. Hence, \( \|u_m - u\|_{C^{0,\gamma}(\Omega)} \to 0. \)

**Exercise**

The space \( C^{0,\gamma}(\Omega) \) norm is not separable.

For an unbounded open set \( \Omega \), we consider a sequence of exhaustion compact sets and make the space \( C^{0,\gamma}(\Omega) \) a Fréchet space.
Compact Operator and Compact Imbedding

**Definition**

Let $X$ and $Y$ be Banach spaces. A continuous (bounded) linear operator $L : X \to Y$ is said to be compact if $L$ maps every bounded subset of $X$ to precompact (closure compact) subsets of $Y$. Equivalently, $L$ maps bounded sequences of $X$ to sequences in $Y$ that admit convergent subsequences.

**Definition**

Let $X$ and $Y$ be Banach spaces such that $X \subset Y$. We say $X$ is continuously imbedded in $Y$ (denoted as $X \hookrightarrow Y$), if there is a constant such that

$$\|x\|_Y \leq C \|x\|_X \quad \forall x \in X.$$

Further, we say $X$ is compactly imbedded in $Y$ (denoted as $X \hookrightarrow\rightarrow Y$) if in addition to being continuously imbedded in $Y$, every bounded set in $X$ (w.r.t the norm in $X$) is compact in $Y$ (w.r.t the norm in $Y$).
Theorem

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \). For any \( 0 < \gamma < \delta \leq 1 \), the inclusion map \( I : C^{0,\delta}(\Omega) \rightarrow C^{0,\gamma}(\Omega) \) is continuous and compact. Further, the inclusion map \( I : C^{0,\gamma}(\Omega) \rightarrow C(\Omega) \) is compact, for all \( 0 < \gamma \leq 1 \).

Proof: The continuity of \( I \) follows from the inequality proved in Exercise 48 and hence \( I \) is a bounded linear operator. We need to show \( I \) is compact. Let \( \{u_m\} \) be a sequence bounded in \( C^{0,\delta}(\Omega) \). Without loss of generality, we can assume that \( \|u_m\|_{C^{0,\delta}(\Omega)} \leq 1 \). Therefore, \( p_\delta(u_m) \leq 1 \) for all \( m \) which implies that \( \{u_m\} \) is an equicontinuous sequence in \( C(\Omega) \). By Arzelà-Ascoli theorem, there exists a subsequence \( \{u_{m_k}\} \) of \( \{u_m\} \) and a \( u \in C(\Omega) \) such that \( \|u_{m_k} - u\|_{\infty} \rightarrow 0 \), as \( k \rightarrow \infty \). The \( u \in C(\Omega) \) is, in fact, in \( C^{0,\delta}(\Omega) \subset C^{0,\gamma}(\Omega) \) because

\[
\frac{|u(x) - u(y)|}{|x - y|^{\delta}} = \lim_k \frac{|u_{m_k}(x) - u_{m_k}(y)|}{|x - y|^{\delta}} \leq 1
\]

and, hence, \( p_\delta(u) \leq 1 \).
Proof Continued...

We now show that the $u_{mk}$ converges to $u$ in $C^{0,\gamma}(\Omega)$. For simplicity, set $v_k := u_{mk} - u$ and we will show that $\{v_k\}$ converges to 0 in $C^{0,\gamma}(\Omega)$. Obviously, $\|v_k\|_\infty \to 0$ thus it is enough to show that $p_\gamma(v_k) \to 0$. Note that, for every given $\varepsilon > 0$, we have $p_\gamma(v_k) \leq S_\varepsilon + T_\varepsilon$, where

$$
S_\varepsilon = \sup_{x,y \in \Omega, \ x \neq y; |x-y| \leq \varepsilon} \left\{ \frac{|v_k(x) - v_k(y)|}{|x - y|^{\gamma}} \right\}
$$

and

$$
T_\varepsilon = \sup_{x,y \in \Omega, \ |x-y| > \varepsilon} \left\{ \frac{|v_k(x) - v_k(y)|}{|x - y|^{\gamma}} \right\}.
$$

Consider

$$
S_\varepsilon = \sup_{x,y \in \Omega, \ x \neq y; |x-y| \leq \varepsilon} \left\{ \frac{|v_k(x) - v_k(y)|}{|x - y|^{\delta}} |x - y|^{\delta-\gamma} \right\} \leq \varepsilon^{\delta-\gamma} p_\delta(v_k) \leq 2\varepsilon^{\delta-\gamma}.
$$
Proof Continued...

Similarly,

\[ T_k^\varepsilon \leq 2\varepsilon^{-\gamma} \| v_k \|_\infty. \]

Hence, \( \limsup_k p_{\gamma}(v_k) \leq 2\varepsilon^{\delta-\gamma} + 2\varepsilon^{-\gamma} \limsup_k \| v_k \|_\infty = 2\varepsilon^{\delta-\gamma} + 0. \) Since \( \varepsilon \) can be made as small as possible, we have \( p_{\gamma}(v_k) \rightarrow 0. \)
Differentiable Hölder Class

- We denote by $C^{k,\gamma}(\Omega)$ the space of all $C^k(\Omega)$, $k$-times continuously differentiable, functions such that $D^\alpha u$ is Hölder continuous with exponent $\gamma$, i.e., $D^\alpha u \in C^{0,\gamma}(\Omega)$ for all $|\alpha| = k$.

- For a bounded open set $\Omega$, we give the $\gamma$-th Hölder norm on $C^{k,\gamma}(\Omega)$ as

  $$ \|u\|_{C^{k,\gamma}(\Omega)} := \sum_{|\alpha|=0}^{k} \|D^\alpha u\|_{\infty} + \sum_{|\alpha|=k} p_\gamma(D^\alpha u). \quad (8.3) $$

- It is enough to consider the Hölder coefficient of only the $k$-th derivative because this is enough to complete the space.
Exercise

For any bounded open set $\Omega \subset \mathbb{R}^n$, show that $\| \cdot \|_{C^{k,\gamma}(\Omega)}$ is a norm on $C^{k,\gamma}(\Omega)$. and the space $C^{k,\gamma}(\Omega)$ is a Banach space with norm $\| \cdot \|_{C^{k,\gamma}(\Omega)}$.

Proof.

Let $\{u_m\}$ be a Cauchy sequence in $C^{k,\gamma}(\Omega)$. Then, $\{u_m\} \subset C^k(\Omega)$ is Cauchy in the supremum norm. Thus, there is a $u \in C^k(\Omega)$ such that $\|u_m - u\|_{C^k(\Omega)} \to 0$, as $m \to \infty$. The fact that $u \in C^{k,\gamma}(\Omega)$ and that $u$ is a limit, in the $\gamma$-Hölder norm, of the Cauchy sequence is similar to case $k = 0$ proved in Theorem 20.

As seen before, for an unbounded open set $\Omega$, we consider a sequence of exhaustion compact sets and make the space $C^{k,\gamma}(\Omega)$ a Fréchet space.
Theorem

Let $\Omega$ be a bounded, convex open subset of $\mathbb{R}^n$. For any $0 < \gamma, \delta \leq 1$ and $k, \ell \in \mathbb{N} \cup \{0\}$ such that $k + \gamma < \ell + \delta$, then the inclusion map $I : C^{\ell,\delta}(\Omega) \to C^{k,\gamma}(\Omega)$ is continuous and compact.

Proof.

If $\gamma = \delta$ and $k = \ell$ then there is nothing to prove. Alternately, without loss of generality we assume $0 < \gamma < \delta \leq 1$ and $k < \ell$. Then, we know from Theorem 21 that $C^{\ell,\delta}(\Omega) \subset C^{\ell,\gamma}(\Omega)$ and the inclusion is continuous and compact. Let $u \in C^{\ell,\gamma}$. We need to show that $u \in C^{k,\gamma}(\Omega)$. i.e., $p_\gamma(D^k u) < \infty$. Note that it is enough to show that $p_1(D^k u) < \infty$. The fact that $u \in C^{\ell,\gamma}$ implies that $u \in C^{\ell}(\Omega)$ and $D^{\ell-1} u$ is bounded. Thus, by Theorem 18, we have $u \in C^{\ell-1,1}(\Omega)$ and hence is in $C^{\ell-1,\gamma}(\Omega)$. By repeating the argument for each derivative of $D^{\ell-i}$ for each $i = 1, 2, \ldots, k$, we have the required result. The inclusion $C^{\ell,\gamma}(\Omega) \subset C^{k,\gamma}(\Omega)$ is continuous. The composition of a continuous and compact operator is compact. Thus, the inclusion $I$ is compact.

$\square$
For each $1 \leq p \leq \infty$, we define its *conjugate* exponent $q$ to be,

$$q = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < +\infty \\ +\infty & \text{if } p = 1 \\ 1 & \text{if } p = +\infty. \end{cases}$$

Note that for, $1 < p < +\infty$, $q$ is the number for which $1/p + 1/q = 1$.

Recall that every $u \in L^p(\Omega)$ being locally integrable induces a distribution $T_u$. Further, the distribution $T_u$ is differentiable for all multi-indices $\alpha$. But we have already seen that $T_u$ need not be weakly differentiable.

For a fixed multi-index $\alpha$, if there exists a $v_\alpha \in L^p(\Omega)$ such that $T_{v_\alpha} = D^\alpha T_u$, then we denote the $v_\alpha$ as $D^\alpha u$.

We know that such a $v_\alpha$ is unique up to a set of measure zero.
Definition

Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is the subclass of all $u \in L^p(\Omega)$ such that there exists a $v_\alpha \in L^p(\Omega)$, for all $0 \leq |\alpha| \leq k$, such that

$$\int_\Omega v_\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx \quad \forall \phi \in D(\Omega).$$

Equivalently,

$$W^{k,p}(\Omega) := \{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \quad \forall 0 \leq |\alpha| \leq k \}.$$

- With this convention, we have $W^{0,p}(\Omega) = L^p(\Omega)$.
- Note that, by definition, if $u \in W^{k,p}(\Omega)$ then every function in the equivalence class of $u$ is also in $W^{k,p}(\Omega)$ and they all have their derivative as $v_\alpha$ up to a set of measure zero.
Let $\Omega = (-a, a) \subset \mathbb{R}$, for a positive real number $a$. Recall that the function $u(x) = |x|$ is not in $C^1(-a, a)$. But $u \in W^{1,p}(-a, a)$. Consider the $v \in L^p(-a, a)$ defined as

$$v(x) = \begin{cases} 
1 & x \in [0, a) \\
-1 & x \in (-a, 0).
\end{cases}$$

We shall show that $v$ is the weak derivative of $u$. Consider, for $\phi \in \mathcal{D}(-a, a)$,

$$\int_{-a}^{a} v \phi \, dx = -\int_{-a}^{0} \phi \, dx + \int_{0}^{a} \phi \, dx = \int_{-a}^{0} x \phi' \, dx - \int_{0}^{a} x \phi' \, dx = -\int_{-a}^{a} |x| \phi' \, dx.$$  

More generally, any continuous function on $[-a, a]$ which is piecewise differentiable, i.e., piecewise $C^1$ on $(-a, a)$ is in $W^{1,p}(-a, a)$. 
Example

The function $v$ defined in above example is not in $W^{1,p}(-a, a)$. The argument is similar for $w$ defined as

$$w(x) = \begin{cases} 
1 & x \in [0, a) \\
0 & x \in (-a, 0). 
\end{cases}$$

Note that $w \in L^p(-a, a)$, for all $p$. However, the distributional derivative of $w$ (and $v$) is the Dirac measure at 0, $\delta_0$. Hence $w \not\in W^{k,p}(\Omega)$, for all $k > 0$.

The function $u$ defined as

$$u(x) = \begin{cases} 
x & x \in (0, a) \\
0 & x \in (-a, 0] 
\end{cases}$$

is in $W^{1,p}(-a, a)$, for all $p \in [1, \infty]$, since both $u$ and its distributional derivative $Du = w$ are in $L^p(-a, a)$. But $u \not\in W^{k,p}(\Omega)$, for all $k \geq 2$. 
Note that we have the inclusion $W^{k,p}(\Omega) \subsetneq W^{\ell,p}(\Omega)$ for all $\ell < k$. The inclusion is strict as seen from above examples.

**Exercise**

Find values of $\gamma \in \mathbb{R}$ such that $|x|^\gamma \in W^{1,p}(B_1(0))$ for a fixed $p \in [1, \infty)$ and $B_1(0)$ is the open unit ball in $\mathbb{R}^n$.

**Proof.**

Let $u(x) = |x|^{\gamma}$. Note that $D^e_i u(x) = \gamma x_i |x|^{\gamma - 2}$ and $|\nabla u(x)| = \gamma |x|^{\gamma - 1}$.

Consider

$$\|\nabla u\|_p = \gamma^p \int_{\Omega} |x|^{p(\gamma - 1)} \, dx = \gamma^p \omega_n \int_0^1 r^{p\gamma - p + n - 1} \, dr.$$ 

The last quantity is finite iff $p(\gamma - 1) + n - 1 > -1$, i.e., $\gamma > 1 - \frac{n}{p}$ (also $\gamma = 0$, if not already included in the inequality condition) and $|x|^{\gamma} \in W^{1,p}(\Omega)$.
Using the above exercise, we have an example of a function in \( L^p(B_1(0)) \) but not in \( W^{1,p}(B_1(0)) \). For instance, if \( \Omega = B_1(0) \) in \( \mathbb{R}^n \) and 
\[ u(x) = |x|^\gamma, \]
for non-zero \( \gamma \), such that \( -\frac{n}{p} < \gamma \leq 1 - \frac{n}{p} \), is not in \( W^{1,p}(B_1(0)) \) but is in \( L^p(B_1(0)) \).

**Exercise**

Find values of \( \gamma \in \mathbb{R} \) such that \( |x|^\gamma \in W^{2,p}(B_1(0)) \), for a fixed \( p \in [1, \infty) \).

**Proof.**

Let \( u(x) = |x|^\gamma \). Note that

\[
|u_{x_i,x_j}| \leq |\gamma(\gamma - 2)||x|^{\gamma - 2} + |\gamma||x|^{\gamma - 2}.
\]

Hence, \( u_{x_i,x_j} \in L^p(B_1(0)) \) if \( \gamma - 2 > -\frac{n}{p} \), or \( \gamma > 2 - \frac{n}{p} \). Thus

\( u \in W^{2,p}(B_1(0)) \) if \( \gamma > 2 - \frac{n}{p} \) and \( \gamma = 0 \).
Exercise

Show that the spaces $W^{k,p}(\Omega)$ are all (real) vector spaces.

Exercise

Show that $u \mapsto \sum_{|\alpha|=k} \|D^\alpha u\|_p$ defines a semi-norm on $W^{k,p}(\Omega)$ for $k > 0$. We denote the semi-norm by $|u|_{k,p,\Omega}$.

For $1 \leq p < \infty$, we endow the space $W^{k,p}(\Omega)$ with the natural norm,

$$\|u\|_{k,p,\Omega} := \sum_{|\alpha|=0}^{k} \|D^\alpha u\|_p = \sum_{|\alpha|=0}^{k} \left( \int_{\Omega} |D^\alpha u|^p \right)^{1/p}.$$ 

For $p = \infty$, we define the norm on $W^{k,\infty}(\Omega)$ to be,

$$\|u\|_{k,\infty,\Omega} = \sum_{|\alpha|=0}^{k} \text{ess sup}_{\Omega} |D^\alpha u|.$$ 

Observe that $\|u\|_{0,p,\Omega} = |u|_{0,p,\Omega} = \|u\|_p$, the usual $L^p$-norm.
Exercise

Show that the norm $\| u \|_{k,p,\Omega}$ defined above is equivalent to the norm

$$\left( \sum_{|\alpha|=0}^{k} \| D^\alpha u \|_p^p \right)^{1/p} = \left( \sum_{|\alpha|=0}^{k} \int_\Omega | D^\alpha u |^p \right)^{1/p}, \quad \text{for } 1 < p < \infty.$$ 

For $p = 1$, the norms are same.

Frequently, we may skip the domain subscript in the norm where there is no confusion on the domain of function. Also, we shall tend to use the symbol $\| \cdot \|_p$ for $\| \cdot \|_{0,p}$ in $L^p$-spaces.
**$H^k$-spaces**

We set $H^k(\Omega) = W^{k,2}(\Omega)$. For $u \in H^k(\Omega)$, we shall denote the norm as $\| \cdot \|_{k,\Omega}$.

**Exercise**

Show that

$$\langle u, v \rangle_{k,\Omega} = \sum_{|\alpha|=0}^{k} \int_{\Omega} D^\alpha u D^\alpha v, \quad \forall u, v \in H^k(\Omega)$$

defines an inner-product in $H^k(\Omega)$. 
Theorem

For every $1 \leq p \leq \infty$, the space $W^{k,p}(\Omega)$ is a Banach space. If $1 < p < \infty$, it is reflexive and if $1 \leq p < \infty$, it is separable. In particular, $H^k(\Omega)$ is a separable Hilbert space.

Proof: Let $\{u_m\}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. Thus, by the definition of the norm on $W^{k,p}(\Omega)$, $\{u_m\}$ and $\{D^\alpha u_m\}$, for $1 \leq |\alpha| \leq k$ are Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exist function $u, v_\alpha \in L^p(\Omega)$ such that

$$u_m \to u \quad \text{and} \quad D^\alpha u_m \to v_\alpha \quad \forall 1 \leq |\alpha| \leq k.$$

To show that $W^{k,p}(\Omega)$ is complete, it is enough to show that $D^\alpha u = v_\alpha$, for $1 \leq |\alpha| \leq k$. 
Let $\phi \in \mathcal{D}(\Omega)$, then for each $\alpha$ such that $1 \leq |\alpha| \leq k$, we have

$$
\int_{\Omega} u D^{\alpha} \phi \, dx = \lim_{m \to \infty} \int_{\Omega} u_m D^{\alpha} \phi \, dx
$$

$$
= \lim_{m \to \infty} (-1)^{|\alpha|} \int_{\Omega} D^{\alpha} u_m \phi \, dx
$$

$$
= (-1)^{|\alpha|} \int_{\Omega} v^{\alpha} \phi \, dx
$$

Thus, $D^{\alpha} u = v^{\alpha}$, i.e., $v^{\alpha}$ is the $\alpha$ weak derivative of $u$.

Note that we have an isometry from $W^{k,p}(\Omega)$ to $(L^p(\Omega))^\beta$, where $\beta := \sum_{i=0}^kn^i$ and the norm endowed on $(L^p(\Omega))^\beta$ is

$$
\|u\| := \left(\sum_{i=1}^\beta \|u_i\|^p \right)^{1/p} \quad \text{with } u = (u_1, u_2, \ldots, u_\beta). \quad \text{The image of}
$$

$W^{k,p}(\Omega)$ under this isometry is a closed subspace of $(L^p(\Omega))^\beta$. The reflexivity and separability of $W^{k,p}(\Omega)$ is inherited from reflexivity and separability of $(L^p(\Omega))^\beta$, for $1 < p < \infty$ and $1 \leq p < \infty$, respectively.

**Remark:** The proof above uses an important fact which is worth noting. If $u_m \to u$ in $L^p(\Omega)$ and $D^{\alpha} u_m$, for all $|\alpha| = 1$, is bounded in $L^p(\Omega)$ then $u \in W^{1,p}(\Omega)$.
Approximations by Smooth Functions

- Recall that in Theorem 6, we proved that \( C_c^\infty(\Omega) \) is dense in \( L^p(\Omega) \) for any open subset \( \Omega \subseteq \mathbb{R}^n \).

- We are now interested in knowing if the density of \( C_c^\infty(\Omega) \) holds true in the Sobolev space \( W^{k,p}(\Omega) \), for any \( k \geq 1 \). (The case \( k = 0 \) is precisely the result of Theorem 6).

**Theorem**

For \( 1 \leq p < \infty \), \( C_c^\infty(\mathbb{R}^n) \) is dense in \( W^{k,p}(\mathbb{R}^n) \), for all \( k \geq 1 \).

**Proof:** Let \( k \geq 1 \) and \( u \in W^{k,p}(\mathbb{R}^n) \). The \( D^\alpha u \in L^p(\mathbb{R}^n) \) for all \( 0 \leq |\alpha| \leq k \). By Theorem 5, we know that \( \{\rho_m \ast u\} \subset C^\infty(\mathbb{R}^n) \) converges to \( u \) in the \( W^{k,p}(\Omega) \)-norm, for all \( k \geq 0 \). Choose a function \( \phi \in C_c^\infty(\mathbb{R}^n) \) such that \( \phi \equiv 1 \) on \( B(0;1) \), \( \phi \equiv 0 \) on \( \mathbb{R}^n \setminus B(0;2) \) and \( 0 \leq \phi \leq 1 \) on \( B(0;2) \setminus B(0;1) \). Note that \( |D^\alpha \phi| \) is bounded. In fact, \( |D^\alpha \phi(x)| \leq 1 \) for all \( 0 \leq |\alpha| \leq k \). Now set \( \phi_m = \phi(x/m) \). This is one choice of cut-off functions as introduced in Theorem 6.
Set $u_m := \phi_m(\rho_m * u)$ in $\mathbb{R}^n$, then

$$\|D^\alpha u_m - D^\alpha u\|_p \leq \sum_{\beta < \alpha} \|C_\alpha (D^{\alpha-\beta} \phi_m)(\rho_m * D^\beta u)\|_p$$

$$+ \sum_{\beta = \alpha} \|\phi_m(\rho_m * D^\alpha u) - D^\alpha u\|_p$$

$$\leq \sum_{\beta < \alpha} |C_\alpha| \|D^{\alpha-\beta} \phi_m\|_\infty \|\rho_m * D^\beta u\|_p$$

$$+ \sum_{\beta = \alpha} \|\phi_m(\rho_m * D^\alpha u - D^\alpha u)\|_p$$

$$+ \sum_{\beta = \alpha} \|\phi_mD^\alpha u - D^\alpha u\|_p$$

$$\leq \frac{M}{m^{\alpha-\beta}} \sum_{\beta < \alpha} |C_\alpha| \|D^\beta u\|_p$$

$$+ \sum_{\beta = \alpha} \left(\|\rho_m * D^\alpha u - D^\alpha u\|_p + \|\phi_mD^\alpha u - D^\alpha u\|_p\right).$$
The first term with $1/m$ goes to zero, the second term tends to zero by Theorem 5 and the last term goes to zero by Dominated convergence theorem. Since

$$
\|u_m - u\|_{k,p} = \sum_{|\alpha|=0}^{k} \|D^\alpha u_m - D^\alpha u\|_p
$$

$$
= \sum_{|\alpha|=0}^{k} \left\| \sum_{\beta \leq \alpha} C_\alpha (D^{\alpha-\beta} \phi_m)(\rho_m * D^\beta u) - D^\alpha u \right\|_p
$$

and each term in the sum goes to zero as $m$ increases, we have $u_m$ converges to $u$ in $W^{k,p}(\mathbb{R}^n)$ norm.
Theorem

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$. Then $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{k,p}_{loc}(\Omega)$, i.e., for each $u \in W^{k,p}(\Omega)$ there is a sequence $v_m \in C^\infty_c(\mathbb{R}^n)$ such that $\|v_m - u\|_{k,p,\omega}$, for all $k \geq 1$ and for all $\omega \subset \subset \Omega$ (relatively compact in $\Omega$).

Proof: Let $u \in W^{k,p}(\Omega)$. Fix $\omega$ relatively compact subset of $\Omega$ and a $\alpha$ such that $1 \leq |\alpha| \leq k$. Let $\tilde{u}$ denote the extension of $u$ by zero in $\Omega^c$. Choose a $\phi \in C^\infty_c(\Omega)$ such that $\phi \equiv 1$ on $\omega$ and $0 \leq \phi \leq 1$ in $\Omega$. Then $\text{supp}(\phi u) \subset \Omega$ and we extend $\phi u$ to all of $\mathbb{R}^n$ by zero in $\Omega^c$ and denote it as $\tilde{\phi}u$. Set $v = \tilde{\phi}u$ on $\mathbb{R}^n$. Note that $v \in W^{k,p}(\mathbb{R}^n)$. By Theorem 24, we have the sequence $v_m := \phi_m(\rho_m * v) \in C^\infty_c(\mathbb{R}^n)$ converging to $v$ in $\alpha$ Sobolev norm. Since on $\omega$, $v = u$, we have

$$\|v_m - u\|_{k,p,\omega} \leq \|v_m - v\|_{k,p,\mathbb{R}^n} \to 0.$$ 

This is true for all relatively compact subset $\omega$ of $\Omega$. 

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Note that the restriction ‘relatively compact set’ is only for \( k \geq 1 \).
For the case \( k = 0 \), the restriction to \( \Omega \) works, as seen in Theorem 6.

The density of \( C_c^\infty(\Omega) \) in \( W^{k,p}(\Omega) \) may fail to generalise for an arbitrary proper subset \( \Omega \subset \mathbb{R}^n \), because a “bad” derivative may be introduced at the boundary while extending by zero outside \( \Omega \).

**Example**

Let \( \Omega = (0, 1) \subset \mathbb{R} \) and \( u \equiv 1 \) on \( \Omega \). Then \( u \in W^{1,p}(0, 1) \). Setting \( \tilde{u} = 0 \) in \( \mathbb{R} \setminus (0, 1) \), we see that \( \tilde{u} \in L^p(\mathbb{R}) \) but not in \( W^{1,p}(\mathbb{R}) \). Because \( D\tilde{u} = \delta_0 - \delta_1 \) is not in \( L_{\text{loc}}^1(\mathbb{R}) \).

Recall the inclusion \( C_c^\infty(\Omega) \subset C^\infty(\overline{\Omega}) \subset C^\infty(\Omega) \). The \( C^\infty(\overline{\Omega}) \) denotes all functions in \( C^\infty(\Omega) \) such that all its derivatives can be extended continuously to \( \overline{\Omega} \).
For an arbitrary subset $\Omega \subseteq \mathbb{R}^n$ the best one can do is the following density result.

**Theorem (Meyers-Serrin)**

Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$. For $u \in W^{k,p}(\Omega)$ and any $\varepsilon > 0$, there is a $\phi \in C^\infty(\Omega)$ such that $\|\phi\|_{k,p,\Omega} < \infty$ and $\|u - \phi\|_{k,p,\Omega} < \varepsilon$.

**Proof:** For each $m \in \mathbb{N}$, consider the sets

$$\omega_m := \{x \in \Omega \mid |x| < m \text{ and } \text{dist}(x, \partial \Omega) > \frac{1}{m}\}$$

and set $\omega_0 = \emptyset$. Define the collection of open sets $\{U_m\}$ as $U_m := \omega_{m+1} \cap (\overline{\omega}_{m-1})^c$. Note that $\Omega = \bigcup_m U_m$ is an open covering of $\Omega$. Thus, we choose the $C^\infty$ locally finite partition of unity $\{\phi_m\} \subset C^\infty_c(\Omega)$ such that $\text{supp}(\phi_m) \subset U_m$, $0 \leq \phi_m \leq 1$ and $\sum_m \phi_m = 1$. For the given $u \in W^{k,p}(\Omega)$, note that $\phi_m u \in W^{k,p}(U_m)$ with $\text{supp}(\phi_m u) \subset U_m$. 

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Proof Continued...

We extend $\phi_m u$ to all of $\mathbb{R}^n$ by zero outside $U_m$, i.e.,

$$\tilde{\phi}_m u(x) = \begin{cases} 
\phi_m u(x) & x \in U_m \\
0 & x \in \mathbb{R}^n \setminus U_m.
\end{cases}$$

Observe that $\tilde{\phi}_m u \in W^{k,p}(\mathbb{R}^n)$. Let $\rho_\delta$ be the sequence of mollifiers and consider the sequence $\rho_\delta \ast \phi_m u$ in $C^\infty(\mathbb{R}^n)$. Support of $\rho_\delta \ast \phi_m u \subset U_m + B(0; \delta)$. Note that for all $x \in U_m$, $1/(m + 1) < \text{dist}(x, \partial \Omega) < 1/(m - 1)$. Thus, for all $0 < \delta < 1/(m + 1)(m + 2)$, (The choice of this range for $\delta$ is motivated from the fact that $\left(\frac{1}{m+1} - \frac{1}{(m+1)(m+2)}\right) = \frac{1}{(m+2)}$),

$$\text{supp}(\rho_\delta \ast \phi_m u) \subset \omega_{m+2} \cap (\bar{\omega}_m - 2)^c,$$

which is compactly contained in $\Omega$. Since $\phi_m u \in W^{k,p}(\Omega)$, we can choose a subsequence $\{\delta_m\}$ going to zero in $(0, 1/(m + 1)(m + 2))$ such that

$$\|\rho_{\delta_m} \ast \phi_m u - \phi_m u\|_{k,p,\Omega} = \|\rho_{\delta_m} \ast \tilde{\phi}_m u - \phi_m u\|_{k,p,\mathbb{R}^n} < \frac{\varepsilon}{2^m}.$$
Set \( \phi = \sum_{m} \rho \delta_m \ast \phi_m u \). Note that \( \phi \in C^\infty(\Omega) \) and \( \| \phi \|_{k,p,\Omega} < \infty \). Since every \( U_m \) intersects \( U_{m-1} \) and \( U_{m+1} \), we at most have three non-zero terms in the sum for each \( x \in U_m \), i.e.,

\[
\phi(x) = \sum_{i=-1}^{1} (\rho \delta_{m+i} \ast \phi_{m+i})(x) \quad x \in U_m.
\]

Therefore,

\[
\| u - \phi \|_{k,p,\Omega} = \| \sum_{m} (u \phi_m - \rho \delta_m \ast \phi_m u) \|_{k,p,\Omega} \\
\leq \sum_{m} \| \phi_m u - \rho \delta_m \ast \phi_m u \|_{k,p,\Omega} < \varepsilon.
\]

This proves the density of \( C^\infty(\Omega) \) in \( W^{k,p}(\Omega) \).
Let $C^{k,p}(\Omega)$ denote the closure of $E := \{ \phi \in C^k(\Omega) \mid \| \phi \|_{k,p,\Omega} < \infty \}$ w.r.t the Sobolev norm $\| \cdot \|_{k,p,\Omega}$. The space $C^{k,p}(\Omega)$ is a subspace of $W^{k,p}(\Omega)$ because classical derivative (continuous) and distributional derivative coincide (because integration by parts is valid). Since $W^{k,p}(\Omega)$ is a Banach space, we have $C^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$. In fact, a consequence of above result is that the Sobolev space $W^{k,p}(\Omega) = C^{k,p}(\Omega)$, a result due to Meyers and Serrin proved in 1964.

Corollary (Meyers-Serrin)

Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$, then $C^{k,p}(\Omega) = W^{k,p}(\Omega)$.

Proof.

It is enough to show that $E$ is dense in $W^{k,p}(\Omega)$ because, as a consequence, $W^{k,p}(\Omega) = \overline{E} = C^{k,p}(\Omega)$. For each given $\varepsilon > 0$ and $u \in W^{k,p}(\Omega)$, we need to show the existence of $\phi \in E$ such that $\| u - \phi \|_{k,p,\Omega} < \varepsilon$. By Theorem 26 there is a $\phi \in C^\infty(\Omega) \cap E$ such that $\| u - \phi \|_{k,p,\Omega} < \varepsilon$. Hence proved.
The density of $C^\infty(\Omega)$ is not true, in general, and fails for some “bad” domains as seen in examples below. This, in turn, means that $C^\infty_c(\Omega)$ cannot, in general, be dense in $W^{k,p}(\Omega)$.

**Example**

Let $\Omega := \{(x, y) \in \mathbb{R}^2 \mid 0 < |x| < 1 \text{ and } 0 < y < 1\}$. Consider the function $u : \Omega \to \mathbb{R}$ defined as

$$u(x, y) = \begin{cases} 
1 & x > 0 \\
0 & x < 0.
\end{cases}$$

Then for any fixed $\varepsilon > 0$ there exists no $\phi \in C^1(\Omega)$ such that $\|u - \phi\|_{1,p} < \varepsilon$.

**Example**

Let $\Omega := \{(r, \theta) \in \mathbb{R}^2 \mid 1 < r^2 < 2 \text{ and } \theta \neq 0\}$. Consider $u(r, \theta) = \theta$.

Then there exists no $\phi \in C^1(\Omega)$ such that $\|u - \phi\|_{1,1} < 2\pi$. 
The trouble with domains in above examples is that they lie on both sides of the boundary \( \partial \Omega \) which becomes the main handicap while trying to approximate \( W^{k,p}(\Omega) \) by \( C^\infty(\overline{\Omega}) \) functions.

**Definition**

A subset \( \Omega \subseteq \mathbb{R}^n \) is said to satisfy the segment property if for every \( x \in \partial \Omega \), there is a open set \( B_x \), containing \( x \), and a non-zero unit vector \( e_x \) such that \( z + te_x \in \Omega \), for all \( 0 < t < 1 \) whenever \( z \in \overline{\Omega} \cap B_x \).

Observe that a domain with segment property cannot lie on both sides of its boundary.

**Theorem**

Let \( 1 \leq p < \infty \) and \( \Omega \subseteq \mathbb{R}^n \) satisfy the segment property. Then, for every \( u \in W^{k,p}(\Omega) \) there exists a sequence \( \{v_m\} \subseteq C^\infty_c(\mathbb{R}^n) \) such that \( \|v_m - u\|_{k,p,\Omega} \to 0 \), for all \( k \geq 1 \), i.e., \( C^\infty_c(\mathbb{R}^n) \) is dense in \( W^{k,p}(\Omega) \).
Theorem 24 is a particular case of the above result, because $\mathbb{R}^n$ trivially satisfies the segment property as it has ‘no boundary’. A stronger version of above result will be proved later in the course (Theorem 28).

**Corollary**

Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$ satisfy the segment property. Then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$. 
In general, the closure of $C_c^\infty(\Omega)$ is a proper subspace of $W^{k,p}(\Omega)$ w.r.t the $\|\cdot\|_{k,p,\Omega}$.

**Definition**

Let $W_0^{k,p}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. For $p = 2$, we denote $W_0^{k,p}(\Omega)$ by $H^k_0(\Omega)$.

**Exercise**

Show that $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$.

In general, $W_0^{k,p}(\Omega)$ is a strict subspace of $W^{k,p}(\Omega)$. However, as observed in Theorem 24, for $\Omega = \mathbb{R}^n$, $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ for all $1 \leq p < \infty$ and $k \geq 0$. In fact, $W^{k,p}(\Omega) = W_0^{k,p}(\Omega)$ iff $\text{cap}_p(\mathbb{R}^n \setminus \Omega) = 0^1$.

---

$^1(k,p)$ polar sets
The density results discussed in this section is not true for $p = \infty$ case as seen from the example below.

**Example**

Let $\Omega = (-1, 1)$ and

$$u(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

Then its distributional derivative is $u'(x) = 1_{(0, \infty)}$. Let $\phi \in C^\infty(\Omega)$ such that $\|\phi' - u'\|_\infty < \varepsilon$. Thus, if $x < 0$, $|\phi'(x)| < \varepsilon$ and if $x > 0$, $|\phi'(x) - 1| < \varepsilon$. In particular, $\phi'(x) > 1 - \varepsilon$. By continuity, $\phi'(0) < \varepsilon$ and $\phi'(0) > 1 - \varepsilon$ which is impossible if $\varepsilon < 1/2$. Hence, $u$ cannot be approximated by smooth functions in $W^{1,\infty}(\Omega)$ norm.
In general, many properties of $W^{k,p}(\Omega)$ can be inherited from $W^{k,p}(\mathbb{R}^n)$ provided the domain is “nice”. We classify such classes of “nice” domain.

**Definition**

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We say $P$ is an $(k, p)$-extension operator for $\Omega$, if $P : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ is a bounded linear operator, i.e., there is a constant $C > 0$ (depending on $\Omega$, $k$ and $p$) such that

$$\|Pu\|_{k,p,\mathbb{R}^n} \leq C\|u\|_{k,p,\Omega} \quad \forall u \in W^{k,p}(\Omega)$$

and $Pu|_\Omega = u$ a.e. for every $u \in W^{k,p}(\Omega)$. If $P$ is same for all $1 \leq p < \infty$ and $0 \leq m \leq k$, then $P$ is called strong $k$-extension operator. If $P$ is a strong $k$-extension operator for all $k$ then $P$ is called total extension operator.
Example

There is a natural extension operator $P : W_0^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ which is the extension by zero. Define

$$Pu := \tilde{u} = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Obviously, $\tilde{u} |_{\Omega} = u$ and $\|\tilde{u}\|_{0,p,\mathbb{R}^n} = \|u\|_{0,p,\Omega}$. We shall show that $\|\tilde{u}\|_{k,p,\mathbb{R}^n} = \|u\|_{k,p,\Omega}$. Since $u \in W_0^{k,p}(\Omega)$, there is a sequence $\{\phi_m\} \subset C^\infty_c(\Omega)$ converging to $u$ in $\| \cdot \|_{k,p,\Omega}$. For any $\phi \in C^\infty_c(\mathbb{R}^n)$ and $|\alpha| \leq k$, we have

$$D^\alpha \tilde{u}(\phi) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \tilde{u} D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \phi \, dx$$

$$= (-1)^{|\alpha|} \lim_{m \to \infty} \int_{\Omega} \phi_m D^\alpha \phi \, dx = \lim_{m \to \infty} \int_{\Omega} D^\alpha \phi_m \phi \, dx$$

$$= \int_{\Omega} D^\alpha u \phi \, dx = \tilde{D}^\alpha u(\phi).$$
Example Continued...

Thus, $D^\alpha \tilde{u} = \widehat{D^\alpha u}$ and therefore

$$\| \tilde{u} \|_{k,p,\mathbb{R}^n} = \sum_{|\alpha|=0}^{k} \| D^\alpha \tilde{u} \|_{p,\mathbb{R}^n} = \sum_{|\alpha|=0}^{k} \| \widehat{D^\alpha u} \|_{p,\mathbb{R}^n}$$

$$= \sum_{|\alpha|=0}^{k} \| D^\alpha u \|_{p,\Omega} = \| u \|_{k,p,\Omega}.$$
Theorem

Let $1 \leq p < \infty$. If $\Omega$ is an open subset of $\mathbb{R}^n$ such that there is a $(k, p)$-extension operator $P : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$, then $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{k,p}(\Omega)$. In particular, $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ w.r.t $\| \cdot \|_{k,p,\Omega}$.

Proof.

For each $u \in W^{k,p}(\Omega)$, we choose the sequence $\phi_m(\rho_m * Pu)$ in $C^\infty_c(\mathbb{R}^n)$ which converges to $Pu$ in $W^{k,p}(\mathbb{R}^n)$ and their restriction to $\Omega$ is in $C^\infty(\overline{\Omega})$ and converges in $W^{k,p}(\Omega)$. 

□
Above result is a particular case of Theorem 27. A natural question provoked by Theorem 28 is: For what classes of open sets $\Omega$ can one expect an extension operator $P$.

**Definition**

For an open set $\Omega \subset \mathbb{R}^n$ we say that its boundary $\partial \Omega$ is $C^k$ ($k \geq 1$), if for every point $x \in \partial \Omega$, there is a $r > 0$ and a $C^k$ diffeomorphism $\gamma : B_r(x) \to B_1(0)$ (i.e. $\gamma^{-1}$ exists and both $\gamma$ and $\gamma^{-1}$ are $k$-times continuously differentiable) such that

1. $\gamma(\partial \Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n | x_n = 0\}$ and
2. $\gamma(\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n | x_n > 0\}$

We say $\partial \Omega$ is $C^\infty$ if $\partial \Omega$ is $C^k$ for all $k = 1, 2, \ldots$ and $\partial \Omega$ is analytic if $\gamma$ is analytic.
Equivalently, a workable definition of $C^k$ boundary would be the following: if for every point $x \in \partial \Omega$, there exists a neighbourhood $B_x$ of $x$ and a $C^k$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$
\Omega \cap B_x = \{ x \in B_x \mid x_n > \gamma(x_1, x_2, \ldots, x_{n-1}) \}.
$$

To keep the illustration simple, we shall restrict ourselves to $k = 1$. We begin by constructing an extension operator for the half-space and then use it along with partition of unity to construct an extension operator for $C^1$ boundary domains.

**Theorem**

Let $\mathbb{R}_+^n := \{ x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0 \}$ where $x' = (x_1, x_2, \ldots, x_{n-1})$. Given $u \in W^{1,p}(\mathbb{R}_+^n)$, we define the extension to $\mathbb{R}^n$ as

$$
Pu = u^*(x) := \begin{cases} 
u(x', x_n) & x_n > 0 \\ u(x', -x_n) & x_n < 0. \end{cases}
$$

Then $\|u^*\|_{1,p,\mathbb{R}^n} \leq 2\|u\|_{1,p,\mathbb{R}_+^n}$ and $u^* \in W^{1,p}(\mathbb{R}^n)$. 

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Proof

Observe that \( u^* \in L^p(\mathbb{R}^n) \) because

\[
\| u^* \|_{p, \mathbb{R}^n}^p = \int_{\mathbb{R}^n_+} |u(x', x_n)|^p \, dx + \int_{\mathbb{R}^n_-} |u(x', -x_n)|^p \, dx \\
= \int_{\mathbb{R}^n_+} |u(x', x_n)|^p \, dx + \int_{\mathbb{R}^{n-1}} \int_0^\infty |u(x', y_n)|^p \, dx' \, dy_n \\
= 2 \int_{\mathbb{R}^n_+} |u(x)|^p \, dx = 2 \| u \|_{p, \mathbb{R}^n_+}^p.
\]

Thus, \( \| u^* \|_{p, \mathbb{R}^n} = 2^{1/p} \| u \|_{p, \mathbb{R}^n_+}. \) We now show that for \( \alpha = e_i, 1 \leq i \leq n - 1, \) \( D^\alpha u^* = (D^\alpha u)^*. \)
Consider, for $\phi \in D(\mathbb{R}^n)$,

$$D^\alpha u^*(\phi) = -\int_{\mathbb{R}^n} u^* D^\alpha \phi \, dx$$

$$= -\int_{\mathbb{R}^n_+} u(x', x_n) D^\alpha \phi \, dx - \int_{\mathbb{R}^n_-} u(x', -x_n) D^\alpha \phi \, dx$$

$$= -\int_{\mathbb{R}^n_+} u(x', x_n) D^\alpha \phi(x', x_n) \, dx$$

$$- \int_{\mathbb{R}^n_+} u(x', x_n) D^\alpha \phi(x', -x_n) \, dx.$$

Hence,

$$D^\alpha u^*(\phi) = -\int_{\mathbb{R}^n_+} u D^\alpha \psi(x) \, dx, \quad (10.1)$$

where $\psi(x', x_n) = \phi(x', x_n) + \phi(x', -x_n)$ when $x_n > 0$. In general, $\psi \notin D(\mathbb{R}^n_+)$. Thus, we shall multiply $\psi$ by a suitable cut-off function so that the product is in $D(\mathbb{R}_+^n)$.
Choose a $\{\zeta_m\} \in C^\infty(\mathbb{R})$ such that

$$
\zeta_m(t) = \begin{cases} 
0 & \text{if } t < 1/2m \\
1 & \text{if } t > 1/m,
\end{cases}
$$

then $\zeta_m(x_n)\psi(x) \in D(\mathbb{R}_n^+)$. Since $\zeta_m$ is independent of $x_i$, for $1 \leq i \leq n - 1$, we have

$$
D^\alpha u(\zeta_m \psi) = -\int_{\mathbb{R}_n^+} u(x)D^\alpha (\zeta_m(x_n)\psi(x)) \, dx
$$

$$
= -\int_{\mathbb{R}_n^+} u(x)\zeta_m(x_n)D^\alpha \psi(x) \, dx.
$$

Passing to limit, as $m \to \infty$ both sides, we get

$$
\int_{\mathbb{R}_n^+} D^\alpha u(x)\psi(x) \, dx = -\int_{\mathbb{R}_n^+} u(x)D^\alpha \psi(x) \, dx.
$$

The RHS in above equation is same as the RHS obtained in (10.1). Hence, we have $D^\alpha u^*(\phi) = \int_{\mathbb{R}_n^+} D^\alpha u(x)\psi(x) \, dx$. 

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By setting

\[
(D^\alpha u)^*(x', x_n) := \begin{cases} 
D^\alpha u(x', x_n) & x_n > 0 \\
D^\alpha u(x', -x_n) & x_n < 0 
\end{cases}
\]

we get \(D^\alpha u^* = (D^\alpha u)^*\). We now show a similar result for \(\alpha = e_n\).

Consider, for \(\phi \in \mathcal{D}(\mathbb{R}^n)\),

\[
D^{en} u^*(\phi) = - \int_{\mathbb{R}^n} u^* D^{en} \phi \, dx \\
= - \int_{\mathbb{R}^n} u(x', -x_n) D^{en} \phi \, dx - \int_{\mathbb{R}^n} u(x', x_n) D^{en} \phi \, dx \\
= \int_{\mathbb{R}^n} u(x', x_n) D^{en} \phi(x', -x_n) \, dx \\
- \int_{\mathbb{R}^n} u(x', x_n) D^{en} \phi(x', x_n) \, dx.
\]
Hence,
\[ D^e u^*(\phi) = - \int_{\mathbb{R}^n_+} u D^e \psi(x) \, dx, \]  
where \( \psi(x', x_n) = \phi(x', x_n) - \phi(x', -x_n) \) when \( x_n > 0 \). Note that \( \psi(x', 0) = 0 \). Thus, by Mean value theorem, \(|\psi(x', x_n)| \leq C|x_n|\). As before, in general, \( \psi \notin \mathcal{D}(\mathbb{R}^n_+) \), so \( \zeta_m(x_n)\psi(x) \in \mathcal{D}(\mathbb{R}^n_+) \). One such choice of \( \zeta_m \) is by choosing \( \zeta_m(t) = \zeta(mt) \) where
\[
\zeta(t) = \begin{cases} 
0 & \text{if } t < \frac{1}{2} \\
1 & \text{if } t > 1.
\end{cases}
\]
Consider,
\[
D^e u(\zeta_m \psi) = - \int_{\mathbb{R}^n_+} u(x) D^e (\zeta_m(x_n) \psi(x)) \, dx
= - \int_{\mathbb{R}^n_+} u(x) \zeta'_m(x_n) \psi(x) \, dx - \int_{\mathbb{R}^n_+} u(x) \zeta_m(x_n) D^e \psi(x) \, dx.
\]
Passing to limit, as $m \to \infty$ both sides, we get
\[
\int_{\mathbb{R}^n_+} D^e u(x) \psi(x) \, dx = - \lim_{m \to \infty} \int_{\mathbb{R}^n_+} u(x) \zeta'_m(x_n) \psi(x) \, dx
\]
\[- \int_{\mathbb{R}^n_+} u(x) D^e \psi(x) \, dx.
\]

Let us handle the first term in RHS. Note that
\[
\left| \int_{\mathbb{R}^n_+} u(x) \zeta'_m(x_n) \psi(x) \, dx \right| = m \left| \int_{\mathbb{R}^n_+} u(x) \zeta'(mx_n) \psi(x) \, dx \right|
\]
\[\leq mC \int_{\mathbb{R}^n_+} |u(x)||\zeta'| |x_n| \, dx
\]
\[\leq C \|\zeta\|_{\infty,[0,1]} \int_{\mathbb{R}^{n-1}} \int_{0<x_n<1/m} |u(x)| \, dx.
\]

Therefore, $\lim_{m \to \infty} \int_{\mathbb{R}^n_+} u(x) \zeta'_m(x_n) \psi(x) \, dx = 0$ and hence
\[
\int_{\mathbb{R}^n_+} D^e u(x) \psi(x) \, dx = - \int_{\mathbb{R}^n_+} u(x) D^e \psi(x) \, dx.
\]
The RHS in above equation is same as the RHS obtained in (10.2). Hence, we have \( D^e_n u^*(\phi) = \int_{\mathbb{R}_+^n} D^e_n u(x) \psi(x) \, dx \). By setting

\[
(D^e_n u)^\#(x', x_n) := \begin{cases} 
D^e_n u(x', x_n) & x_n > 0 \\
-D^e_n u(x', -x_n) & x_n < 0,
\end{cases}
\]

we get \( D^e_n u^* = (D^e_n u)^\# \).

Therefore, we have the estimate,

\[
\| u^* \|_{1,p, \mathbb{R}^n} = \| u^* \|_{p, \mathbb{R}^n} + \sum_{|\alpha| = 1} \| D^\alpha u^* \|_{p, \mathbb{R}^n} \\
= 2^{1/p} \| u \|_{p, \mathbb{R}_+^n} + \sum_{i=1}^{n-1} \| (D^e_i u)^* \|_{p, \mathbb{R}^n} + \| (D^e_n u)^\# \|_{p, \mathbb{R}^n} \\
= 2^{1/p} \| u \|_{1,p, \mathbb{R}_+^n}.
\]

Hence, \( u^* \in W^{1,p}(\mathbb{R}^n) \).
Theorem

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^1$ boundary. Then there is an extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.

Corollary

For $1 \leq p < \infty$ and $\Omega$ be bounded open set with $C^1$ boundary, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$, for all $k \geq 1$. 
Topological Dual of Sobolev Spaces

We shall now introduce the topological dual of Sobolev spaces and motivate its notation. For any $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$ and $\Omega$ be an open subset of $\mathbb{R}^n$. Let $X_{k,p}(\Omega)$ be the topological dual of $W_{0}^{k,p}(\Omega)$. Thus, if $F \in X_{k,p}(\Omega)$ then $F$ is a continuous linear functional on $W_{0}^{k,p}(\Omega)$ and the norm of $F$ is given as,

$$
\|F\|_{X_{k,p}(\Omega)} := \sup_{\substack{u \in W_{0}^{k,p}(\Omega) \\
u \neq 0}} \frac{|F(u)|}{\|u\|_{k,p}}.
$$

Note that the dual space is considered for $W_{0}^{k,p}(\Omega)$ and not for $W^{k,p}(\Omega)$. The reason is that $\mathcal{D}(\Omega)$ is dense in $W_{0}^{k,p}(\Omega)$ and hence $W_{0}^{k,p}(\Omega)$ will have a unique continuous extension (by Hahn-Banach) for any continuous linear functional defined on $\mathcal{D}(\Omega)$.
Example

In general the dual of $W^{k,p}(\Omega)$ may not even be a distribution. Note that, in general, $\mathcal{D}(\Omega)$ is not dense in $W^{k,p}(\Omega)$. Thus, its dual $[W^{k,p}(\Omega)]^*$ is not in the space of distributions $\mathcal{D}'(\Omega)$. Of course, the restriction to $\mathcal{D}(\Omega)$ of every $T \in [W^{k,p}(\Omega)]^*$ is a distribution but this restriction may not identify with $T$. For instance, consider $f \in [L^2(\Omega)]^n$ with $|f| \geq c > 0$ a.e. and $\text{div}(f) = 0$. Define

$$T(\phi) := \int_{\Omega} f \cdot \nabla \phi \, dx.$$ 

Since $|T(\phi)| \leq \|f\|_2 \|\nabla \phi\|_2$, we infer that $T \in [H^1(\Omega)]^*$. However, the restriction of $T$ to $\mathcal{D}(\Omega)$ is the zero operator of $\mathcal{D}'(\Omega)$ because

$$\langle T, \phi \rangle = -\langle \text{div}(f), \phi \rangle = 0 \quad \forall \phi \in \mathcal{D}(\Omega).$$
Theorem (Characterisation of $X_{1,p}(\Omega)$)

Let $1 \leq p < \infty$ and let $F \in X_{1,p}(\Omega)$. Then there exist functions $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ such that

$$F(u) = \int_{\Omega} f_0 u \, dx + \sum_{i=1}^n \int_{\Omega} f_i \frac{\partial u}{\partial x_i} \quad \forall u \in W_0^{1,p}(\Omega)$$

and $\|F\|_{X_{1,p}(\Omega)} = \max_{0 \leq i \leq n} \|f_i\|_q$. Further, if $\Omega$ is bounded, one may assume $f_0 = 0$.

Proof: Recall that (cf. proof of Theorem 23) $\mathcal{T} : W_0^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ defined as

$$\mathcal{T}(u) = \left( u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$$

is an isometry. The norm in $(L^p(\Omega))^{n+1}$ is defined as

$$\|u\| := \left( \sum_{i=1}^{n+1} \|u_i\|_p^p \right)^{1/p}, \text{ where } u = (u_1, u_2, \ldots, u_{n+1}).$$
Let $E = \mathcal{I}(W_0^{1,p}(\Omega)) \subset (L^p(\Omega))^{n+1}$. Observe that $F \circ \mathcal{I}^{-1}$ is a continuous linear functional on $E$. By Hahn Banach theorem, there is a continuous extension $S$ of $F \circ \mathcal{I}^{-1}$ to all of $(L^p(\Omega))^{n+1}$. Now, by Riesz representation theorem, there exist $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ such that

$$S(\nu) = \int_{\Omega} f_0 \nu_0 \, dx + \sum_{i=1}^{n} \int_{\Omega} f_i \nu_i \, dx \quad \forall \nu = (\nu_0, \nu_1, \ldots, \nu_n) \in (L^p(\Omega))^{n+1}$$

and $\|S\| = \|F \circ \mathcal{I}^{-1}\|$. Now, for any $u \in W_0^{1,p}(\Omega)$,

$$F(u) = F \circ \mathcal{I}^{-1} \left( u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$$

$$= S \left( u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$$

$$= \int_{\Omega} f_0 u \, dx + \sum_{i=1}^{n} \int_{\Omega} f_i \frac{\partial u}{\partial x_i} \, dx.$$
Also, $\|F\| = \|F \circ \mathcal{T}^{-1}\|$, by isometry of $\mathcal{T}$ and hence $\|F\| = \|S\| = \max_{0 \leq i \leq n} \|f_i\|^q$.

Further, if $\Omega$ is bounded we have by Poincaré inequality that the seminorm $\|\nabla u\|_p$ becomes a norm in $W^{1,p}_0(\Omega)$, thus by using the gradient map as an isometry from $W^{1,p}_0(\Omega)$ to $(L^p(\Omega))^n$ and arguing as above, we see that $f_0$ can be chosen to be zero.\qed
Remark

Let $F \in X_{1,p}(\Omega)$ and $\phi \in D(\Omega)$, then

$$F(\phi) = \int_{\Omega} f_0 \phi \, dx + \sum_{i=1}^{n} \int_{\Omega} f_i \frac{\partial \phi}{\partial x_i} \, dx$$

$$= \int_{\Omega} f_0 \phi \, dx - \sum_{i=1}^{n} \int_{\Omega} \phi \frac{\partial f_i}{\partial x_i} \, dx.$$ 

Now, since $D(\Omega)$ is dense in $W_{0}^{1,p}(\Omega)$, the extension of $F$ to $W_{0}^{1,p}(\Omega)$ should be unique. Henceforth, we shall identify any element $F \in X_{1,p}(\Omega)$ with the distribution

$$f_0 - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$
The above remark motivates the right notation for the space $X_{k,p}(\Omega)$. Observe that if $u \in W^{k,p}(\Omega)$, then the first derivatives of $\frac{\partial u}{\partial x_i}$, for all $i$, are in $W^{k-1,p}$. To carry forward this feature in our notation, the above remark motivates to rewrite $X_{k,p}(\Omega)$ as $W^{-k,q}(\Omega)$, where $q$ is the conjugate exponent corresponding to $p$.

Let us observe that the representation of $F$ in terms $f_i$ is not unique. Let $g \in W^{1,q}(\Omega)$ such that $\Delta g = 0$, i.e., $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial g}{\partial x_i} \right) = 0$. Then

$$F = f_0 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( f_i + \frac{\partial g}{\partial x_i} \right)$$

is also a representation of $F$.

**Exercise**

Show that

$$\|F\|_{W^{-1,q}(\Omega)} = \inf_{F=f_0-\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}} \left\{ \left( \sum_{i=0}^{n} \|f_i\|_q \right)^{1/q} \right\}.$$
Let $1 \leq p < \infty$ and $0 < \sigma < 1$. The Sobolev spaces $W^{\sigma,p}(\Omega)$, for non-integral $\sigma$, is defined as:

$$W^{\sigma,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \frac{|u(x) - u(y)|}{|x - y|^\sigma \left(\frac{n}{p}\right)} \in L^p(\Omega \times \Omega) \right\}$$

with the obvious norm. For any positive real number $s$, set $k := \lfloor s \rfloor$, integral part and $\sigma := s$ is the fractional part. Note that $0 < \sigma < 1$.

$$W^{s,p}(\Omega) = \{ u \in W^{k,p}(\Omega) \mid D^\alpha u \in W^{\sigma,p}(\Omega) \quad \text{for all } |\alpha| = k \}. \quad (10.4)$$

We denote by $W^{s,p}_0(\Omega)$, the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$ is dual of $W^{s,p}_0(\Omega)$. 
We begin by giving a characterisation of the space $H^1(\mathbb{R}^n)$ in terms of Fourier transform. Recall that for any $u \in L^1(\mathbb{R}^n)$ its Fourier transform $\hat{u}$ is defined as
\[
\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi \xi \cdot x} u(x) \, dx.
\] (10.5)

Recall that $\hat{u} \in C_0(\mathbb{R}^n)$ and if $\hat{u} \in L^1(\mathbb{R}^n)$, one can invert the Fourier transform to obtain $u$ from $\hat{u}$ by the following formula:
\[
u(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \hat{u}(\xi) \, d\xi.
\] (10.6)

In particular, by Fourier transform, if $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{u} \in L^2(\mathbb{R}^n)$ and
\[
\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.
\] (10.7)
Theorem

The Sobolev space

\[ H^1(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) \mid (1 + |\xi|^2)^{\frac{1}{2}} \hat{u} \in L^2(\mathbb{R}^n) \}. \]

Further, the \( H^1 \) is given as

\[ \| u \|_{H^1(\mathbb{R}^n)} = \| (1 + |\xi|^2)^{\frac{1}{2}} \hat{u} \|_{2,\mathbb{R}^n}. \]

Proof: Consider \( u \in \mathcal{D}(\mathbb{R}^n) \). By definition,

\[ \frac{\hat{\partial u}}{\partial x_k}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi \xi \cdot x} \frac{\partial u}{\partial x_k}(x) \, dx. \]

Using integration by parts and the fact that \( u \) has compact support, we obtain

\[ \frac{\hat{\partial u}}{\partial x_k}(\xi) = 2\pi i \xi_k \hat{u}(\xi). \] (10.8)
By the density of $\mathcal{D}(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$ (cf. Theorem 24), for any $u \in H^1(\mathbb{R}^n)$, there exists a sequence $\{u_m\}_{m \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ such that $u_m \to u$ in the norm topology of $H^1(\mathbb{R}^n)$. So, for each $m \in \mathbb{N}$,

$$
\hat{\partial u_m / \partial x_k}(\xi) = 2\pi i \xi_k \hat{u}_m(\xi).
$$

(10.9)

Since $u_m \to u$ in the $H^1(\mathbb{R}^n)$ norm, $u_m \to u$ in $L^2(\mathbb{R}^n)$ and $\partial u_m / \partial x_k \to \partial u / \partial x_k$ in $L^2(\mathbb{R}^n)$, we can extract a subsequence $u_{m\ell}$ such that

$$
\hat{u}_{m\ell}(\xi) \to \hat{u}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n
$$

and

$$
\hat{\partial u_{m\ell} / \partial x_k}(\xi) \to \hat{\partial u / \partial x_k}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n.
$$

By the continuity of Fourier transform, we obtain

$$
\hat{\partial u / \partial x_k}(\xi) = 2\pi i \xi_k \hat{u}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n.
$$

(10.10)

Now, if $u \in H^1(\mathbb{R}^n)$ then $\partial u / \partial x_k \in L^2(\mathbb{R}^n)$, for all $k = 1, 2, \ldots, n$. 

Thus, the Fourier transform of $\frac{\partial u}{\partial x_k}$ is well-defined and

$$\hat{\frac{\partial u}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{u}(\xi).$$

Hence, $\xi_k \hat{u}(\xi) \in L^2(\mathbb{R}^n)$, for all $k = 1, 2, \ldots, n$. Conversely, if $u \in L^2(\mathbb{R}^n)$ such that $\xi_k \hat{u}(\xi) \in L^2(\mathbb{R}^n)$, for all $k = 1, 2, \ldots, n$, then $\frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n)$ for all $k = 1, 2, \ldots, n$ and, hence, $u \in H^1(\mathbb{R}^n)$. Therefore, $u \in H^1(\mathbb{R}^n)$ if and only if $\hat{u} \in L^2(\mathbb{R}^n)$ and $\xi_k \hat{u} \in L^2(\mathbb{R}^n)$, for all $k = 1, 2, \ldots, n$. This is equivalent to saying that $(1 + |\xi|^2)^{\frac{1}{2}} \hat{u} \in L^2(\mathbb{R}^n)$ (cf. Lemma 14). Further,

$$\| u \|_{H^1(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left( u^2(x) + \sum_{k=1}^n \left| \frac{\partial u}{\partial x_k}(x) \right|^2 \right) \, dx$$

$$= \int_{\mathbb{R}^n} \left( |\hat{u}(\xi)|^2 + \sum_{k=1}^n \left| \hat{\frac{\partial u}{\partial x_k}}(\xi) \right|^2 \right) \, d\xi$$

$$= \int_{\mathbb{R}^n} (1 + |\xi|^2) |\hat{u}(\xi)|^2 \, d\xi = \|(1 + |\xi|^2)^{\frac{1}{2}} \hat{u}\|_{L^2(\mathbb{R}^n)}^2.$$
Lemma

There exist positive constants $C_1$ and $C_2$ depending only on $k$ and $n$ such that

$$C_1(1 + |\xi|^2)^k \leq \sum_{|\alpha| \leq k} |\xi^{\alpha}|^2 \leq C_2(1 + |\xi|^2)^k \quad \forall \xi \in \mathbb{R}^n.$$ 

Proof.

Note that $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$ and $|\xi^{\alpha}| = |\xi_1|^{\alpha_1} \ldots |\xi_n|^{\alpha_n}$. By induction argument on $k$, we can see that same powers of $\xi$ occur in $(1 + |\xi|^2)^k$ and $\sum_{|\alpha| \leq k} |\xi^{\alpha}|^2$, albeit with different coefficients, which depend only on $n$ and $k$. Since the number of terms is finite and depends again only on $n$ and $k$, the result follows.

Owing to the above lemma one can define the space $H^k(\mathbb{R}^n)$ as follows:

$$H^k(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) | (1 + |\xi|^2)^{\frac{k}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \}$$

and $\| u \|_{H^k(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$
A major interest of the above approach is that it suggests a natural definition of the space \( H^s(\mathbb{R}^n) \), for \( s \in \mathbb{R} \). The central point being that the integral \( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \) is finite, for any \( s \in \mathbb{R} \), since \( a^s \) makes sense for any \( s \in \mathbb{R} \) when \( a > 0 \). In this case, \( a = 1 + |\xi|^2 \) is positive.

**Definition**

Let \( s \geq 0 \) be non-negative real number. We define

\[
H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) \mid (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n) \},
\]

equipped with the scalar product, for any \( u, v \in H^s(\mathbb{R}^n) \)

\[
\langle u, v \rangle_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}}(\xi) \, d\xi
\]

and the corresponding norm,

\[
\| u \|_{H^s(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
\]
Theorem

For any \( s \in [0, \infty) \), \( H^s(\mathbb{R}^n) \) is a Hilbert space. If \( s = k \in \mathbb{N} \) then \( H^s(\mathbb{R}^n) = H^k(\mathbb{R}^n) = W^{k,2}(\Omega) \) is the classical Sobolev space.

Proof.

Recall that the Fourier transform is an isomorphism from \( H^s(\mathbb{R}^n) \) onto Lebesgue space \( L^2_\mu(\mathbb{R}^n) \) where \( \mu := (1 + |\xi|^2)^s \, dx \) is the measure with density \( (1 + |\xi|^2)^s \) w.r.t the Lebesgue measure \( dx \) on \( \mathbb{R}^n \). Moreover, the Fourier transform is an isometry and the Hilbert structure of the weighted Lebesgue space \( L^2_\mu(\mathbb{R}^n) \) is passed on to \( H^s(\mathbb{R}^n) \). Hence, \( H^s(\mathbb{R}^n) \) is isomorphic to \( L^2_\mu(\mathbb{R}^n) \).
For $s > 0$, we define $H^{-s}(\mathbb{R}^n)$ as the dual of $H^s(\mathbb{R}^n)$. The negative order Sobolev spaces has the following characterization:

**Theorem**

Let $s \in (0, \infty)$. Then

$$H^{-s}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid (1 + |\xi|^2)^{-\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \}$$  \hspace{1cm} (10.11)

**Proof:** We shall give the proof for $s = 1$. If $u \in H^{-1}(\mathbb{R}^n)$ then

$$u = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \quad f_0, f_1 \ldots, f_n \in L^2(\mathbb{R}^n).$$

Hence, $u$ is a tempered distribution and

$$\hat{u} = \hat{f}_0 + \sum_{i=1}^{n} (2\pi i)^i \xi_i \hat{f}_i.$$
Then \((1 + |\xi|^2)^{-\frac{1}{2}} \hat{u} \in L^2(\mathbb{R}^n)\), proving one inclusion in (10.11). To prove the reverse inclusion, consider \(u \in S'(\mathbb{R}^n)\) such that
\[(1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n).\]
Let \(\phi \in \mathcal{D}(\mathbb{R}^n)\). Then there exists \(\psi \in S(\mathbb{R}^n)\) such that \(\phi = \psi\). Set
\[k(\xi) := (1 + |\xi|^2)^{\frac{1}{2}} \quad \text{and} \quad k_-(\xi) := (1 + |\xi|^2)^{-\frac{1}{2}}.\]
Note that both \(k\) and \(k_-\) are in \(C^\infty(\mathbb{R}^n)\), we write
\[u(\phi) = u(\hat{\psi}) = \hat{u}(\psi) = (kk_-)\hat{u}(\psi) = k_- \hat{u}(k\psi).\]
But \(k_- \hat{u} \in L^2(\mathbb{R}^n)\) and, hence,
\[u(\phi) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi)(1 + |\xi|^2)^{\frac{1}{2}} \psi(\xi) \, d\xi.\]
Therefore,
\[|u(\phi)| \leq |(1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi)|_{0, \mathbb{R}^n} |(1 + |\xi|^2)^{\frac{1}{2}} \psi(\xi)|_{0, \mathbb{R}^n}.\]
But

\[
\left|(1 + |\xi|^2)^{\frac{1}{2}} \psi(\xi)\right|_{0, \mathbb{R}^n}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2) \psi^2(\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} (1 + |\xi|^2) \psi^2(-\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} (1 + |\xi|^2) \hat{\psi}^2(\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} (1 + |\xi|^2)(\hat{\phi}(\xi))^2 \, d\xi = \|\phi\|^2_{H^1(\mathbb{R}^n)}.
\]

Thus \( u \) defines a continuous linear functional on \( H^1(\mathbb{R}^n) \) and so \( u \in H^{-1}(\mathbb{R}^n) \). Also,

\[
\|u\|_{H^{-1}(\mathbb{R}^n)} = |(1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi)|_{0, \mathbb{R}^n}.
\]
For an open subset \( \Omega \) of \( \mathbb{R}^n \), we may define the Sobolev spaces \( H^s(\Omega) \), for real \( s \), as the restrictions to \( \Omega \) of elements of \( H^s(\mathbb{R}^n) \).

**Example**

If \( \delta_0 \) is the Dirac distribution, we know that \( \hat{\delta}_0 \equiv 1 \) and, hence, \( \delta_0 \in H^{-s}(\mathbb{R}^n) \) if and only if \( (1 + |\xi|^2)^{-s/2} \in L^2(\mathbb{R}^n) \). This is true for \( s > \frac{n}{2} \) since the integral in polar coordinates is

\[
\int_0^\infty \frac{r^{n-1}}{(1 + r^2)^s} \, dr
\]

is finite only when \( s > \frac{n}{2} \).
Let us make a remark on the intuitive importance of Sobolev norms. Recall that the uniform norm or essential supremum on the space of bounded continuous functions $C_b(\Omega)$. One can, intuitively, think of essential supremum norm as capturing the “height” of the function.

In a similar sense, the $L^p$ norms, for $p < \infty$, capture the “height” and “width” of a function. In mathematical terms, “width” is same as the measure of the support of the function.

The Sobolev norms captures “height”, “width” and “oscillations”. The Fourier transform measures oscillation (or frequency or wavelength) by the decay of the Fourier transform, i.e., the “oscillation” of a function is translated to “decay” of the its Fourier transform. Sobolev norms measures “oscillation” via its derivatives (or regularity).

Thus, Sobolev imbedding results are precisely statements about functions after incorporating its “oscillation” information.
We shall restrict ourselves to $W^{1,p}$, for all $1 \leq p \leq \infty$, to make the presentation clear and later state the results for the spaces $W^{k,p}$, $k \geq 2$. Recall that the $L^p$ spaces are actually equivalence classes of functions with equivalence relation being “equality almost everywhere”. This motivates the following definition.

**Definition**

For any $u \in L^p$ ($1 \leq p \leq \infty$), we say $u^*$ is a representative\(^a\) of $u$ if $u = u^*$ a.e.

\(^a\)This is not a standard usage in literature and is introduced by the author of this manuscript for convenience sake
Theorem (One dimensional case)

Let \((a, b) \subseteq \mathbb{R}\) be an open interval and \(1 \leq p \leq \infty\). If \(u \in W^{1,p}(a, b)\) then there is a representative of \(u\), \(u^*\), which is absolutely continuous (is in \(AC(a, b)\)).

Proof: Since \(u \in W^{1,p}(a, b)\), \(u\) is weakly differentiable and \(u' \in L^p(a, b)\). For each \(x \in (a, b)\), we define \(v : (a, b) \rightarrow \mathbb{R}\) as

\[
v(x) := \int_a^x u'(t) \, dt.
\]

By definition, \(v \in BV(a, b)\) and is differentiable a.e. Hence \(v' = u'\) and \((v - u)' = 0\) a.e. Thus, \(u = v - c\) a.e., where \(c\) is some constant. Set \(u^* := v - c\). We claim that \(u^*\) is absolutely continuous because

\[
u^*(b) - u^*(a) = v(b) - v(a) = \int_a^b u'(t) \, dt = \int_a^b v'(t) \, dt = \int_a^b (u^*)'(t) \, dt.
\]

Thus, \(u^*\) is the absolutely continuous representative of \(u\).
Example

In the above result, we have shown that $W^{1,p}(a, b) \subset AC(a, b)$, in the sense of representatives. Let us give an example of a continuous function on a bounded interval $I \subset \mathbb{R}$ which does not belong to $H^1(I)$. Take $I = (-1, 1)$. For non-zero $\gamma$ in the range $-\frac{1}{2} < \gamma \leq \frac{1}{2}$, the function $|x|^\gamma \notin H^1(I)$ but is in $L^2(I)$. If $0 < \gamma \leq \frac{1}{2}$ then $|x|^\gamma$ is continuous function and not in $H^1(I)$.
If there exists a positive constant $C > 0$ such that

$$\|u\|_{r, \mathbb{R}^n} \leq C \|\nabla u\|_{p, \mathbb{R}^n} \quad \forall u \in W^{1,p}(\mathbb{R}^n) \quad (11.1)$$

for some $r \in [1, \infty)$ and $1 \leq p < \infty$, then we have a continuous imbedding of $W^{1,p}(\Omega)$ in to $L^r(\mathbb{R}^n)$ because

$$\|u\|_{r, \mathbb{R}^n} \leq C \|\nabla u\|_{p, \mathbb{R}^n} \leq C \|\nabla u\|_{p, \mathbb{R}^n} + C \|u\|_{p, \mathbb{R}^n} = C \|u\|_{1,p, \mathbb{R}^n}.$$

However, it is obvious that (11.1) is not a sufficient condition for continuous imbedding. The equation (11.1) is a Sobolev inequality, a stronger necessary condition for continuous imbedding.

Before we prove an inequality like (11.1), let’s check its validity. When can we expect such an inequality?

If $u \in W^{1,p}(\mathbb{R}^n)$ satisfies (11.1) for some $C > 0$ and $r$, then $u_{\lambda}(x) := u(\lambda x)$, for any $\lambda > 0$, also satisfies (11.1) for the same $C$ and $r$. 

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Since, for $1 \leq p < \infty$, $\|u_{\lambda}\|_p = \frac{1}{\lambda^{n/p}} \|u\|_p$ and $\|\nabla u_{\lambda}\|_p = \frac{\lambda}{\lambda^{n/p}} \|\nabla u\|_p$, we have

$$\frac{1}{\lambda^{n/r}} \|u\|_r \leq C \frac{\lambda}{\lambda^{n/p}} \|\nabla u\|_p$$

$$\|u\|_r \leq C \lambda^{1 + \frac{n}{r} - \frac{n}{p}} \|\nabla u\|_p.$$

The above obtained inequality being true for $\lambda > 0$ would contradict (11.1) except when $1 + \frac{n}{r} - \frac{n}{p} = 0$. Consequently, to expect an inequality of the kind (11.1), we need to have $\frac{1}{r} = \frac{1}{p} - \frac{1}{n}$. Therefore, $\frac{1}{p} - \frac{1}{n} > 0$ and hence $1 \leq p < n$ and $r = \frac{np}{n-p}$.

**Definition**

*If $1 \leq p < n$, the Sobolev conjugate of $p$ is defined as*

$$p^* := \frac{np}{n-p}$$

*Equivalently, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Also, $p^* > p$.*
Lemma (Loomis-Whitney Inequality)

Let \( n \geq 2 \). Let \( f_1, f_2, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1}) \). For \( x \in \mathbb{R}^n \), set \( \hat{x}_i = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1} \) for \( 1 \leq i \leq n \). Define \( f(x) = f_1(\hat{x}_1) \ldots f_n(\hat{x}_n) \) for \( x \in \mathbb{R}^n \). Then \( f \in L^1(\mathbb{R}^n) \) and

\[
\|f\|_{1,\mathbb{R}^n} \leq \prod_{i=1}^{n} \|f_i\|_{n-1,\mathbb{R}^{n-1}}.
\]

Proof: Let \( n = 2 \), then

\[
\|f\|_{1,\mathbb{R}^2} = \int_{\mathbb{R}^2} |f(x_1, x_2)| \, dx_1 \, dx_2 = \int_{\mathbb{R}^2} |f_1(x_2)||f_2(x_1)| \, dx_1 \, dx_2 = \int_{\mathbb{R}} |f_2(x_1)| \, dx_1 \int_{\mathbb{R}} |f_1(x_2)| \, dx_2 = \|f_2\|_{1,\mathbb{R}} \|f_1\|_{1,\mathbb{R}}.
\]
Let $n = 3$, then using Cauchy-Schwarz inequality twice we get,

\[
\|f(x)\|_{1, \mathbb{R}^3} = \int_{\mathbb{R}^3} |f_1(x_2, x_3)||f_2(x_1, x_3)||f_3(x_1, x_2)| \, dx_1 \, dx_2 \, dx_3
\]

\[
= \int_{\mathbb{R}^2} |f_3(x_1, x_2)| \left( \int_{\mathbb{R}} |f_1(x_2, x_3)||f_2(x_1, x_3)| \, dx_3 \right) \, dx_1 \, dx_2
\]

\[
\leq \int_{\mathbb{R}^2} |f_3(x_1, x_2)| \left[ \prod_{i=1}^{2} \left( \int_{\mathbb{R}} |f_i(\hat{x}_i)|^2 \, dx_3 \right)^{\frac{1}{2}} \right] \, dx_1 \, dx_2
\]

\[
= \int_{\mathbb{R}^2} |f_3(x_1, x_2)| g(x_2)^{1/2} h(x_1)^{1/2} \, dx_1 \, dx_2
\]

\[
\leq \sqrt{\left( \int_{\mathbb{R}^2} |f_3(x_1, x_2)|^2 \, dx_1 \, dx_2 \right) \left( \int_{\mathbb{R}^2} g(x_2)h(x_1) \, dx_1 \, dx_2 \right)}
\]

\[
= \|f_3\|_{2, \mathbb{R}^2} \|f_2\|_{2, \mathbb{R}^2} \|f_1\|_{2, \mathbb{R}^2}. \]
The general case will be proved by induction. Assume the result for $n$. Let $x \in \mathbb{R}^{n+1}$. Fix $x_{n+1}$ and $x' = (x_1, \ldots, x_n)$, then by Hölder’s inequality,

$$
\int_{\mathbb{R}^n} |f(x)| \, dx' \leq \|f_{n+1}\|_{n,\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f_1 f_2 \cdots f_n|^{n'} \, dx_1 \, dx_2 \cdots dx_n \right)^{1/n'} \quad (11.2)
$$

where $n' = \frac{n}{n-1}$ is the conjugate exponent of $n$. Recall that $f_1, \ldots, f_n \in L^n(\mathbb{R}^n)$. Thus, by treating $x_{n+1}$ as a fixed parameter, $|f_1|^{n'}, \ldots, |f_n|^{n'} \in L^{n-1}(\mathbb{R}^{n-1})$. Therefore, by induction hypothesis,

$$
\int_{\mathbb{R}^n} |f_1 f_2 \cdots f_n|^{n'} \, dx_1 \cdots dx_n \leq \prod_{i=1}^{n} \left( \int_{\mathbb{R}^{n-1}} |f_i|^{n'(n-1)} \, dx_1 \cdots dx_n \right)^{1/(n-1)}
$$

$$
= \prod_{i=1}^{n} \left( \int_{\mathbb{R}^{n-1}} |f_i|^n \, dx_1 \cdots dx_n \right)^{n'/n}
$$

$$
= \prod_{i=1}^{n} \|f_i\|_{n,\mathbb{R}^{n-1}}^{n'}
$$
Now, substituting above inequality in (11.2), we get

$$\int_{\mathbb{R}^n} |f(x)| \, dx' \leq \left\| f_{n+1} \right\|_{n, \mathbb{R}^n} \prod_{i=1}^{n} \left\| f_i \right\|_{n, \mathbb{R}^{n-1}}.$$  

Integrate both sides with respect to $x_{n+1}$. We get,

$$\int_{\mathbb{R}^{n+1}} |f(x)| \, dx \leq \left\| f_{n+1} \right\|_{n, \mathbb{R}^n} \prod_{i=1}^{n} \left\| f_i \right\|_{n, \mathbb{R}^n}.$$
Theorem (Gagliardo-Nirenberg-Sobolev Inequality)

Let $1 \leq p < n$. Then there exists a constant $C > 0$ (depending on $p$ and $n$) such that

$$\|u\|_{p^*,\mathbb{R}^n} \leq C\|\nabla u\|_{p,\mathbb{R}^n} \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

In particular, we have the continuous imbedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n).$$

Proof: We begin by proving for the case $p = 1$. Note that $p^* = 1^* = \frac{n}{n-1}$. We first prove the result for the space of test functions which is dense in $W^{1,p}(\mathbb{R}^n)$. 

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Let \( \phi \in D(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \) and \( 1 \leq i \leq n \), then

\[
\phi(x) = \int_{-\infty}^{x_i} D^e_i \phi(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \, dt
\]

\[
|\phi(x)| \leq \int_{-\infty}^{\infty} |D^e_i \phi(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)| \, dt =: f_i(\hat{x}_i)
\]

\[
|\phi(x)|^n \leq \prod_{i=1}^{n} f_i(\hat{x}_i)
\]

\[
|\phi(x)|^{n/n-1} \leq \prod_{i=1}^{n} |f_i(\hat{x}_i)|^{1/n-1}.
\]

Now, integrating both sides with respect to \( x \), we have

\[
\int_{\mathbb{R}^n} |\phi|^{n/n-1} \, dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} |f_i|^{1/n-1} \, dx. \quad (11.3)
\]

Observe that \( g_i := |f_i|^{1/n-1} \in L^{n-1}(\mathbb{R}^{n-1}) \) for each \( 1 \leq i \leq n \).
Hence, by Lemma 15, we have

\[
\int_{\mathbb{R}^n} \prod_{i=1}^{n} |f_i|^{1/n-1} \, dx = \int_{\mathbb{R}^n} \prod_{i=1}^{n} |g_i| \, dx \leq \prod_{i=1}^{n} \left( \int_{\mathbb{R}^{n-1}} |g_i|^{n-1} \, dx \right)^{1/n-1} = \prod_{i=1}^{n} \left( \int_{\mathbb{R}^{n-1}} |f_i| \, dx \right)^{1/n-1} = \prod_{i=1}^{n} \|D_i \phi\|_{1,\mathbb{R}^{n-1}}^{1/n-1} = \prod_{i=1}^{n} \|D_i \phi\|_{1,\mathbb{R}^{n}}^{1/n-1}.
\]

Thus, substituting above inequality in (11.3), we get

\[
\|\phi\|_{1^*,\mathbb{R}^n} \leq \prod_{i=1}^{n} \|D_i \phi\|_{1,\mathbb{R}^{n}}^{1/n}
\]

and consequently, we get

\[
\|\phi\|_{1^*,\mathbb{R}^n} \leq \prod_{i=1}^{n} \|D_i \phi\|_{1,\mathbb{R}^{n}}^{1/n} \leq \prod_{i=1}^{n} \|\nabla \phi\|_{1,\mathbb{R}^{n}}^{1/n} = \|\nabla \phi\|_{1,\mathbb{R}^{n}}.
\]

Hence the result proved for \( p = 1 \).
Let $\psi := |\phi|^\gamma$, where $\gamma > 1$ will be chosen appropriately during the subsequent steps of the proof. Also, if $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\psi \in C^1_c(\mathbb{R}^n)$. We shall apply the $p = 1$ result to $\psi$. Therefore,

$$
\left\| \psi \right\|_{1^*, \mathbb{R}^n} \leq \left\| \nabla \psi \right\|_{1, \mathbb{R}^n}
$$

$$
\left( \int_{\mathbb{R}^n} |\phi|^n \frac{n\gamma}{n-1} \, dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla|\phi|^\gamma | \, dx
$$

$$
= \gamma \int_{\mathbb{R}^n} |\phi|^\gamma \frac{1}{\nabla} \phi \, dx
$$

$$
\leq \gamma \left( \int_{\mathbb{R}^n} |\phi|^{(\gamma-1)q} \, dx \right)^{1/q} \left( \int_{\mathbb{R}^n} |\nabla \phi|^p \, dx \right)^{1/p}
$$

(using Hölder’s Inequality).

Since we want only the gradient term on the RHS, we would like to bring the $q$ norm term to LHS. If we choose $\gamma$ such that $\frac{n\gamma}{n-1} = (\gamma - 1)q$, then we can club their powers. Thus, we get $\gamma := \frac{p(n-1)}{n-p}$. The fact that $p > 1$ implies that $\gamma > 1$, as we had demanded.
Thus, the inequality obtained above reduces to

\[ \| \phi \|_{p^*, \mathbb{R}^n} \leq \frac{p(n-1)}{n-p} \| \nabla \phi \|_{p, \mathbb{R}^n} \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n). \]

Now, for any \( u \in W^{1, p}(\mathbb{R}^n) \), there is a sequence \( \{ u_m \} \subset \mathcal{D}(\mathbb{R}^n) \) (cf. Theorem 24) such that \( u_m \to u \) in \( W^{1, p}(\mathbb{R}^n) \). Therefore, \( u_m \) is Cauchy in \( W^{1, p}(\mathbb{R}^n) \) and hence is Cauchy in \( L^{p^*}(\mathbb{R}^n) \) by above inequality. Since \( L^{p^*}(\mathbb{R}^n) \) is complete, \( u_m \) converges in \( L^{p^*}(\mathbb{R}^n) \) and should converge to \( u \), since \( u_m = \phi_m(\rho_m * u) \) (cf. Theorem 24). Thus, \( u \in L^{p^*}(\mathbb{R}^n) \) and the inequality is satisfied. Hence the theorem is proved for any \( 1 \leq p < n \). In fact, in the proof we have obtained the constant \( C \) to be \( C = \frac{p(n-1)}{n-p} \) (and this is not the best constant).
For any $1 \leq p < n$, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ is continuously imbedded, for all $r \in [p, p^*]$.

**Proof:** Let $u \in W^{1,p}(\mathbb{R}^n)$. By Theorem 36, we have $u \in L^{p^*}(\mathbb{R}^n)$. We need to show that $u \in L^r(\mathbb{R}^n)$ for any $r \in (p, p^*)$. Since $1/r \in [1/p^*, 1/p]$, there is a $0 \leq \lambda \leq 1$ such that $1/r = \lambda/p + (1 - \lambda)/p^*$. Consider,

\[
\|u\|_r^r = \int_{\mathbb{R}^n} |u|^r \, dx = \int_{\mathbb{R}^n} |u|^{\lambda r} |u|^{(1-\lambda)r} \, dx \\
\leq \left( \int_{\mathbb{R}^n} |u|^p \, dx \right)^{\frac{\lambda r}{p}} \left( \int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{\frac{(1-\lambda)r}{p^*}} \quad \text{(by Hölder inequality)}
\]

\[
\|u\|_r \leq \|u\|_p^\lambda \|u\|_{p^*}^{1-\lambda} \\
\leq \lambda \|u\|_p + (1 - \lambda)\|u\|_{p^*} \quad \text{(By generalised AM-GM inequality)}
\]

\[
\|u\|_r \leq \|u\|_p + \|u\|_{p^*} \leq \|u\|_p + C\|\nabla u\|_p \quad \text{(By Theorem 36)}
\]

\[
\leq \max\{C, 1\} \|u\|_{1,p}.
\]

Hence the continuous imbedding is shown for all $r \in [p, p^*]$.

\[\square\]
We now extend the Sobolev inequality for a proper subset $\Omega \subset \mathbb{R}^n$ with smooth boundary, using the extension operator.

**Theorem (Sobolev Inequality for a Subset)**

Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^1$ boundary. Also, let $1 \leq p < n$. Then $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where the constant $C$ obtained depends on $p$, $n$ and $\Omega$. Further, $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [p, p^*]$.

**Proof:** Let $C$ denote a generic constant in this proof. Since $\Omega$ has a $C^1$ boundary, by Theorem 30, there is an extension operator $P$ such that for some constant $C > 0$ (depending on $\Omega$ and $p$)

$$\|Pu\|_{1,p,\mathbb{R}^n} \leq C\|u\|_{1,p,\Omega} \quad \forall u \in W^{1,p}(\Omega).$$

Moreover, by Theorem 36, there exists a constant $C > 0$ (depending on $p$ and $n$) such that

$$\|Pu\|_{p^*,\mathbb{R}^n} \leq C\|\nabla(Pu)\|_{p,\mathbb{R}^n} \quad \forall u \in W^{1,p}(\Omega).$$
Let \( u \in W^{1,p}(\Omega) \), then

\[
\|u\|_{p^*,\Omega} \leq \|Pu\|_{p^*,\mathbb{R}^n} \leq C\|\nabla(Pu)\|_{p,\mathbb{R}^n} \leq C\|Pu\|_{1,p,\mathbb{R}^n} \leq C\|u\|_{1,p,\Omega}.
\]

where the final constant \( C \) is dependent on \( p, n \) and \( \Omega \). Similar ideas work to prove the continuous imbedding in \( L^r(\Omega) \) for \( r \in [p, p^*] \).

Note that the above result says

\[
\|u\|_{p^*} \leq C\|u\|_{1,p} \quad \forall u \in W^{1,p}(\Omega)
\]

and not

\[
\|u\|_{p^*} \leq C\|\nabla u\|_p \quad \forall u \in W^{1,p}(\Omega)
\]

because constant functions may belong to \( W^{1,p}(\Omega) \) (as happens for bounded subsets \( \Omega \)) whose derivatives are zero.
Corollary (For $W^{1,p}_0(\Omega)$)

Let $\Omega$ be an open subset of $\mathbb{R}^n$ (not necessarily bounded) and $1 \leq p < n$. Then there is a constant $C > 0$ (depending on $p$ and $n$) such that

$$\|u\|_{p^*} \leq C \|\nabla u\|_p \quad \forall u \in W^{1,p}_0(\Omega)$$

and

$$\|u\|_r \leq C \|u\|_{1,p} \quad \forall u \in W^{1,p}_0(\Omega), \ r \in [p, p^*].$$

Proof.

Use the fact that $\mathcal{D}(\Omega)$ is dense in $W^{1,p}_0(\Omega)$ and follow the last step of the proof of Theorem 36 to get

$$\|u\|_{p^*} \leq C \|\nabla u\|_p \quad \forall u \in W^{1,p}_0(\Omega).$$

The proof of the second inequality remains unchanged from the case of $\mathbb{R}^n$. One can also extend functions in $W^{1,p}_0(\Omega)$ by zero outside $\Omega$ and use the results proved for $\mathbb{R}^n$ and then restrict back to $\Omega$. 

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Note that the second inequality in the statement of the above corollary involves the $W^{1,p}$-norm of $u$ and not the $L^p$-norm of the gradient of $u$. However, for bounded sets one can hope to get the inequality involving gradient of $u$.

**Corollary**

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. For $1 \leq p < n$ there is a constant $C$ (depending on $p$, $n$, $r$ and $\Omega$) such that

$$
\|u\|_r \leq C\|\nabla u\|_p \quad \forall u \in W^{1,p}_0(\Omega) \quad \forall r \in [1, p^*].$

**Proof.**

Let $1 \leq p < n$ and $u \in W^{1,p}_0(\Omega)$. Then, by previous corollary, there is a constant $C > 0$ (depending on $p$ and $n$) $\|u\|_{p^*} \leq C\|\nabla u\|_p$. For any $1 \leq r \leq p^*$, there is a constant $C > 0$ (depending on $r$ and $\Omega$) such that $\|u\|_r \leq C\|u\|_{p^*}$ (since $\Omega$ is bounded). Therefore, we have a constant $C > 0$ (depending on $p$, $n$, $r$ and $\Omega$) $\|u\|_r \leq C\|\nabla u\|_p$ for all $r \in [1, p^*]$. $\square$
Exercise (Poincaré Inequality)

Using the corollary proved above show that, for $1 \leq p \leq \infty$, then there is a constant $C$ (depending on $p$ and $\Omega$) such that

$$
\|u\|_p \leq C \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).
$$

The Poincaré inequality makes the norm $\|\nabla u\|_p$ equivalent to $\|u\|_{1,p}$ in $W_0^{1,p}(\Omega)$.

Remark

Poincaré inequality is not true for $u \in W^{1,p}(\Omega)$. For instance, if $u \equiv c$, a constant, then $\nabla u = 0$ and hence $\|\nabla u\|_p = 0$ while $\|u\|_p > 0$. However, if $u \in W^{1,p}(\Omega)$ such that $u = 0$ on $\Gamma \subset \partial \Omega$, then Poincaré inequality remains valid for such $u$'s.
Remark

Poincaré inequality is not true for unbounded domains. However, one can relax the bounded-ness hypothesis on $\Omega$ to bounded-ness along one particular direction, as seen from the proof.

The Poincaré inequality proved above can be directly proved without using the Sobolev inequality, as shown below.

**Theorem (Poincaré Inequality)**

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, then there is a constant $C$ (depending on $p$ and $\Omega$) such that

$$
\|u\|_{p, \Omega} \leq C \|\nabla u\|_{p, \Omega} \quad \forall u \in W_0^{1,p}(\Omega). 
$$

(11.4)

**Proof:** Let $a > 0$ and suppose $\Omega = (-a, a)^n$. Let $u \in \mathcal{D}(\Omega)$ and $x = (x', x_n) \in \mathbb{R}^n$. Then

$$
u(x) = \int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x', t) \, dt. $$
Moreover, \( u(x', -a) = 0 \). Thus, by Hölder’s inequality,

\[
|u(x)| \leq \left( \int_{-a}^{x_n} \left| \frac{\partial u}{\partial x_n}(x', t) \right|^p \, dt \right)^{1/p} |x_n + a|^{1/q}
\]

\[
|u(x)|^p \leq |x_n + a|^{p/q} \int_{-a}^{a} \left| \frac{\partial u}{\partial x_n}(x', t) \right|^p \, dt
\]

\[
|u(x)|^p \leq (2a)^{p/q} \int_{-a}^{a} \left| \frac{\partial u}{\partial x_n}(x', t) \right|^p \, dt.
\]

First integrating w.r.t \( x' \) and then integrating w.r.t \( x_n \) we get

\[
\|u\|_{p, \Omega}^p \leq (2a)^{(p/q)+1} \left\| \frac{\partial u}{\partial x_n} \right\|_{p, \Omega}^p, \quad \forall u \in \mathcal{D}(\Omega)
\]

and taking \((1/p)\)-th power both sides, we get

\[
\|u\|_{p, \Omega} \leq 2a \left\| \frac{\partial u}{\partial x_n} \right\|_{p, \Omega}, \quad \forall u \in \mathcal{D}(\Omega).
\]
Thus,

\[ \| u \|_{p, \Omega} \leq 2a \left\| \frac{\partial u}{\partial x_n} \right\|_{p, \Omega} \leq 2a \| \nabla u \|_{p, \Omega}, \quad \forall u \in D(\Omega). \]

By the density of \( D(\Omega) \) in \( W^{1,p}_0(\Omega) \), we get the result for all \( u \in W^{1,p}_0(\Omega) \). Now, suppose \( \Omega \) is not of the form \((-a, a)\), then \( \Omega \subset (-a, a) \) for some \( a > 0 \), since \( \Omega \) is bounded. Then any \( u \in W^{1,p}_0(\Omega) \) can be extended to \( W^{1,p}_0(-a, a) \) and use the result proved above.
Theorem

Let \( n \geq 2 \), \( \mathcal{W}^{1,n}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n) \) for all \( r \in [n, \infty) \).

Proof: Let \( u \in \mathcal{D}(\mathbb{R}^n) \). Observe that the conjugate exponent of \( n \) is same as \( 1^* \), the Sobolev conjugate of 1. Thus, we adopt the initial part of the proof of Theorem 36 to obtain \( \| u \|_{1^*,\mathbb{R}^n} \leq \| \nabla u \|_{1,\mathbb{R}^n} \) where \( 1^* = \frac{n}{n-1} \), the conjugate exponent of \( n \). Applying this inequality to \( |u|^\gamma \), for some \( \gamma > 1 \) as in Theorem 36, we get

\[
\left( \int_{\mathbb{R}^n} |u|^\frac{n\gamma}{n-1} \, dx \right)^{\frac{n-1}{n}} \leq \gamma \left( \int_{\mathbb{R}^n} |u|^\frac{(\gamma-1)n}{n-1} \, dx \right)^{\frac{n-1}{n}} \left( \int_{\mathbb{R}^n} |\nabla u|^n \, dx \right)^{\frac{1}{n}}
\]

\[
\| u \|_{\frac{n\gamma}{n-1}} \leq \gamma \| u \|_{\frac{n(\gamma-1)}{n-1}} \| \nabla u \|_n
\]

\[
\leq \left( \| u \|_{\frac{n(\gamma-1)}{n-1}} + \| \nabla u \|_n \right)^\gamma
\]

Using \( (a + b)^\gamma \geq \gamma a^{\gamma-1} b \) for \( a, b \geq 0 \)

\[
\| u \|_{\frac{n\gamma}{n-1}} \leq \| u \|_{\frac{n(\gamma-1)}{n-1}} + \| \nabla u \|_n.
\]
Now, by putting $\gamma = n$, we get

$$\|u\|_{\frac{n^2}{n-1}} \leq \|u\|_n + \|\nabla u\|_n = \|u\|_{1,n}.$$ 

and extending the argument, as done in Corollary 12, we get

$$\|u\|_r \leq C\|u\|_{1,n} \quad \forall r \in \left[n, \frac{n^2}{n-1}\right].$$

Now, repeating the argument for $\gamma = n + 1$, we get

$$\|u\|_r \leq C\|u\|_{1,n} \quad \forall r \in \left[\frac{n^2}{n-1}, \frac{n(n+1)}{n-1}\right].$$

Thus, continuing in similar manner for all $\gamma = n + 2, n + 3, \ldots$ we get

$$\|u\|_r \leq C\|u\|_{1,n} \quad \forall r \in [n, \infty).$$

The result extends to $W^{1,n}(\mathbb{R}^n)$ by similar density arguments of $\mathcal{D}(\mathbb{R}^n)$, as done for the $1 \leq p < n$ case.
We now extend the results to proper subsets of $\mathbb{R}^n$. The proofs are similar to the equivalent statements from previous section.

**Corollary (For Subset)**

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^1$ boundary and let $n \geq 2$. Then $W^{1,n}(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [n, \infty)$.

**Corollary**

Let $\Omega$ be an open subset of $\mathbb{R}^n$ (not necessarily bounded) and $n \geq 2$. Then $W^{1,n}_0(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [n, \infty)$. Further, if $\Omega$ is bounded, $W^{1,n}_0(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [1, \infty)$. 
Theorem (Morrey’s Inequality)

Let \( n < p < \infty \), then \( W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \). Moreover, for any \( u \in W^{1,p}(\mathbb{R}^n) \), there is a representative of \( u \), \( u^* \), which is Hölder continuous with exponent \( 1 - n/p \) and there is a constant \( C > 0 \) (depending only on \( p \) and \( n \)) such that

\[
\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}
\]

where \( \gamma := 1 - n/p \).

Proof: Let \( u \in \mathcal{D}(\mathbb{R}^n) \) and let \( E \) be a cube of side \( a \) containing the origin and each of its sides being parallel to the coordinate axes of \( \mathbb{R}^n \).
Let $x \in E$. We have

\[
|u(x) - u(0)| = \left| \int_0^1 \frac{d}{dt}(u(tx)) \, dt \right| = \left| \int_0^1 \nabla u(tx) \cdot x \, dt \right|
\]

\[
\leq \int_0^1 |\nabla u(tx)| \cdot |x| \, dt = \int_0^1 \sum_{i=1}^n |x_i| \left| \frac{\partial}{\partial x_i} u(tx) \right| \, dt
\]

\[
\leq \int_0^1 \sum_{i=1}^n a \left| \frac{\partial}{\partial x_i} u(tx) \right| \, dt = a \int_0^1 \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u(tx) \right| \, dt
\]

Let $\bar{u}$ denote the average of $u$ over the cube $E$,

\[
\bar{u} = \frac{1}{|E|} \int_E u(x) \, dx = \frac{1}{a^n} \int_E u(x) \, dx.
\]

Consider,

\[
|\bar{u} - u(0)| = \left| \frac{1}{a^n} \int_E u(x) \, dx - u(0) \right| \leq \frac{1}{a^n} \int_E |u(x) - u(0)| \, dx
\]
\[
\leq \frac{a}{a^n} \int_E \int_0^1 \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u(tx) \right| \, dt \, dx \\
= \frac{1}{a^{n-1}} \sum_{i=1}^n \int_0^1 \int_E \left| \frac{\partial}{\partial x_i} u(tx) \right| \, dx \, dt \quad \text{(Fubini's Theorem)} \\
= \frac{1}{a^{n-1}} \sum_{i=1}^n \int_0^1 t^{-n} \int_{tE} \left| \frac{\partial}{\partial x_i} u(y) \right| \, dy \, dt \quad \text{(Change of variable)} \\
\leq \frac{1}{a^{n-1}} \sum_{i=1}^n \int_0^1 t^{-n} \left\| \frac{\partial u}{\partial x_i} \right\|_{p,E} (|tE|)^{1/q} \, dt \\
\quad \text{(By Hölder inequality and } tE \subset E \text{ for } 0 \leq t \leq 1) \\
= \frac{a^{n/q}}{a^{n-1}} \left\| \nabla u \right\|_{p,E} \int_0^1 t^{-n} t^{n/q} \, dt \quad \text{(} q \text{ is conjugate exponent of } p) \\
= a^{1-n/p} \left\| \nabla u \right\|_{p,E} \int_0^1 t^{-n/p} \, dt = \frac{a^{1-n/p}}{1 - n/p} \left\| \nabla u \right\|_{p,E}.
\]
The above inequality is then true for any cube $E$ of side length $a$ with sides parallel to axes, by translating it in $\mathbb{R}^n$. Therefore, for any cube $E$ of side $a$ and $x \in E$, we have

$$|\overline{u} - u(x)| \leq \frac{a^\gamma}{\gamma} \|\nabla u\|_{p,E}$$

(11.5)

where $\gamma := 1 - n/p$. Consider,

$$|u(x)| = |u(x) - \overline{u} + \overline{u}| \leq \frac{1}{\gamma} \|\nabla u\|_{p,E} + \|u\|_{1,E}$$

$$\leq \frac{1}{\gamma} \|\nabla u\|_{p,E} + \|u\|_{p,E} \quad \text{(by Hölder’s inequality)}$$

where we have used (11.5), in particular, for a unit cube $E$. Then

$$\|u\|_{\infty,\mathbb{R}^n} \leq \frac{1}{\gamma} \|\nabla u\|_{p,\mathbb{R}^n} + \|u\|_{p,\mathbb{R}^n}.$$

Therefore,

$$\|u\|_{\infty,\mathbb{R}^n} \leq C \|u\|_{1,p,\mathbb{R}^n},$$

(11.6)

where $C = \max\{1, \frac{1}{\gamma}\}$, and hence $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. 
Further, it follows from (11.5) that for any \( x, y \in E \),

\[
|u(x) - u(y)| \leq \frac{2a^\gamma}{\gamma} \|\nabla u\|_{p,E}.
\]

Now, for any given \( x, y \in \mathbb{R}^n \), one can always choose a cube \( E \) whose side \( a = 2|x - y| \) and applying the above inequality, we get

\[
|u(x) - u(y)| \leq \frac{2^{\gamma+1}}{\gamma} |x - y|\gamma \|\nabla u\|_{p,E} \leq \frac{2^{\gamma+1}}{\gamma} |x - y|\gamma \|\nabla u\|_{p,\mathbb{R}^n}.
\]

Thus, \( u \) is Hölder continuous and its Hölder seminorm \( p_{\gamma}(.) \) (cf.(8.2)) is bounded as below,

\[
p_{\gamma}(u) \leq \frac{2^{\gamma+1}}{\gamma} \|\nabla u\|_{p,\mathbb{R}^n}.
\]

and this together with (11.6) gives the bound for the \( \gamma \)-th Hölder norm,

\[
\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{1,p,\mathbb{R}^n}.
\]

By the density of \( \mathcal{D}(\mathbb{R}^n) \) in \( W^{1,p}(\mathbb{R}^n) \) we have a sequence \( u_\varepsilon \rightarrow u \) in \( W^{1,p}(\mathbb{R}^n) \). By the bound on Hölder norm, we find the sequence is also Cauchy in \( C^{0,\gamma}(\mathbb{R}^n) \) and should converge to a representative of \( u \), \( u^* \), in the \( \gamma \)-th Hölder norm.
Remark

As usual, the results can be extended to $W^{1,p}(\Omega)$ for $\Omega$ bounded with $C^1$ smooth boundary and to $W^{1,p}_0(\Omega)$ for any open subset $\Omega$.

Theorem (Characterisation of $W^{1,\infty}$)

For any $u \in W^{1,\infty}(\mathbb{R}^n)$ there is a representative $u^*$ which is Lipschitz continuous from $\mathbb{R}^n$ to $\mathbb{R}$.

Example

Let $p < n$. Consider the function $|x|^\delta$, for any choice of $\delta$ in $1 - \frac{n}{p} < \delta < 0$, is in $W^{1,p}(\Omega)$ which has no continuous representative.
Example

Let $p = n$ and $n \geq 2$. We shall give an example of a function in $W^{1,n}(\Omega)$ which has no continuous representative. We shall given an example for the case $n = 2$. Let $\Omega := \{ x \in \mathbb{R}^n : |x| < R \}$ and $u(x) = (-\ln |x|)^{\delta}$ for $x \neq 0$. We have, using polar coordinates,

$$
\int_{B_R(0)} u^n \, dx = R^{n-1}\omega_n \int_0^R (-\ln r)^{n\delta} r \, dr.
$$

Using the change of variable $t = -\ln r$, we get

$$
\int_{B_R(0)} u^n \, dx = R^{n-1}\omega_n \int_{-\ln R}^{+\infty} t^{n\delta} e^{-2t} \, dt < \infty \quad \text{for all } \delta.
$$

Thus, $u \in L^p(\Omega)$ for every $\delta \in \mathbb{R}$. Further, for each $i = 1, 2, \ldots, n$,

$$
u_{x_i} = -\delta x_i |x|^{-2} (-\ln |x|)^{\delta-1}
$$

and, therefore,

$$|
\nabla u | = \left| \delta (-\ln |x|)^{\delta-1} \right| |x|^{-1}.
$$
Thus, using polar coordinates, we get
\[
\int_{B_R(0)} |\nabla u|^n \, dx = R^{n-1} \omega_n |\delta|^n \int_0^R |\ln r|^{n\delta-n} r^{1-n} \, dr.
\]

Using the change of variable \( t = -\ln r \), we get
\[
\int_{B_R(0)} |\nabla u|^n \, dx = R^{n-1} \omega_n |\delta|^n \int_{-\ln R}^\infty |t|^{n(\delta-1)} e^{t(n-2)} \, dt.
\]

If \( n = 2 \) then the integral is finite iff \( n(1 - \delta) > 1 \) or \( \delta < 1 - \frac{1}{n} = \frac{1}{2} \). In particular, \( \nabla u \) represents the gradient of \( u \) in the distribution sense as well. We conclude that \( u \in H^1(\Omega) \) iff \( \delta < 1/2 \). We point out that when \( \delta > 0 \), \( u \) is **unbounded** near 0. Thus, by taking \( 0 < \delta < \frac{1}{2} \), we obtain a function belonging to \( H^1(\Omega) \) which blows up to \( +\infty \) at the origin and which does not have a continuous representative.
Generalised Sobolev Imbedding

We shall now generalise the results of previous section to all derivative orders of $k \geq 2$.

**Theorem**

Let $k \geq 1$ be an integer and $1 \leq p < \infty$. Then

1. If $p < n/k$, then $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ for all $r \in [p, np/(n - pk)]$.
2. If $p = n/k$, then $W^{k,n/k}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ for all $r \in [n/k, \infty)$.
3. If $p > n/k$, then $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and further there is a representative of $u$, say $u^*$, whose $k$-th partial derivative is Hölder continuous with exponent $\gamma$ and there is a constant $C > 0$ (depending only on $p, k$ and $n$) such that

$$
\|u^*\|_{C^{k-1-[n/p],\gamma}(\mathbb{R}^n)} \leq C \|u\|_{k,p}
$$

where $\gamma := k - n/p - [k - n/p]$ and $[l]$ is the largest integer such that $[l] \leq l$. 

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**Proof**

**Step-1:** If \( k = 1 \) we know the above results are true. Let \( k = 2 \) and \( p < n/2 \). Then, for any \( u \in W^{2,p}(\mathbb{R}^n) \), both \( u, D^1 u \in W^{1,p}(\mathbb{R}^n) \) and since \( p < n/2 < n \), using Sobolev inequality, we have both \( u, D^1 u \in L^{p^*}(\mathbb{R}^n) \) with \( p^* = np/(n-p) \). Thus, \( u \in W^{1,p^*}(\mathbb{R}^n) \). Now, since \( p < n/2 \), we have \( p^* < n \). Thus, using Sobolev inequality again again, we get \( u \in L^{(p^*)^*}(\mathbb{R}^n) \). But \( 1/(p^*)^* = 1/p - 2/n \). Extending similar arguments for each case, we get the result. Note that when we say \( D^1 u \), we actually mean \( D^\alpha u \) for each \(|\alpha| = 1\). Such convention will be used throughout this proof.

**Step-2:** We know the result for \( k = 1 \). Let \( k = 2 \) and \( u \in W^{2,n/2}(\mathbb{R}^n) \). Then, \( u, D^1 u \in W^{1,n/2}(\mathbb{R}^n) \). Since \( n/2 < n \) and \((n/2)^* = n\), we have, using Sobolev inequality, \( u \in W^{1,r}(\mathbb{R}^n) \) for all \( r \in [n/2, n] \). Thus, \( u \in W^{1,n}(\mathbb{R}^n) \), which is continuously imbedded in \( L^r(\mathbb{R}^n) \) for all \( r \in [n, \infty) \).
Step-3: Let $k = 2$ and $p > n/2$. $W^{2,p}(\mathbb{R}^n)$ is continuously imbedded in $W^{1,p}(\mathbb{R}^n)$. If $p > n > n/2$, using Morrey’s inequality, we have $W^{1,p}(\mathbb{R}^n)$ is continuously imbedded in $L^\infty(\mathbb{R}^n)$ and the Hölder norm estimate is true for both $u$ and $\nabla u$. If $n/2 < p < n$ then $p^* > n$ and, by Sobolev inequality, any $u \in W^{2,p}(\mathbb{R}^n)$ is also in $W^{1,r}(\mathbb{R}^n)$ for all $r \in [p, p^*]$. Now, by Morrey’s inequality, $W^{1,p^*}(\mathbb{R}^n)$ is continuously imbedded in $L^\infty(\mathbb{R}^n)$ and the Hölder norm estimate is true for both $u$ and $\nabla u$. Let $p = n > n/2$ then for any $u \in W^{2,n}(\mathbb{R}^n)$, we have $u \in W^{1,r}(\mathbb{R}^n)$ for all $r \in [n, \infty)$. Thus for $r > n$, we have the required imbedding in $L^\infty(\mathbb{R}^n)$. It now only remains to show the Hölder estimate.

Remark

As before, the results can be extended to $W^{1,p}(\Omega)$ for $\Omega$ bounded with $C^k$ smooth boundary and to $W^{1,p}_0(\Omega)$ for any open subset $\Omega$. 

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Compact Imbedding

We now identify those continuous imbeddings which are also compact. We first note that we cannot expect compact imbedding for unbounded domains.

**Example:** We shall construct a bounded sequence in $W^{1,p}(\mathbb{R})$ and show that it can not converge in $L^r(\mathbb{R})$, for all those $r$ for which the imbedding is continuous. Let $I = (0, 1) \subset \mathbb{R}$ and $I_j := (j, j + 1)$, for all $j = 1, 2, \ldots$. Choose any $f \in C^1(\mathbb{R})$ with support in $I$ and set $f_j(x) := f(x - j)$. Thus, $f_j$ is same as $f$ except that its support is now contained in $I_j$. Hence $\|f\|_{1,p} = \|f_j\|_{1,p}$ for all $j$. Now, set $g := \frac{f}{\|f\|_{1,p}}$ and $g_j := \frac{f_j}{\|f_j\|_{1,p}}$. Note that $\{g_j\}$ is a bounded sequence (norm being one) in $W^{1,p}(\mathbb{R})$. We know that $W^{1,p}$ is continuously imbedded in $L^\infty(\mathbb{R})$ for $1 < p < \infty$ and with $p = 1$ is imbedded in $L^r(\mathbb{R})$ for all $1 \leq r < \infty$. Since $g_j$ have compact support, $\{g_j\} \subset L^r(\mathbb{R})$ for all $1 \leq r \leq \infty$ (depending on $p$), by the continuous imbedding.
We will show that \( \{g_j\} \) do not converge strongly in \( L^r(\mathbb{R}) \). Note that 
\[
\|g_j\|_{r,\mathbb{R}} = \|g_j\|_{r,I_j} = \|g\|_{r,I} = c > 0.
\]
Consider for any \( i \neq j \),
\[
\|g_i - g_j\|_{r,\mathbb{R}} = \|g_i\|_{r,I_i} + \|g_j\|_{r,I_j} + \|g_i - g_j\|_{r,\mathbb{R}\setminus(I_i \cup I_j)} = 2c^r.
\]
Thus, the sequence is not Cauchy in \( L^r(\mathbb{R}) \) (as seen by choosing \( \varepsilon < 2^{1/r} c \) for all \( i, j \)). The arguments can be generalised to \( \mathbb{R}^n \).

\[\square\]

**Theorem**

Let \( 1 \leq p \leq \infty \). For all \( u \in W^{1,p}(\mathbb{R}^n) \) the following inequality holds:
\[
\|\tau_h u - u\|_p \leq \|\nabla u\|_p |h|, \quad \forall h \in \mathbb{R}^n.
\]

**Proof:** It is enough to prove the inequality for \( u \in \mathcal{D}(\mathbb{R}^n) \) due to the density of \( \mathcal{D}(\mathbb{R}^n) \) in \( W^{1,p}(\mathbb{R}^n) \). Consider
\[
(\tau_h u)(x) - u(x) = u(x - h) - u(x)
= - \int_0^1 [\nabla u(x - th)] h \, dt.
\]
Hence, by the Cauchy-Schwarz inequality

\[ |(\tau_h u)(x) - u(x)| \leq \int_0^1 |\nabla u(x - th)| h \, dt, \]

and then, by Hölder’s inequality,

\[ |(\tau_h u)(x) - u(x)|^p \leq |h|^p \int_0^1 |\nabla u(x - th)|^p \, dt. \]

Integrating over \( \mathbb{R}^n \) we have

\[ \int_{\mathbb{R}^n} |(\tau_h u)(x) - u(x)|^p \, dx \leq |h|^p \int_{\mathbb{R}^n} \left( \int_0^1 |\nabla u(x - th)|^p \, dt \right) \, dx. \]

By Fubini-Tonelli theorem and the invariance, under translation, of the Lebesgue measure in \( \mathbb{R}^n \) to obtain

\[ \int_{\mathbb{R}^n} |\tau_h u - u|^p \, dx \leq |h|^p \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx. \]
Theorem (Kolmogorov Compactness Criteria)

Let $p \in [1, \infty)$ and let $A$ be a subset of $L^p(\mathbb{R}^n)$. Then $A$ is relatively compact in $L^p(\mathbb{R}^n)$ iff the following conditions are satisfied:

1. $A$ is bounded in $L^p(\mathbb{R}^n)$;
2. $\lim_{r \to +\infty} \int_{\{|x| > r\}} |f(x)|^p \, dx = 0$ uniformly with respect to $f \in A$;
3. $\lim_{h \to 0} \| \tau_h f - f \|_p = 0$ uniformly with respect to $f \in A$, where $\tau_h f$ is the translated function $(\tau_h f)(x) := f(x - h)$.
Theorem (Rellich-Kondrasov)

Let $1 \leq p < \infty$ and let $\Omega$ be a bounded domain with $C^1$ boundary, then

(i) If $p < n$, then $W^{1,p}(\Omega) \subset \subset L^r(\Omega)$ for all $r \in [1, p^*)$.

(ii) If $p = n$, then $W^{1,n}(\Omega) \subset \subset L^r(\Omega)$ for all $r \in [1, \infty)$.

(iii) If $p > n$, then $W^{1,p}(\Omega) \subset \subset C(\overline{\Omega})$.

Proof: (i) We first prove the case $p < n$. Let $B$ be the unit ball in $W^{1,p}(\Omega)$. We shall verify conditions (i) and (ii) of Theorem 44. Let $1 \leq q < p^*$. Then choose $\alpha$ such that $0 < \alpha \leq 1$ and

$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1 - \alpha}{p^*}.$$

If $u \in B$, $\Omega' \subset \subset \Omega$ and $h \in \mathbb{R}^n$ such that $|h| < \text{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\|\tau_{-h}u - u\|_{q,\Omega'} \leq \|\tau_{-h}u - u\|_{1,\Omega'}^{\alpha} \|\tau_{-h}u - u\|_{p^*,\Omega'}^{1-\alpha},$$

$$\leq (|h|^\alpha \|\nabla u\|_{1,\Omega}^{\alpha}) (2 \|u\|_{p^*,\Omega})^{1-\alpha},$$

$$\leq C|h|^\alpha.$$

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We choose $h$ small enough such that $C|h|^\alpha < \varepsilon$. This will verify condition (i) of Theorem 44. Now, if $u \in B$ and $\Omega' \subset \subset \Omega$, it follows by Hölder’s inequality that

$$
\|u\|_{q, \Omega \setminus \overline{\Omega}'} \leq \|u\|_{p^*, \Omega \setminus \overline{\Omega}' |\Omega \setminus \overline{\Omega}|^{1-(q/p^*)}}
\leq C|\Omega \setminus \overline{\Omega}'|^{1-(q/p^*)}
$$

which can be made less than any given $\varepsilon > 0$ by choosing $\Omega' \subset \subset \Omega$ to be 'as closely filling $\Omega'$ as needed. This verifies condition (ii). Thus $B$ is relatively compact in $L^q(\Omega)$ for $1 \leq q < p^*$.

(ii) Assume for the moment that the result is true for $p < n$. Notice that as $p \to n$, $p^* \to \infty$. Since $\Omega$ is bounded, $W^{1,n}(\Omega) \subset W^{1,n-\varepsilon}(\Omega)$, for every $\varepsilon > 0$. Since $n - \varepsilon < n$, using the $p < n$ case, we get $W^{1,n-\varepsilon}(\Omega)$ is compactly imbedded in $L^r(\Omega)$ for all $r \in [1, (n - \varepsilon)^*)$. Note that as $\varepsilon \to 0$, $(n - \varepsilon)^* \to \infty$. Therefore, for any $r < \infty$ we can find small enough $\varepsilon > 0$ such that $1 \leq r < (n - \varepsilon)^*$. We deduce that $W^{1,n}(\Omega)$ is compactly imbedded in $L^r(\Omega)$ for any $1 \leq r < \infty$. 

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(iii) For \( p > n \), the functions of \( W^{1,p}(\Omega) \) are Hölder continuous. If \( B \) is the unit ball in \( W^{1,p}(\Omega) \) then the functions in \( B \) are uniformly bounded and equicontinuous in \( C(\bar{\Omega}) \). Thus \( B \) is relatively compact in \( C(\bar{\Omega}) \) by the Ascoli-Arzela Theorem.

**Remark**

Note that the continuous inclusion for the \( r = p^* \) case is not compact. The above result can be extended to \( W^{1,p}_0(\Omega) \) provided \( \Omega \) is bounded and is a connected open subset (bounded domain) of \( \mathbb{R}^n \).

**Corollary (Compact subsets of \( W^{1,p}(\mathbb{R}^n) \))**

Let \( A \) be a subset of \( W^{1,p}(\mathbb{R}^n) \), \( 1 \leq p < +\infty \) which satisfies the two following conditions:

- **(i)** \( A \) is bounded in \( W^{1,p}(\mathbb{R}^n) \), i.e., \( \sup_{f \in A} \| f \|_{W^{1,p}(\mathbb{R}^n)} < +\infty \).

- **(ii)** \( A \) is \( L^p \)-equi-integrable at infinity, i.e.,
  \[
  \lim_{r \to +\infty} \int_{\{ |x| > r \}} |f(x)|^p \, dx = 0 \text{ uniformly with respect to } f \in A.
  \]

Then, \( A \) is relatively compact in \( L^p(\mathbb{R}^n) \).
**Exercise**

If $\Omega$ is a connected subset of $\mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ such that $\nabla u = 0$ a.e. in $\Omega$, then show that $u$ is constant a.e. in $\Omega$.

**Theorem (Poincaré-Wirtinger Inequality)**

Let $\Omega$ be a bounded, connected open subset of $\mathbb{R}^n$ with $C^1$ smooth boundary and let $1 \leq p \leq \infty$. Then there is a constant $C > 0$ (depending on $p$, $n$ and $\Omega$) such that

$$\|u - \overline{u}\|_p \leq C\|\nabla u\|_p \quad \forall u \in W^{1,p}(\Omega),$$

where $\overline{u} := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx$ is the average of $u$ in $\Omega$. 
Proof

Suppose the inequality was false, then for each positive integer $m$ we have a $u_m \in W^{1,p}(\Omega)$ such that

$$\|u_m - \overline{u_m}\|_p > m\|\nabla u_m\|_p \quad \forall u \in W^{1,p}(\Omega). \quad (12.1)$$

Set for all $m$,

$$v_m := \frac{u_m - \overline{u_m}}{\|u_m - \overline{u_m}\|_p}.$$

Thus, $\|v_m\|_p = 1$ and $\overline{v_m} = 0$. Hence, by (12.1), we have

$$\|\nabla v_m\|_p < 1/m.$$

Therefore, $\{v_m\}$ are bounded in $W^{1,p}(\Omega)$ and, by Rellich-Kondrasov compact imbedding, there is a subsequence of $\{v_m\}$ (still denoted by $m$) and a function $v \in L^p(\Omega)$ such that $v_m \to v$ in $L^p(\Omega)$. Therefore, $\overline{v} = 0$ and $\|v\|_p = 1$. 
Also, for any $\phi \in \mathcal{D}(\Omega)$,
\[
\int_{\Omega} v \frac{\partial \phi}{\partial x_i} \, dx = \lim_{m \to \infty} \int_{\Omega} v_m \frac{\partial \phi}{\partial x_i} \, dx
\]
\[
= - \lim_{m \to \infty} \int_{\Omega} \frac{\partial v_m}{\partial x_i} \phi \, dx
\]
\[
\to 0.
\]
Thus, $\nabla v = 0$ a.e. and $v \in W^{1,p}(\Omega)$. Moreover $v$ is constant, since $\Omega$ is connected. But $v = 0$ implies that $v \equiv 0$ which contradicts the fact that $\|v\|_p = 1$. 

\[\square\]

**Theorem (Compactness for measures)**

$\mathcal{M}_b(\Omega)$ is compactly imbedded in $W^{-1,r}(\Omega)$ for all $r \in \left[1, \frac{n}{n-1}\right)$.

**Proof:** Let $\{\mu_k\}_{k=1}^{\infty}$ be a bounded sequence in $\mathcal{M}_b(\Omega)$. By weak compactness, we extract a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty} \subset \{\mu_k\}_{k=1}^{\infty}$ such that $\mu_{k_j} \rightharpoonup \mu$ weak-* converges in $\mathcal{M}_b(\Omega)$, for some measure $\mu \in \mathcal{M}_b(\Omega)$. 

Proof Continued ...

Set $s = \frac{r}{r-1}$ and denote by $B$ the closed unit ball in $W_{0}^{1,s}(\Omega)$. Since $1 \leq r < \frac{n}{n-1}$, we have $s > n$ and so $B$ is compact in $C_{0}(\overline{\Omega})$. Thus, given $\varepsilon > 0$ there exist $\{\phi_{i}\}_{i=1}^{N(\varepsilon)} \subset C_{0}(\overline{\Omega})$ such that

$$\min_{1 \leq i \leq N(\varepsilon)} \|\phi - \phi_{i}\|_{C(\overline{\Omega})} < \varepsilon$$

for each $\phi \in B$. Therefore, if $\phi \in B$,

$$\left| \int_{\Omega} \phi \, d\mu_{k_{j}} - \int_{\Omega} \phi \, d\mu \right| \leq 2\varepsilon \sup_{j} |\mu_{k_{j}}|(\Omega) + \left| \int_{\Omega} \phi_{i} \, d\mu_{k_{j}} - \int_{\Omega} \phi_{i} \, d\mu \right|$$

for some index $1 \leq i \leq N(\varepsilon)$. Consequently,

$$\lim_{j \to \infty} \sup_{\phi \in B} \left| \int_{\Omega} \phi \, d\mu_{k_{j}} - \int_{\Omega} \phi \, d\mu \right| = 0$$

and, hence, $\mu_{k_{j}} \to \mu$ strongly in $W^{-1,r}(\Omega)$. 

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Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous function, where $\Omega$ is an open subset of $\mathbb{R}^n$. The trace of $u$ on the boundary $\partial \Omega$ is the continuous function $\gamma(u) : \partial \Omega \to \mathbb{R}$ defined by $\gamma(u)(x) = u(x)$ for all $x \in \partial \Omega$. Thus, we have a linear map $\gamma : C(\overline{\Omega}) \to C(\partial \Omega)$ which is the restriction to the boundary.

If $u \in L^2(\Omega)$ then there is no sufficient information to talk about $u$ on $\partial \Omega$, because the Lebesgue measure of $\partial \Omega$ is zero.

However, an additional information on $u$, viz., "$\frac{\partial u}{\partial x_i}$ belongs to $L^2(\Omega)$ for any $i = 1, \ldots, n$", can give meaning to $u$ restricted on $\partial \Omega$. Thus, in the Sobolev space $W^{1,p}(\Omega)$, where $\Omega$ is a domain (open connected set) in $\mathbb{R}^n$, the notion of trace or restriction to boundary can be defined on $\partial \Omega$, even for functions not continuous on $\overline{\Omega}$. 

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Trace of a Function

The basis for this extension is the following observation: Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( 1 \leq p < n \). Then one can show that the linear mapping

\[
\gamma : C^\infty(\overline{\Omega}) \to L^{p^\#}(\partial\Omega)
\]

is well defined and continuous if the space \( C^\infty(\overline{\Omega}) \) is endowed with the \( W^{1,p} \)-norm where \( p^\# > 1 \) is defined as

\[
\frac{1}{p^\#} := \frac{1}{p} - \frac{p - 1}{p(n - 1)} \quad \text{for } 1 \leq p < n.
\]

Since the space \( C^\infty(\overline{\Omega}) \) is dense in \( W^{1,p}(\Omega) \) (cf. Corollary 11) and \( L^{p^\#}(\partial\Omega) \) is complete, there exists a unique continuous linear extension of \( \gamma \) from \( C^\infty(\overline{\Omega}) \) to \( W^{1,p}(\Omega) \). This extension, still denoted as \( \gamma \), is called the trace operator and each \( \gamma(u) \in L^{p^\#}(\partial\Omega) \) is called the trace of the function \( u \in W^{1,p}(\Omega) \).
Theorem

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $1 \leq p < \infty$. The trace operator $\gamma$ satisfies the following:

(a) $\gamma : W^{1,p}(\Omega) \rightarrow L^{p^\#}(\partial \Omega)$ is a continuous linear operator, for $1 \leq p < n$.

(b) For $p = n$, $\gamma : W^{1,n}(\Omega) \rightarrow L^q(\partial \Omega)$ is a continuous linear operator, for all $q \in [1, \infty)$.

(c) For $p > n$, $\gamma : W^{1,p}(\Omega) \rightarrow C(\partial \Omega)$ is a continuous linear operator.

Further, if $1 < p < n$ then the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^q(\partial \Omega)$ is compact for all $q \in [1, p^\#)$.

A consequence of the above Theorem is that, independent of the dimension $n$, the trace map $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial \Omega)$ is a continuous linear map for all $1 \leq p < \infty$. 
Lemma

Let $\Omega := \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$. Then, for any $1 \leq p < +\infty$, and $u \in \mathcal{D}(\mathbb{R}^n_+)$

$$\|\gamma(u)\|_{L^p(\mathbb{R}^{n-1})} \leq p^{\frac{1}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n_+)}.$$  

Proof: Let $u \in \mathcal{D}(\mathbb{R}^n_+)$. For any $x' \in \mathbb{R}^{n-1}$ we have

$$|u(x', 0)|^p = -\int_0^{+\infty} \frac{\partial}{\partial x_n} |u(x', x_n)|^p dx_n \leq p \int_0^{+\infty} |u(x', x_n)|^{p-1} \left| \frac{\partial u}{\partial x_n}(x', x_n) \right| dx_n.$$  

Using Young’s convexity inequality $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ to the following situation:

$$a = \left| \frac{\partial u}{\partial x_n}(x', x_n) \right| \quad \text{and} \quad b = |u(x', x_n)|^{p-1}.$$  

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We obtain
\[
|u(x', 0)|^p \leq p \left[ \int_0^{+\infty} \left( \frac{1}{p} \left| \frac{\partial u}{\partial x_n}(x', x_n) \right|^p + \frac{1}{q} |u(x', x_n)|^{(p-1)q} \right) \, dx_n \right].
\]

By using the relation \((p - 1)q = p\) we obtain
\[
|u(x', 0)|^p \leq (p - 1) \int_0^{+\infty} |v(x', x_n)|^p \, dx_n + \int_0^{+\infty} \left| \frac{\partial u}{\partial x_n}(x', x_n) \right|^p \, dx_n.
\]

Integrating over \(\mathbb{R}^{n-1}\) yields
\[
\int_{\mathbb{R}^{n-1}} |u(x', 0)|^p \, dx' \leq (p - 1) \int_{\mathbb{R}^n_+} |u(x)|^p \, dx + \int_{\mathbb{R}^n_+} \left| \frac{\partial u}{\partial x_n}(x) \right|^p \, dx
\]
\[
\leq (p - 1) \int_{\mathbb{R}^n_+} |u(x)|^p \, dx + \int_{\mathbb{R}^n_+} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p \, dx
\]
\[
\leq p \int_{\mathbb{R}^n_+} \left( |u(x)|^p + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p \right) \, dx.
\]

Hence
\[
\|u(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \leq p^{\frac{1}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n_+)}.
\]
Theorem

Let $\Omega$ be bounded and $\partial \Omega$ is in the class $C^k$. Then there is a bounded linear operator $\gamma : W^{k,p}(\Omega) \to L^p(\partial \Omega)$ such that

1. $\gamma(u) = u \mid_{\partial \Omega}$ if $u \in W^{k,p}(\Omega) \cap C(\overline{\Omega})$,
2. $\|\gamma(u)\|_{p,\partial \Omega} \leq C\|u\|_{1,p,\Omega}$ for all $u \in W^{k,p}(\Omega)$ and $C$ depending on $p$ and $\Omega$.

We call $\gamma(u)$ to be the trace of $u$ on $\partial \Omega$.

Proof: By the regularity property of the boundary $\partial \Omega$, we know that $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. For any $v \in \mathcal{D}(\overline{\Omega})$, we can define the restriction of $u$ to $\partial \Omega$, setting $\gamma(u) := u \mid_{\partial \Omega}$. Suppose we prove that

$$\gamma : (\mathcal{D}(\overline{\Omega}), \| \cdot \|_{W^{1,p}(\Omega)}) \longrightarrow (L^p(\partial \Omega), \| \cdot \|_{L^p(\partial \Omega)})$$

is continuous, then, by the linearity of $\gamma$, it is uniformly continuous. The space $L^p(\partial \Omega)$ is a Banach space. Therefore, there exists a unique linear and continuous extension of $\gamma$

$$\gamma : W^{1,p}(\Omega) \longrightarrow L^p(\partial \Omega).$$
Thus, we just need to prove that $\gamma$ is continuous. By Lemma 16, the result is true for half-space $\Omega = \mathbb{R}^n_+$. Let $\bar{\Omega} \subset \bigcup_{i=0}^{k} G_i$ with $\bar{G}_0 \subset \bar{\Omega}$, $G_i$ open for all $i = 0, \ldots, k$, while $\phi_i : B(0,1) \to G_i$, $i = 1, 2, \ldots, k$, are the local coordinates $\{\alpha_0, \ldots, \alpha_i\}$, is an associated partition of unity, i.e., $\alpha_i \in \mathcal{D}(G_i)$, $\alpha_i \leq 0$, $\sum_{i=0}^{k} \alpha_i = 1$ on $\bar{\Omega}$.

$$w_i = \begin{cases} (\alpha_i v) \circ \phi_i & \text{on } B_+, \\ 0 & \text{on } \mathbb{R}^n_+ \setminus B_+. \end{cases}$$

Clearly $w_i$ belongs to $\mathcal{D}(\mathbb{R}^n_+)$. By Lemma 16, we have

$$\|w_i(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \leq \rho^\frac{1}{p} \|w_i\|_{W^{1,p}(\mathbb{R}^n_+)}. \quad (12.2)$$

By using classical differential calculus rules (note that all the functions $\alpha_i, v, \phi_i$ are continuously differentiable), one obtains the existence, for any $i = 1, \ldots, k$, of a constant $C_i$ such that

$$\|w_i\|_{W^{1,p}(\mathbb{R}^n_+)} \leq C_i \|v\|_{W^{1,p}(\Omega)}. \quad (12.3)$$

Combining the two inequalities (12.2) and (12.3), we obtain

$$\|w_i(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \leq C_i \rho^\frac{1}{p} \|v\|_{W^{1,p}(\Omega)}. \quad (12.4)$$
We now use the definition of the $L^p(\partial \Omega)$ norm which is based on the use of local coordinates. One can show that an equivalent norm to the $L^p(\partial \Omega)$ norm can be obtained by using local coordinates: denoting by $\sim$ the extension by zero outside of $\mathbb{R}^{n-1}\setminus\{y \in \mathbb{R}^{n-1} | |y| < 1\}$, we have that

$$L^p(\partial \Omega) = \{ u : \partial \Omega \to \mathbb{R} | (\tilde{\alpha}_i u) \circ \phi_i(\cdot, 0) \in L^p(\mathbb{R}^{n-1}), 1 \leq i \leq k \}$$

and

$$u \mapsto \left( \sum_{i=1}^{k} \| (\tilde{\alpha}_i u) \circ \phi_i \|^p_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{1}{p}}$$

(12.5)

is an equivalent norm to the $L^p(\partial \Omega)$ norm. This definition of the $L^p(\partial \Omega)$ norm and the inequality (12.4) (note that $w_i = (\tilde{\alpha}_i v) \circ \phi_i$) yield

$$\| u \|_{L^p(\partial \Omega)} \leq C(p, n, \Omega) \| u \|_{W^{1,p}(\Omega)}$$

for some constant $C(p, n, \Omega)$. Thus, $\gamma_0$ is continuous.
**Theorem (range of $\gamma$)**

Let $\Omega$ be an open bounded set in $\mathbb{R}^n$ whose boundary $\partial \Omega$ is of class $C^1$. Then the trace operator $\gamma$ is linear continuous and surjective from $H^1(\Omega) \to H^{\frac{1}{2}}(\partial \Omega)$.

**Proof:** The definition of $H^{\frac{1}{2}}(\partial \Omega)$ is obtained by local coordinates. Thus, it is enough to prove when $\Omega = \mathbb{R}^n_+$ and $\partial \Omega = \mathbb{R}^{n-1}$. We do the proof in three steps.

First step: Let $w(x') = v(x', 0)$. We first relate the Fourier transform (in $\mathbb{R}^{n-1}$) of $w$ to that of $v$ (in $\mathbb{R}^n$). We denote by $\tilde{w}$ the Fourier transform of $w$ in $\mathbb{R}^{n-1}$. By the Fourier inversion formula, if $v \in \mathcal{D}(\mathbb{R}^n)$,

$$
\begin{align*}
v(x', 0) &= \int_{\mathbb{R}^n} e^{2\pi i x' \cdot \xi'} \hat{v}(\xi) \, d\xi, \quad \xi = (\xi', \xi_n) \\
&= \int_{\mathbb{R}^{n-1}} e^{2\pi i x' \cdot \xi'} \left( \int_{-\infty}^{\infty} \hat{v}(\xi) \, d\xi_n \right) \, d\xi'.
\end{align*}
$$

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Now applying the same formula in $\mathbb{R}^{n-1}$, we get

$$v(x', 0) = w(x') = \int_{\mathbb{R}^{n-1}} e^{2\pi i x' \cdot \xi'} \tilde{w}(\xi') d\xi'.$$

By the uniqueness of this formula (since $w, \tilde{w} \in S(\mathbb{R}^{n-1})$ if $v \in \mathcal{D}(\mathbb{R}^n)$) we deduce that

$$\tilde{w}(\xi') = \int_{-\infty}^{\infty} \hat{v}(\xi) d\xi_n.$$
Second step: To show that $v(x', 0) = w(x') \in H^1/2(\mathbb{R}^{n-1})$ we need to show that \((1 + |\xi'|^2)^{1/2} |\widetilde{w}(\xi')|^2\) is integrable. But

\[
\int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} |\widetilde{w}(\xi')|^2 \, d\xi' \\
= \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} \left| \int_{-\infty}^{\infty} \hat{v}(\xi) \, d\xi_n \right|^2 \, d\xi' \\
= \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} \left| \int_{-\infty}^{\infty} \hat{v}(\xi)(1 + |\xi|^2)^{-1/2} (1 + |\xi|^2)^{1/2} \, d\xi_n \right|^2 \, d\xi' \\
\leq \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} \left[ \int_{-\infty}^{\infty} (1 + |\xi|^2) |\hat{v}(\xi)|^2 \, d\xi_n \int_{-\infty}^{\infty} (1 + |\xi|^2)^{-1} \, d\xi_n \right] \, d\xi' \\
= \pi \int_{\mathbb{R}^n} (1 + |\xi|^2) |\hat{v}(\xi)|^2 \, d\xi = \pi \|v\|^2_{H^1(\mathbb{R}^n)}
\]

since

\[
\int_{-\infty}^{\infty} (1 + |\xi|^2)^{-1} \, d\xi_n = \int_{-\infty}^{\infty} \frac{d\xi_n}{1 + |\xi'|^2 + \xi_n^2} = \pi(1 + |\xi'|^2)^{-1/2}.
\]
The above formula is obtained by introducing the change of variable 
\[\xi_n = \left(1 + |\xi'|^2\right)^{\frac{1}{2}} \tan \theta.\] Then

\[
\frac{1}{\left(1 + |\xi'|^2\right)^{\frac{1}{2}}} \int_{-\pi/2}^{\pi/2} \frac{\sec^2 \theta}{1 + \tan^2 \theta} \, d\theta = (1 + |\xi'|^2)^{-\frac{1}{2}} \int_{-\pi/2}^{\pi/2} \, d\theta = \pi (1 + |\xi'|^2)^{-\frac{1}{2}}.
\]

Thus, if \(v \in \mathcal{D}(\mathbb{R}^n), v(x', 0) \in H_{\frac{1}{2}}(\mathbb{R}^{n-1})\) and by density the result follows for \(v \in H^1(\mathbb{R}^n_+).\)

Third step: We now show that \(\gamma\) is onto \(H_{\frac{1}{2}}(\mathbb{R}^{n-1})\). Let \(h(x') \in H_{\frac{1}{2}}(\mathbb{R}^{n-1})\). Let \(\tilde{h}(\xi')\) be its Fourier transform. We define \(u(x', x_n)\) by

\[
\tilde{u}(\xi', x_n) = e^{-(1 + |\xi'|)x_n} \tilde{h}(\xi').
\]

We must first show that \(u\) then belongs to \(H^1(\mathbb{R}_n^+).\) To see this extend \(u\) by zero outside \(\overline{\mathbb{R}^n_+}.\)
Now

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} u(x) \, dx$$

$$= \int_0^\infty \int_{\mathbb{R}^{n-1}} e^{-2\pi i x' \cdot \xi'} u(x', x_n) e^{-2\pi i x_n \xi} \, dx' \, dx_n$$

$$= \int_0^\infty e^{-2\pi i x_n \xi_n} \hat{u}(\xi', x_n) \, dx_n$$

$$= \tilde{h}(\xi') \int_0^\infty e^{-\left(1+|\xi'|+2\pi i x_n \xi_n\right)} \, dx_n$$

$$= \frac{\tilde{h}(\xi')}{1 + |\xi'| + 2\pi i \xi_n}.$$
Now,

\[
\int_{\mathbb{R}^n} (1 + |\xi'|^2)|\widehat{u}(\xi)|^2 \, d\xi \quad = \quad \int_{\mathbb{R}^n} \frac{(1 + |\xi'|)^2 |\tilde{h}(\xi')|^2}{(1 + |\xi'|)^2 + 4\pi^2 \xi_n^2} \, d\xi \\
\leq \quad \int_{\mathbb{R}^n} \frac{(1 + |\xi'|)^2 |\tilde{h}(\xi')|^2}{1 + |\xi'|^2 + \xi_n^2} \, d\xi \\
= \quad \pi \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{\frac{1}{2}} |\tilde{h}(\xi')|^2 \, d\xi' \quad < \quad +\infty
\]

since \( h \in H^\frac{1}{2}(\mathbb{R}^{n-1}) \).

This proves that \( u \) (extended by zero) and \( \frac{\partial u}{\partial x_i} \) are in \( L^2(\mathbb{R}^n) \), for all \( 1 \leq i \leq n - 1 \). Hence, \( u, \frac{\partial u}{\partial x_i} \) are in \( L^2(\mathbb{R}^n) \), for all \( 1 \leq i \leq n - 1 \). For the case \( i = n \), notice that by differentiating under the integral sign,

\[
\left( \frac{\partial u}{\partial x_n} \right) (\xi', x_n) = \frac{\partial \tilde{u}}{\partial x_n} (\xi', x_n) = -(1 + |\xi'|)\tilde{u} (\xi', x_n).
\]

Extend \( \frac{\partial u}{\partial x_n} \) by zero outside \( \mathbb{R}^n_+ \).
Then as before we get

\[
\left( \frac{\partial u}{\partial x_n} \right) (\xi) = \frac{-(1 + |\xi'|)\tilde{h}(\xi')}{1 + |\xi'| + 2\pi \imath \xi_n}.
\]

Then

\[
\int_{\mathbb{R}^n} \left| \left( \frac{\partial u}{\partial x_n} \right) (\xi) \right|^2 d\xi \leq 2 \int_{\mathbb{R}^n} \frac{(1 + |\xi'|^2)|\tilde{h}(\xi')|^2}{1 + |\xi'|^2 + \xi_n^2} d\xi
\]

\[
= 2\pi \int_{\mathbb{R}^n} (1 + |\xi'|^2)^{1/2}|\tilde{h}(\xi')|^2 d\xi' < \infty.
\]

Thus \( \frac{\partial u}{\partial x_n} \) (extended by zero) is in \( L^2(\mathbb{R}^n) \) and so \( \frac{\partial u}{\partial x_n} \in L^2(\mathbb{R}^n_+) \) and hence \( u \in H^1(\mathbb{R}^n_+) \). Now (by the Fourier inversion formula)

\[
\tilde{u}(\xi', 0) = \tilde{h}(\xi')
\]

implies that \( u(x', 0) = h(x') \) and so \( \gamma(u) = h \).
Remark

If \( \nu \in W^{2,p}(\Omega) \), by a similar argument one can give a meaning to \( \frac{\partial \nu}{\partial \nu} \). Note that \( \nabla \nu \in [W^{1,p}(\Omega)]^n \), and hence the trace of \( \nabla \nu \) on \( \partial \Omega \) belongs to \( [L^p(\partial \Omega)]^n \). One defines

\[
\frac{\partial \nu}{\partial \nu} := \gamma(\nabla \nu) \cdot \nu,
\]

which belongs to \( L^p(\partial \Omega) \). Indeed, one can show that

\[
\frac{\partial \nu}{\partial \nu} \in W^{1-\frac{1}{p},p}(\partial \Omega).
\]

For \( p = 2 \), for \( \nu \in H^2(\Omega) \) we have \( \frac{\partial \nu}{\partial \nu} \in H^{1\frac{1}{2}}(\partial \Omega) \). One can also show that the operator \( \nu \mapsto \{ \nu|_{\partial \Omega}, \frac{\partial \nu}{\partial \nu} \} \) is linear continuous and onto from \( W^{2,p}(\Omega) \) onto \( W^{2-\frac{1}{p},p}(\partial \Omega) \times W^{1-\frac{1}{p},p}(\partial \Omega) \).
The relation \( \gamma \left( W^{1,p}(\Omega) \right) \subseteq L^{p^\#}(\partial \Omega) \) is the basis for defining the trace spaces:

\[
W^{1-\frac{1}{p},p}(\partial \Omega) := \{ \gamma(v) \in L^{p^\#}(\partial \Omega) | v \in W^{1,p}(\Omega) \} \quad \text{for } 1 \leq p < n
\]

which is the traces of all the functions in \( W^{1,p}(\Omega), 1 \leq p < n \).

**Remark**

By appropriate modifications one can easily prove that \( \gamma \) maps \( H^k(\mathbb{R}_+^n) \) onto \( H^{k-1/2}(\mathbb{R}^{n-1}) \). In the same way, if \( u \in H^2(\mathbb{R}_+^n) \) one can prove that \( \frac{\partial u}{\partial x_n}(x',0) \) is in \( L^2(\mathbb{R}^{n-1}) \) and is, in fact, in \( H^{1/2}(\mathbb{R}^{n-1}) \). Similarly, one can extend \( -\frac{\partial u}{\partial x_n}(x',0) \) to a bounded linear map \( \gamma_1 : H^2(\mathbb{R}_+^n) \to L^2(\mathbb{R}^{n-1}) \) whose range is \( H^{1/2}(\mathbb{R}^{n-1}) \). More generally, we have a collection of continuous linear maps \( \{ \gamma_j \} \) into \( L^2(\mathbb{R}^{n-1}) \) such that the map \( \gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{m-1}) \) maps \( H^k(\mathbb{R}_+^n) \) into \([L^2(\mathbb{R}^{n-1})]^k\) and the range in the space

\[
\prod_{j=0}^{k-1} H^{k-j-1/2}(\mathbb{R}^{n-1}).
\]
**Definition**

We shall denote the range of the map $T$ to be $W^{k,p}_2(\partial \Omega)$. For $p = 2$, we denote $W^{k,2}_2$ as $H^{k}_2(\partial \Omega)$.

Thus, instead of defining the fractional power Sobolev spaces using Fourier transform, one can define them as range of trace operator $\gamma$.

**Theorem**

$W^{k,p}_2(\partial \Omega)$ is dense in $L^p(\partial \Omega)$.

**Theorem (Trace zero)**

Let $\Omega \subset \mathbb{R}^n$ be bounded and $\partial \Omega$ is in $C^k$ class. Then $u \in W^{k,p}_0(\Omega)$ iff $\gamma(u) = 0$ on $\partial \Omega$. In particular, ker($\gamma$) = $W^{k,p}_0(\Omega)$.
Proof

We first show the inclusion $W^{1,p}_0(\Omega) \subset \ker \gamma$. Take $v \in W^{1,p}_0(\Omega)$. By definition of $W^{1,p}_0(\Omega)$, there exists a sequence of functions $(v_n)_{n \in \mathbb{N}}$, $v_n \in \mathcal{D}(\Omega)$ such that $v_n \rightarrow v$ in $W^{1,p}(\Omega)$. Since $\gamma_0(v_n) = v_n|_{\partial \Omega} = 0$, by continuity of $\gamma_0$ we obtain that $\gamma_0(v) = 0$, i.e., $v \in \ker \gamma_0$.

The other inclusion is a bit more involved. We shall prove it in a sequence of lemma for $H^1(\mathbb{R}^n_+)$. The idea involved is following: Take $v \in W^{1,p}(\mathbb{R}^n_+)$ such that $\gamma_0(v) = 0$. Prove that $v \in W^{1,p}_0(\mathbb{R}^n_+)$. Let us first extend $v$ by zero outside of $\mathbb{R}^n_+$. By using the information $\gamma_0(v) = 0$ one can verify that the so-obtained extension $\tilde{v}$ belongs to $W^{1,p}(\mathbb{R}^n)$. Then, let us translate $\tilde{v}$, and consider for any $h > 0$

$$
\tau_h \tilde{v}(x', x_n) = \tilde{v}(x', x_n - h).
$$

Finally, one regularizes by convolution the function $\tau_h \tilde{v}$. We have that for $\varepsilon$ sufficiently small, $\rho_{\varepsilon} \ast (\tau_h \tilde{v})$ belongs to $\mathcal{D}(\mathbb{R}^n_+)$ and $\rho_{\varepsilon} \ast (\tau_h \tilde{v})$ tends to $v$ in $W^{1,p}(\mathbb{R}^n_+)$ as $h \rightarrow 0$ and $\varepsilon \rightarrow 0$. Hence $v \in W^{1,p}_0(\mathbb{R}^n_+)$.

\qed
Lemma (Green’s formula)

Let $u, v \in H^1(\mathbb{R}^n_+)$. Then

$$
\int_{\mathbb{R}^n_+} u \frac{\partial v}{\partial x_i} = - \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} v \quad \text{if } 1 \leq i \leq (n - 1) \tag{12.7}
$$

$$
\int_{\mathbb{R}^n_+} u \frac{\partial v}{\partial x_n} = - \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_n} v - \int_{\mathbb{R}^{n-1}} \gamma_0(u) \gamma_0(v). \tag{12.8}
$$

Proof: If $u, v \in \mathcal{D}(\mathbb{R}^n)$, then the relations (12.7) and (12.8) follow by integration by parts. The general case follows by the density of the restrictions of functions of $\mathcal{D}(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n_+)$ and the continuity of the map $\gamma_0 : H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^{n-1})$.

Corollary

If $u, v \in H^1(\mathbb{R}^n_+)$ and at least one of them is in ker $(\gamma_0)$ then (12.7) holds for all $1 \leq i \leq n$. 

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Lemma

Let $v \in \ker(\gamma_0)$. Then its extension by zero outside $\mathbb{R}^n_+$, denoted $\tilde{v}$, is in $H^1(\mathbb{R}^n)$, and

$$\frac{\partial \tilde{v}}{\partial x_i} = \left( \frac{\partial v}{\partial x_i} \right), \quad 1 \leq i \leq n.$$  

Proof.

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then for $1 \leq i \leq n$,

$$\int_{\mathbb{R}^n} \tilde{v} \frac{\partial \phi}{\partial x_i} = \int_{\mathbb{R}^n} v \frac{\partial \phi}{\partial x_i} = - \int_{\mathbb{R}^n} \frac{\partial v}{\partial x_i} \phi = \int_{\mathbb{R}^n} \left( \frac{\partial v}{\partial x_i} \right) \phi$$

by the above corollary. Let $h > 0$ and consider $\bar{h} = he_n \in \mathbb{R}^n$ where $e_n$ is the unit vector $(0, 0, \ldots, 0, 1)$. Consider the function $\tau_{\bar{h}} \tilde{v}$, where $\tilde{v}$ is the extension by zero outside $\mathbb{R}^n_+$ of $v \in \ker(\gamma_0)$. Then $\tau_{\bar{h}} \tilde{v}$ vanishes for all $x \in \mathbb{R}^n$ such that $x_n < h$. (Recall that $\tau_{\bar{h}} \tilde{v}(x) = \tilde{v}(x - \bar{h})$.)
**Lemma**

Let $1 \leq p < \infty$ and $\bar{h} \in \mathbb{R}^n$. Then if $f \in L^p(\mathbb{R}^n)$

$$\lim_{\bar{h} \to 0} |\tau_{\bar{h}} f - f|_{0,p,\mathbb{R}^n} = 0$$

**Proof:** By the translation invariance of the Lebesgue measure, $\tau_{\bar{h}} f \in L^p(\mathbb{R}^n)$ as well. Let $\epsilon > 0$ be given and choose $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that

$$|f - \phi|_{0,p,\mathbb{R}^n} < \frac{\epsilon}{3}. \quad (12.9)$$

Let $a > 0$ such that supp $(\phi) \subset [-a, a]^n$. Since $\phi$ is uniformly continuous, there exists $\delta < 0$ be given and choose $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that

$$|\phi(x - \bar{h}) - \phi(x)| < \frac{\epsilon}{3} (2(a + 1))^{-\frac{n}{p}}.$$
Then
\[ \int_{\mathbb{R}^n} |\phi(x - \overline{h}) - \phi(x)|^p \, dx = \int_{[-(a+1),(a+1)]^n} |\phi(x - \overline{h}) - \phi(x)|^p \, dx < \left( \frac{\epsilon}{3} \right)^p. \]

Thus for $|\overline{h}| < \delta$,
\[ |\tau_{\overline{h}}\phi - \phi|_{0,p,\mathbb{R}^n} < \frac{\epsilon}{3}. \tag{12.10} \]

Finally, again by the translation invariance of the Lebesgue measure, we have
\[ |\tau_{\overline{h}}f - \tau_{\overline{h}}\phi|_{0,p,\mathbb{R}^n} = f - \phi|_{0,p,\mathbb{R}^n} < \frac{\epsilon}{3}. \tag{12.11} \]

The result now follows on combining (12.9), (12.10) and (12.11) by the triangle inequality. \(\square\)
Corollary

If $v \in H^1(\mathbb{R}^n)$ then

$$\lim_{h \to 0} \|\tau_h v - v\|_{1,\mathbb{R}^n} = 0.$$ 

Proof.

Clearly by the preceding lemma $\tau_h v \to v$ in $L^2(\mathbb{R}^n)$. Also it is easy to check that for any $1 \leq i \leq n$,

$$\frac{\partial}{\partial x_i}(\tau_h v) = \tau_h \frac{\partial v}{\partial x_i}.$$ 

Thus again by the lemma, $\frac{\partial(\tau_h v)}{\partial x_i} \to \frac{\partial v}{\partial x_i}$ in $L^2(\mathbb{R}^n)$.

\qed
**Theorem**

\[ \ker(\gamma_0) = H^1_0(\mathbb{R}^n_+) \].

**Proof:** We already have seen that \( H^1_0(\mathbb{R}^n_+) \subset \ker(\gamma_0) \). Let now \( \nu \in \ker(\gamma_0) \). Then we have seen that its extension \( \tilde{\nu} \) by zero is in \( H^1(\mathbb{R}^n) \). Using the cut-off functions \( \{\zeta_k\} \) (cf. Theorem 2.1.3) we have that \( \zeta_k \tilde{\nu} \rightarrow \tilde{\nu} \) as \( k \rightarrow \infty \) in \( H^1(\mathbb{R}^n) \). The functions \( \zeta_k \tilde{\nu} \) have compact support in \( \mathbb{R}^n \) and vanish for \( x_n < 0 \). Now fix such a \( k \) so that

\[ \| \tilde{\nu} - \zeta_k \tilde{\nu} \|_{1, \mathbb{R}^n} < \eta, \]

where \( \eta > 0 \) is a given positive quantity. Again we can choose \( h \) small enough so that if \( \bar{h} = he_n \), then

\[ \| \tau_{\bar{h}}(\zeta_k \tilde{\nu}) - \zeta_k \tilde{\nu} \|_{1, \mathbb{R}^n} < \eta. \]

Now \( \tau_{\bar{h}}(\zeta_k \tilde{\nu}) \) has compact support in \( \mathbb{R}^n_+ \) and vanishes for all \( x \in \mathbb{R}^n \) with \( x_n < h \). Let \( \{\rho_\epsilon\} \) be the family of mollifiers.
If \( \epsilon > 0 \) is chosen small enough then \( \rho_\epsilon \ast \tau_\bar{h}(\zeta_k \tilde{v}) \) will have support contained in the set

\[
B(0; \epsilon) + K \cap \{ x | x_n \geq h > 0 \}
\]

where \( K = \text{supp}(\tau_\bar{h}(\zeta_k \tilde{v})) \) is compact. Thus

\[
\rho_\epsilon \ast \tau_\bar{h}(\zeta_k) \in \mathcal{D}(\mathbb{R}^n_+)\n\]

and we know that as \( \epsilon \downarrow 0 \), \( \rho_\epsilon \ast \tau_\bar{h}(\zeta_k \tilde{v}) \rightarrow \tau_\bar{h}(\zeta_k \tilde{v}) \).

Thus we can choose \( \epsilon \) small enough such that

\[
\| \rho_\epsilon \ast \tau_\bar{h}(\zeta_k \tilde{v}) - \tau_\bar{h}(\zeta_k \tilde{v}) \|_{1, \mathbb{R}^n} < \eta.
\]

Thus we have found a function \( \phi_\eta \in \mathcal{D}(\mathbb{R}^n_+) \),

\[
\phi_\eta = \rho_\epsilon \ast \tau_\bar{h}(\zeta_k \tilde{v})
\]

such that

\[
\| \phi_\eta - v \|_{1, \mathbb{R}^n_+} \leq \| \phi_\eta - \tilde{v} \|_{1, \mathbb{R}^n} < 3\eta.
\]

Thus, as \( \eta \) is arbitrary, it follows that

\[
\ker(\gamma_0) \subset \overline{\mathcal{D}(\mathbb{R}^n_+)} = H^1_0(\mathbb{R}^n_+).
\]
Similarly, it can be proved that if \( \gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{m-1}) \), then the kernel of \( \gamma \) in \( H^m(\mathbb{R}^n_+) \) is precisely the set \( H^1_0(\mathbb{R}^n_+) \).

**Theorem**

Let \( 1 \leq p < \infty \). Then

\[
W^{2,p}_0(\Omega) = \left\{ u \in W^{2,p}(\Omega) \mid \gamma(u) = 0 \text{ and } \sum_{i=1}^{n} \nu_i \gamma\left( \frac{\partial u}{\partial x_i} \right) = 0 \right\},
\]

where \( (\nu_i)_{i=1}^{n} \) denotes the unit outer normal vector field along \( \partial \Omega \).
Theorem (Trace Theorem)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $C^{m+1}$ with boundary $\partial \Omega$. Then there exists a trace map $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{m-1})$ from $H^m(\Omega)$ into $(L^2(\Omega))^m$ such that

1. If $v \in C^\infty(\bar{\Omega})$, Then $\gamma_0(v) = v|_{\partial \Omega}$, $\gamma_1(v) = \frac{\partial v}{\partial \nu}|_{\partial \Omega}$, $\cdots$, and $\gamma_{m-1}(v) = \frac{\partial^{m-1} v}{\partial \nu^{m-1}}|_{\partial \Omega}$, where $\nu$ is the unit exterior normal to the boundary $\partial \Omega$.

2. The range of $\gamma$ is the space

$$\prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\partial \Omega).$$

3. The kernel of $\gamma$ is $H_0^m(\Omega)$.
Proof

Let us now turn to the case of a bounded open set $\Omega$ of class $C^1$. Let
\[ \{ U_j, T_j \}_{j=1}^k \]
be an associated local chart for the boundary $\partial \Omega$ and let
\[ \{ \psi_j \}_{j=1}^k \]
be a partition of unity subordinate to the cover $\{ U_j \}$ of $\partial \Omega$.

If $u \in H^1(\Omega)$, then $(\psi_j u|_{U_j \cap \Omega}) \circ T_j \in H^1(\mathbb{R}^n_+)$ and so we can define its trace as an element of $H^{1/2}(\mathbb{R}^{n-1})$. Coming back by $T_j^{-1}$ we can define the trace on $U_j \cap \partial \Omega$.

Piecing these together we get the trace $\gamma_0 u$ in $L^2(\partial \Omega)$ and the image (by definition of the spaces) will be precisely $H^{1/2}(\partial \Omega)$.

Similarly if the boundary is smoother we can define the higher order traces $\gamma_j$. \qed
Green’s Identity

- Now that we have given a meaning to the functions restricted to the boundary of the domain, we intend to generalise the classical Green’s identities to $H^1(\Omega)$.

- The trace theorem above helps us to obtain Green’s theorem for functions in $H^1(\Omega)$, $\Omega$ of class $C^1$. If $\nu(x)$ denotes the unit outer-normal vector on the boundary $\partial\Omega$ (which is defined uniquely a.e. on $\partial\Omega$), we denote its components along the coordinate axes by $\nu_i(x)$. Thus we write generically,

$$\nu = (\nu_1, \cdots, \nu_n).$$

- For example, if $\Omega = B(0; 1)$ then $\nu(x) = x$ for all $|x| = 1$. Thus $\nu_i(x) = x_i$ in this case. If $\Omega = B(0; R)$, then $\nu(x) = \frac{x}{R}$. If $\Omega$ has a part of its boundary, say, $x_n = 0$, then the unit outer normal is $\pm e_n$ depending on the side on which $\Omega$ lies.
Theorem (Green’s Theorem)

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ set of class $C^1$ lying on the same side of its boundary $\partial \Omega$. Let $u, v \in H^1(\Omega)$. Then for $1 \leq i \leq n$,

$$
\int_{\Omega} u \frac{\partial v}{\partial x_i} = - \int_{\Omega} \frac{\partial u}{\partial x_i} v + \int_{\partial \Omega} (\gamma_0 u)(\gamma_0 v) \nu_i.
$$

(12.12)

Proof.

Recall that $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$. If $u_m, v_m \in C^\infty(\bar{\Omega})$ then we have by the classical Green’s Theorem

$$
\int_{\Omega} u_m \frac{\partial v_m}{\partial x_i} = - \int_{\Omega} \frac{\partial u_m}{\partial x_i} v_m + \int_{\partial \Omega} u_m v_m \nu_i
$$

and choosing $u_m \to u, v_m \to v$ in $H^1(\Omega)$ we deduce (12.12) by the continuity of the trace map $\gamma_0$.

Usually, we rewrite $\gamma_0(v)$ as just $v$ on $\partial \Omega$ and understand it as the trace of $v$ on $\partial \Omega$. Similarly if $\gamma_1 v$ is defined we will write it as $\frac{\partial v}{\partial \nu}$. 

Theorem
Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a $C^1$ boundary. Given a vector field $V = (v_1, \ldots, v_n)$ on $\Omega$ such that $v_i \in H^1(\Omega)$ for all $1 \leq i \leq n$, then

$$
\int_{\Omega} \nabla \cdot V \, dx = \int_{\partial \Omega} V \cdot \nu \, d\sigma.
$$

Proof.
Setting $u \equiv 1$ and $v = v_i \in H^1(\Omega)$, we get from (12.12)

$$
\int_{\Omega} \frac{\partial v_i}{\partial x_i} = \int_{\Omega} v_i \nu_i.
$$

and if $V = (v_i) \in (H^1(\Omega))^n$, we get on summing with respect to $i$,

$$
\int_{\Omega} \text{div} \, (V) = \int_{\partial \Omega} V \cdot \nu \, d\sigma
$$

which is the Gauss Divergence Theorem. \qed
Corollary

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a $C^1$ boundary. Let $u, v \in H^2(\Omega)$, then

(i) \[
\int_{\Omega} (v\Delta u + \nabla v \cdot \nabla u) \, dx = \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} \, d\sigma,
\]

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\Delta := \nabla \cdot \nabla$.

(ii) \[
\int_{\Omega} (v\Delta u - u\Delta v) \, dx = \int_{\partial \Omega} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, d\sigma.
\]

Proof: If we have $u \in H^2(\Omega)$ and use $\frac{\partial u}{\partial x_i}$ in place of $u$ in (12.12), we get

\[
\int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} = - \int_{\Omega} \frac{\partial^2 u}{\partial x_i^2} v + \int_{\partial \Omega} \frac{\partial u}{\partial x_i} v \nu_i.
\]
If $u$ were smooth then \[ \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \nu_i = \frac{\partial u}{\partial \nu}. \] Thus by continuity of the trace $\gamma_1$, we get, for $u \in H^2(\Omega)$, $v \in H^1(\Omega)$,

\[ \int_{\Omega} \nabla u \cdot \nabla v = -\int_{\Omega} (\Delta u) v + \int_{\partial \Omega} v \frac{\partial u}{\partial \nu}. \]

\[ \square \]

**Remark**

The Green’s formula holds for $u \in W^{1,p}(\Omega)$ and $a \in W^{1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.