Data Detection in Single User Massive MIMO with Re-Transmissions

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Introduction

- Single user massive MIMO is used to increase the spectral efficiency
  - Spatial multiplexing
- Large number of antennas in tx and rx in mm-wave freq
  - Antenna size is small
- Operating SNR per bit is not known
  - Nobody can see what is inside a mobile phone anyway

System Model

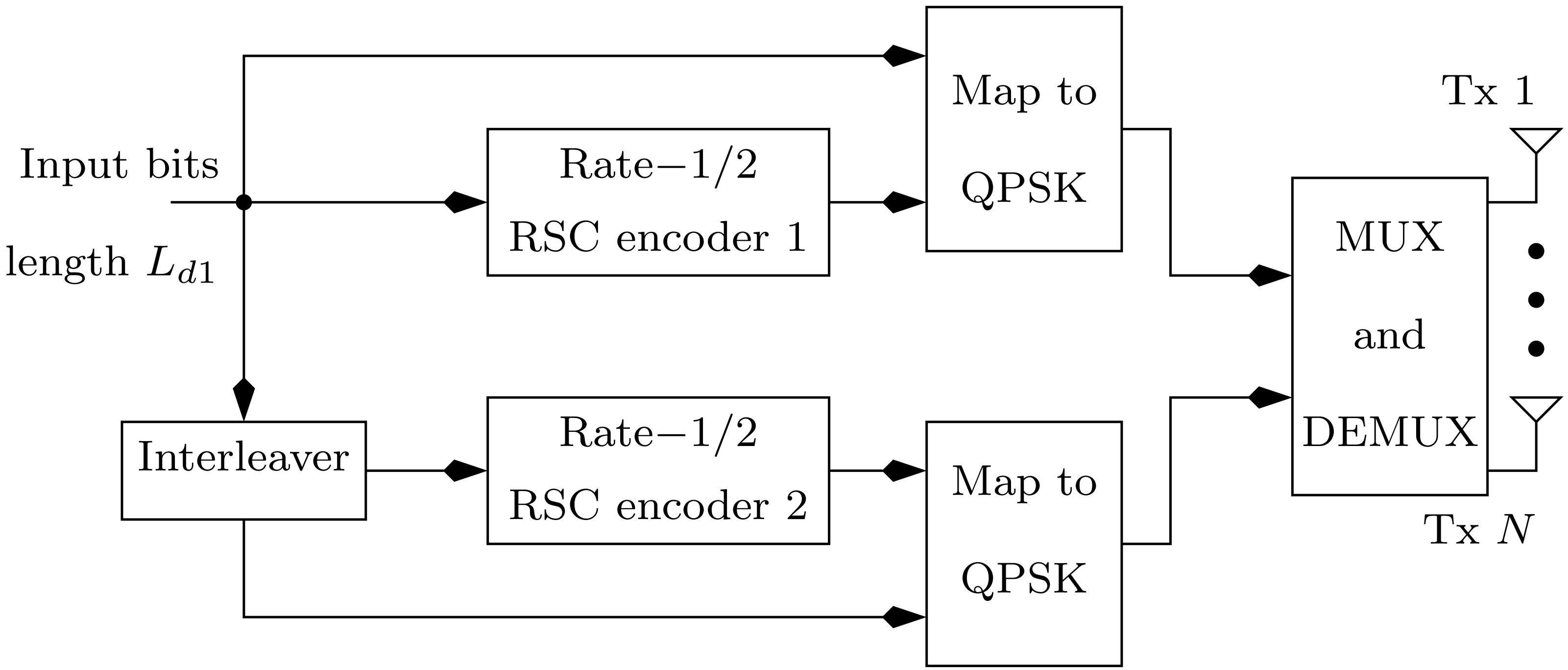

Figure 1: Transmitter.

System Model

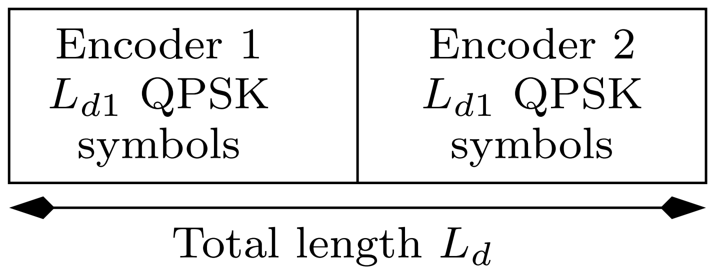

Figure 2: The frame structure.

System Model

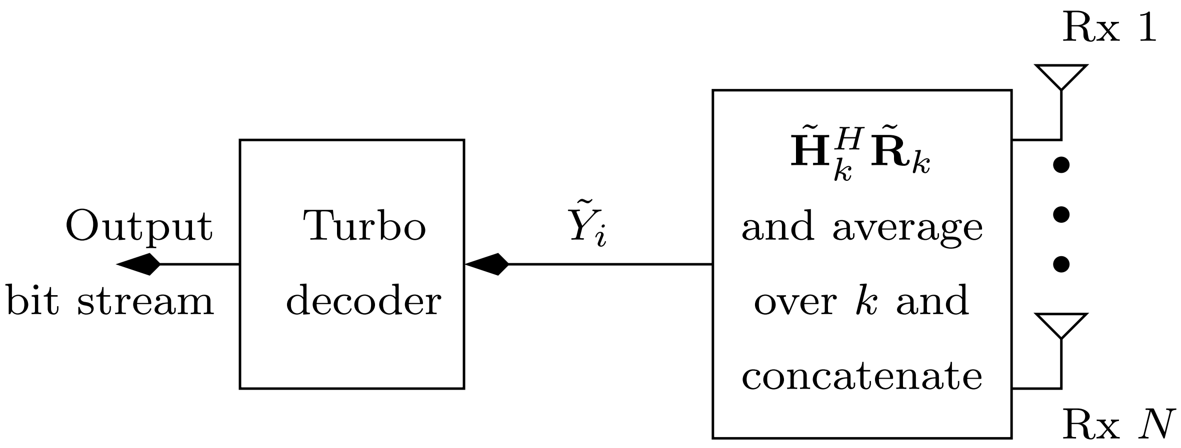

Figure 3: Receiver.

Signal Model

- The received signal in the k^{th} ($0 \leq k \leq N_{rt} - 1$, k is an integer), re-transmission is given by

$$\tilde{\mathbf{R}}_k = \tilde{\mathbf{H}}_k \mathbf{S} + \tilde{\mathbf{W}}_k \tag{1}$$

Signal Model

- $\tilde{\mathbf{R}}_k \in \mathbb{C}^{N\times 1}$ – received vector
- $\tilde{\mathbf{H}}_k \in \mathbb{C}^{N\times N}$ – channel matrix
- $\tilde{\mathbf{W}}_k \in \mathbb{C}^{N\times 1}$ – AWGN vector
- The transmitted symbol vector is $\mathbf{S} \in \mathbb{C}^{N\times 1}$, M-ary constellation
- Boldface letters denote vectors or matrices
- Complex quantities are denoted by a tilde
- Tilde is not used for $\mathbf{S}$

Signal Model

- $$\frac{1}{2}E\left[\left|\tilde{H}_{k,\,i,\,j}\right|^2\right] = \sigma_H^2 \tag{2}$$

- $$\frac{1}{2}E\left[\left|\tilde{W}_{k,\,i}\right|^2\right] = \sigma_W^2 \tag{3}$$

- $$\begin{aligned} \frac{1}{2}E\left[\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_l^H\right] &= N\sigma_H^2\delta_K(k-l)\mathbf{I}_N \\ \frac{1}{2}E\left[\tilde{\mathbf{W}}_k\tilde{\mathbf{W}}_l^H\right] &= \sigma_W^2\delta_K(k-l)\mathbf{I}_N \end{aligned} \tag{4}$$

Problem Statement

- Find $\mathbf{S}$ given $\tilde{\mathbf{R}}_k$
  - ML solution – complexity $\mathscr{O}(M^N)$
  - Inversion of $\tilde{\mathbf{H}}_k$ (zero-forcing solution) requires $\mathscr{O}(N^3)$ operations – noise enhancement
  - Sphere decoding

The Proposed Solution

- Consider

$$\begin{aligned}\tilde{\mathbf{Y}}_k &= \tilde{\mathbf{H}}_k^H \tilde{\mathbf{R}}_k \\ &= \tilde{\mathbf{F}}_k \mathbf{S} + \tilde{\mathbf{V}}_k \end{aligned} \tag{5}$$

where

$$\begin{aligned}\tilde{\mathbf{F}}_k &= \tilde{\mathbf{H}}_k^H \tilde{\mathbf{H}}_k \\ \tilde{\mathbf{V}}_k &= \tilde{\mathbf{H}}_k^H \tilde{\mathbf{W}}_k.\end{aligned} \tag{6}$$

The Proposed Solution

- Observe that

$$\frac{1}{2}E\left[\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_l\right] = N\sigma_H^2\delta_K(k-l)\mathbf{I}_N. \tag{7}$$

- However

$$\frac{1}{2}\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_l \neq N\sigma_H^2\delta_K(k-l)\mathbf{I}_N. \tag{8}$$

- Replace $E[\cdot]$ by time-averaging – using re-transmissions

The Proposed Solution

- Now

$$\tilde{Y}_{k,\,i} = \tilde{F}_{k,\,i,\,i} S_i + \tilde{I}_{k,\,i} + \tilde{V}_{k,\,i} \quad \text{for } 0 \le i \le N-1 \tag{9}$$

The Proposed Solution

where

$$
\begin{aligned}
\tilde{F}_{k,\,i,\,i} &= \sum_{j=1}^{N} \left| \tilde{H}_{k,\,j,\,i} \right|^2 \\
\tilde{I}_{k,\,i} &= \sum_{j=1,\,j\neq i}^{N} \tilde{F}_{k,\,i,\,j} S_j \\
\tilde{F}_{k,\,i,\,j} &= \sum_{l=1}^{N} \tilde{H}^{*}_{k,\,l,\,i} \tilde{H}_{k,\,l,\,j} \\
\tilde{V}_{k,\,i} &= \sum_{j=1}^{N} \tilde{H}^{*}_{k,\,j,\,i} \tilde{W}_{k,\,j}.
\end{aligned}
\tag{10}
$$

The Proposed Solution

- For large of N, $\tilde{I}_{k,\,i}$ and $\tilde{V}_{k,\,i}$ are Gaussian distributed
- Since S_i and $\tilde{W}_{k,\,i}$ are independent, $\tilde{I}_{k,\,i}$ and $\tilde{V}_{k,\,i}$ are uncorrelated

The Proposed Solution

$$
\begin{aligned}
E\left[\tilde{I}_{k,\,i}\tilde{V}^*_{k,\,i}\right] &= E\left[\left(\sum_{j=1,\,j\neq i}^{N}\tilde{F}_{k,\,i,\,j}S_j\right)\left(\sum_{l=1}^{N}\tilde{H}^*_{k,\,l,\,i}\tilde{W}_{k,\,l}\right)^*\right] \\
&= \sum_{j=1,\,j\neq i}^{N}\sum_{l=1}^{N}E\left[\tilde{F}_{k,\,i,\,j}\tilde{H}_{k,\,l,\,i}\right]E\left[S_j\right]E\left[\tilde{W}^*_{k,\,l}\right] \\
&= 0. \qquad (11)
\end{aligned}
$$

The Proposed Solution

- Let

$$\tilde{U}'_{k,\,i} = \tilde{I}_{k,\,i} + \tilde{V}_{k,\,i} \tag{12}$$

- We have

$$\begin{aligned} E\left[\left|\tilde{U}'_{k,\,i}\right|^2\right] &= E\left[\left|\tilde{I}_{k,\,i}\right|^2\right] + E\left[\left|\tilde{V}_{k,\,i}\right|^2\right] \\ &\stackrel{\Delta}{=} \sigma^2_{U'}. \end{aligned} \tag{13}$$

The Proposed Solution

- The noise power is

$$
\begin{aligned}
E\left[\left|\tilde{V}_{k,\,i}\right|^2\right] &= E\left[\left(\sum_{m=1}^{N} \tilde{H}_{k,\,m,\,i}^{*}\tilde{W}_{k,\,m}\right)\left(\sum_{n=1}^{N} \tilde{H}_{k,\,n,\,i}\tilde{W}_{k,\,n}^{*}\right)\right] \\
&= \sum_{m=1}^{N}\sum_{n=1}^{N} E\left[\tilde{H}_{k,\,n,\,i}\tilde{H}_{k,\,m,\,i}^{*}\right] E\left[\tilde{W}_{k,\,m}\tilde{W}_{k,\,n}^{*}\right] \\
&= \sum_{m=1}^{N}\sum_{n=1}^{N} 2\sigma_H^2\delta_K(m-n)2\sigma_W^2\delta_K(m-n) \\
&= 4N\sigma_H^2\sigma_W^2. \qquad (14)
\end{aligned}
$$

The Proposed Solution

$$
\begin{aligned}
E\left[\left|\tilde{I}_{k,\,i}\right|^2\right] &= E\left[\left(\sum_{m=1,\,m\neq i}^{N} \tilde{F}_{k,\,i,\,m} S_m\right)\left(\sum_{n=1,\,n\neq i}^{N} \tilde{F}^*_{k,\,i,\,n} S^*_n\right)\right] \\
&= \sum_{m=1,\,m\neq i}^{N} \sum_{n=1,\,n\neq i}^{N} E\left[\tilde{F}_{k,\,i,\,m} \tilde{F}^*_{k,\,i,\,n}\right] E\left[S_m S^*_n\right] \\
&= \sum_{m=1,\,m\neq i}^{N} \sum_{n=1,\,n\neq i}^{N} E\left[\tilde{F}_{k,\,i,\,m} \tilde{F}^*_{k,\,i,\,n}\right] P_{\mathrm{av}} \delta_K(m-n) \\
&= \sum_{m=1,\,m\neq i}^{N} E\left[\left|\tilde{F}_{k,\,i,\,m}\right|^2\right] P_{\mathrm{av}} \\
&= 8N(N-1)\sigma_H^4 \qquad (15)
\end{aligned}
$$

The Proposed Solution

where

$$
\begin{aligned}
E\left[\left|S_m\right|^2\right] &= P_{\mathrm{av}} \\
&= 2
\end{aligned}
\qquad (16)
$$

The Proposed Solution

$$\begin{aligned}
E\left[\left|\tilde{F}_{k,\,i,\,m}\right|^2\right] &= E\left[\left(\sum_{j=1}^{N}\tilde{H}^*_{k,\,j,\,i}\tilde{H}_{k,\,j,\,m}\right)\left(\sum_{l=1}^{N}\tilde{H}_{k,\,l,\,i}\tilde{H}^*_{k,\,l,\,m}\right)\right] \\
&= \sum_{j=1}^{N}\sum_{l=1}^{N} E\left[\tilde{H}_{k,\,l,\,i}\tilde{H}^*_{k,\,j,\,i}\tilde{H}_{k,\,j,\,m}\tilde{H}^*_{k,\,l,\,m}\right] \\
&= \sum_{j=1}^{N}\sum_{l=1}^{N} E\left[\tilde{H}_{k,\,l,\,i}\tilde{H}^*_{k,\,j,\,i}\right] E\left[\tilde{H}_{k,\,j,\,m}\tilde{H}^*_{k,\,l,\,m}\right] \\
&= \sum_{j=1}^{N}\sum_{l=1}^{N} 2\sigma_H^2\delta_K(l-j)2\sigma_H^2\delta_K(j-l) \\
&= 4N\sigma_H^4. \qquad (17)
\end{aligned}$$

The Proposed Solution

- Hence

$$\sigma^2_{U'} = 4N\sigma^2_H\sigma^2_W + 8N(N-1)\sigma^4_H. \tag{18}$$

- Consider

$$\begin{aligned}\tilde{Y}_i &= \frac{1}{N_{rt}}\sum_{k=0}^{N_{rt}-1}\tilde{Y}_{k,\,i} \\ &= F_i S_i + \tilde{U}_i \qquad \text{for } 0 \le i \le N-1.\end{aligned} \tag{19}$$

The Proposed Solution

where

$$\begin{aligned} F_i &= \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \tilde{F}_{k,\,i,\,i} \\ \tilde{U}_i &= \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \tilde{U}'_{k,\,i}. \end{aligned} \tag{20}$$

The Proposed Solution

- Note that

$$\begin{aligned} E\left[\left|\tilde{U}_i\right|^2\right] &= \frac{\sigma^2_{U'}}{N_{rt}} \\ &= \frac{1}{N_{rt}}\left(4N\sigma^2_H\sigma^2_W + 8N(N-1)\sigma^4_H\right) \\ &\stackrel{\Delta}{=} \sigma^2_U. \end{aligned} \tag{21}$$

The Proposed Solution

- The average SNR-per-bit for each rx antenna is

$$\begin{aligned}
\mathrm{SNR}_{\mathrm{av},\,b} &= 10\log_{10}\left(\frac{E\left[\left|\sum_{j=0}^{N-1}\tilde{H}_{k,\,i,\,j}S_j\right|^2\right]\times 2N_{rt}}{E\left[\left|\tilde{W}_{k,\,i}\right|^2\right]}\right) \\
&= 10\log_{10}\left(\frac{2N\sigma_H^2\times 2\times 2N_{rt}}{2\sigma_W^2}\right) \\
&= 10\log_{10}\left(\frac{4NN_{rt}\sigma_H^2}{\sigma_W^2}\right). \qquad (22)
\end{aligned}$$

The Proposed Solution

- We have

$$\frac{\sigma_W^2}{N_{rt}} = \frac{4N\sigma_H^2}{10^{0.1\,\mathrm{SNR}_{\mathrm{av},\,b}}}. \tag{23}$$

The Proposed Solution

- Thus

$$\begin{aligned}
\sigma_U^2 &= \frac{4N\sigma_H^2\sigma_W^2}{N_{rt}} + \frac{8\sigma_H^4 N(N-1)}{N_{rt}} \\
&= \underbrace{\frac{4N\sigma_H^2 \cdot \left(4N\sigma_H^2\right)}{10^{0.1\,\mathrm{SNR}_{\mathrm{av},\,b}}}}_{\text{Noise power constant for a given SNR}} \\
&+ \underbrace{\frac{8\sigma_H^4 N(N-1)}{N_{rt}}}_{\text{Interference power reduces with increasing } N_{rt}} \\
&= \frac{16N^2\sigma_H^4}{10^{0.1\mathrm{SNR}_{\mathrm{av},\,b}}} + \frac{8\sigma_H^4 N(N-1)}{N_{rt}}. \qquad (24)
\end{aligned}$$

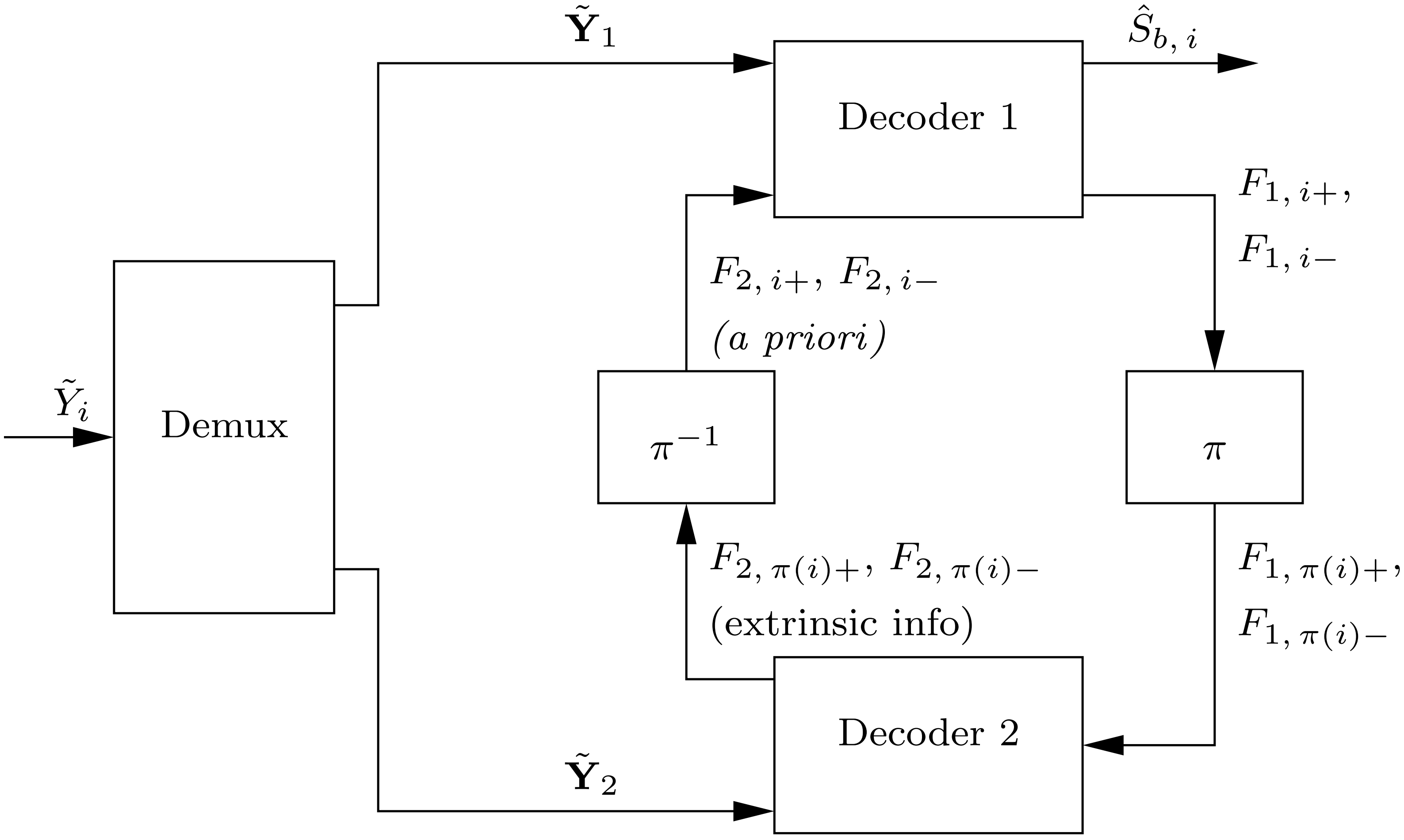

Figure 4: Turbo decoder.

Turbo Decoding – BCJR Algorithm

- We have

$$\begin{aligned}\tilde{\mathbf{Y}}_1 &= \begin{bmatrix} \tilde{Y}_0 & \ldots & \tilde{Y}_{L_{d1}-1} \end{bmatrix} \\ \tilde{\mathbf{Y}}_2 &= \begin{bmatrix} \tilde{Y}_{L_{d1}} & \ldots & \tilde{Y}_{L_d-1} \end{bmatrix}.\end{aligned} \quad (25)$$

- The BCJR algorithm has the following components:
  - Forward recursion
  - Backward recursion
  - Computation of extrinsic information and the final *a posteriori* probabilities.

Turbo Decoding – BCJR Algorithm

- $\mathscr{S}$ is the number of states in the encoder trellis
- $\mathscr{D}_n$ is the set of states that diverge from state n
- $\mathscr{C}_n$ is the set of states that converge to state n
- $\alpha_{i,\,n}$ is the forward SOP at time i $(0 \le i \le L_{d1} - 2)$ at state n
- $\beta_{i,\,n}$ is the backward SOP at time i $(1 \le i \le L_{d1} - 1)$ at state n

Turbo Decoding – BCJR Algorithm

- Forward SOP

$$
\begin{aligned}
\alpha'_{i+1,\,n} &= \sum_{m \in \mathscr{C}_n} \alpha_{i,\,m} \gamma_{1,\,i,\,m,\,n} P\left(S_{b,\,i,\,m,\,n}\right) \\
\alpha_{0,\,n} &= 1 \qquad \text{for } 0 \le n \le \mathscr{S} - 1 \\
\alpha_{i+1,\,n} &= \alpha'_{i+1,\,n} \Bigg/ \left(\sum_{n=0}^{\mathscr{S}-1} \alpha'_{i+1,\,n} \right)
\end{aligned}
\qquad (26)
$$

Turbo Decoding – BCJR Algorithm

- where

$$P\left(S_{b,\,i,\,m,\,n}\right) = \begin{cases} F_{2,\,i+} & \text{if } S_{b,\,i,\,m,\,n} = +1 \\ F_{2,\,i-} & \text{if } S_{b,\,i,\,m,\,n} = -1 \end{cases} \tag{27}$$

$$\gamma_{1,\,i,\,m,\,n} = \exp\left[-\frac{\left|\tilde{Y}_i - F_i S_{m,\,n}\right|^2}{2\sigma_U^2}\right] \tag{28}$$

Turbo Decoding – BCJR Algorithm

- Backward SOP

$$\begin{aligned}
\beta'_{i,\,n} &= \sum_{m \in \mathscr{D}_n} \beta_{i+1,\,m} \gamma_{1,\,i,\,n,\,m} P\left(S_{b,\,i,\,n,\,m}\right) \\
\beta_{L_{d1},\,n} &= 1 \qquad \text{for } 0 \le n \le \mathscr{S} - 1 \\
\beta_{i,\,n} &= \beta'_{i,\,n} \Bigg/ \left(\sum_{n=0}^{\mathscr{S}-1} \beta'_{i,\,n}\right).
\end{aligned} \tag{29}$$

Turbo Decoding – BCJR Algorithm

- $\rho^+(n)$ is the state that is reached from state n when the input symbol is $+1$
- $\rho^-(n)$ is the state that is reached from state n when the input symbol is -1

Turbo Decoding – BCJR Algorithm

- The extrinsic information from decoder 1 to 2 is calculated as follows for $0 \leq i \leq L_{d1} - 1$

$$\begin{aligned} G_{\mathrm{norm},\, i+} &= \sum_{n=0}^{\mathscr{S}-1} \alpha_{i,\, n} \gamma_{1,\, i,\, n,\, \rho^+(n)} \beta_{i+1,\, \rho^+(n)} \\ G_{\mathrm{norm},\, i-} &= \sum_{n=0}^{\mathscr{S}-1} \alpha_{i,\, n} \gamma_{1,\, i,\, n,\, \rho^-(n)} \beta_{i+1,\, \rho^-(n)} \end{aligned} \quad (30)$$

Turbo Decoding – BCJR Algorithm

- which is further normalized to obtain

$$\begin{aligned} F_{1,\,i+} &= G_{\mathrm{norm},\,i+} \Big/ \left(G_{\mathrm{norm},\,i+} + G_{\mathrm{norm},\,i-}\right) \\ F_{1,\,i-} &= G_{\mathrm{norm},\,i-} \Big/ \left(G_{\mathrm{norm},\,i+} + G_{\mathrm{norm},\,i-}\right). \end{aligned} \tag{31}$$

Turbo Decoding – BCJR Algorithm

- The MAP recursions for the second decoder – $\gamma_{1,\, i,\, m,\, n}$ is replaced by

$$\gamma_{2,\, i,\, m,\, n} = \exp\left[-\frac{\left|\tilde{Y}_{i1} - F_{i1} S_{m,\, n}\right|^2}{2\sigma_U^2}\right] \tag{32}$$

where

$$i1 = i + L_{d1} \qquad \text{for } 0 \le i \le L_{d1} - 1. \tag{33}$$

- $F_{1,\, i+}$, $F_{1,\, i-}$ is replaced by $F_{2,\, i+}$, $F_{2,\, i-}$

Turbo Decoding – BCJR Algorithm

- Final *a posteriori* probabilities of the i^{th} data bit (for $0 \leq i \leq L_{d1} - 1$):

$$\begin{aligned} P\left(S_{b,\,i} = +1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right) &= G_{\mathrm{norm},\,i+} F_{2,\,i+} \\ P\left(S_{b,\,i} = -1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right) &= G_{\mathrm{norm},\,i-} F_{2,\,i-} \end{aligned} \tag{34}$$

Turbo Decoding – BCJR Algorithm

- Final estimate of the i^{th} data bit is

$$
\begin{aligned}
&\text{Choose } \hat{S}_{b,\, i} = +1 \text{ if} \\
&P\left(S_{b,\, i} = +1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right) > P\left(S_{b,\, i} = -1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right) \\
&\text{Choose } \hat{S}_{b,\, i} = -1 \text{ if} \\
&P\left(S_{b,\, i} = -1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right) > P\left(S_{b,\, i} = +1 | \tilde{\mathbf{Y}}_1,\, \tilde{\mathbf{Y}}_2\right).
\end{aligned} \tag{35}
$$

Channel Capacity

- Consider the signal

$$\tilde{r}_i = \tilde{x}_i + \tilde{w}_i \tag{36}$$

- The channel capacity is

$$C = \log_2 (1 + \text{SNR}) \qquad \text{bits per transmission} \tag{37}$$

over a complex dimension.

- The average SNR is

$$\text{SNR} = \frac{E\left[\left|\tilde{x}_i\right|^2\right]}{E\left[\left|\tilde{w}_i\right|^2\right]} = \frac{P'_{\text{av}}}{2\sigma_w^2} \tag{38}$$

over a complex dimension.

Channel Capacity

Proposition 0.1 *The channel capacity is additive over the number of complex dimensions. In other words, the channel capacity over N complex dimensions, is equal to the sum of the capacities over each complex dimension, provided the information is independent across the complex dimensions. Independence of information also implies that, the bits transmitted over one complex dimension is not the interleaved version of the bits transmitted over any other complex dimension.*

Channel Capacity

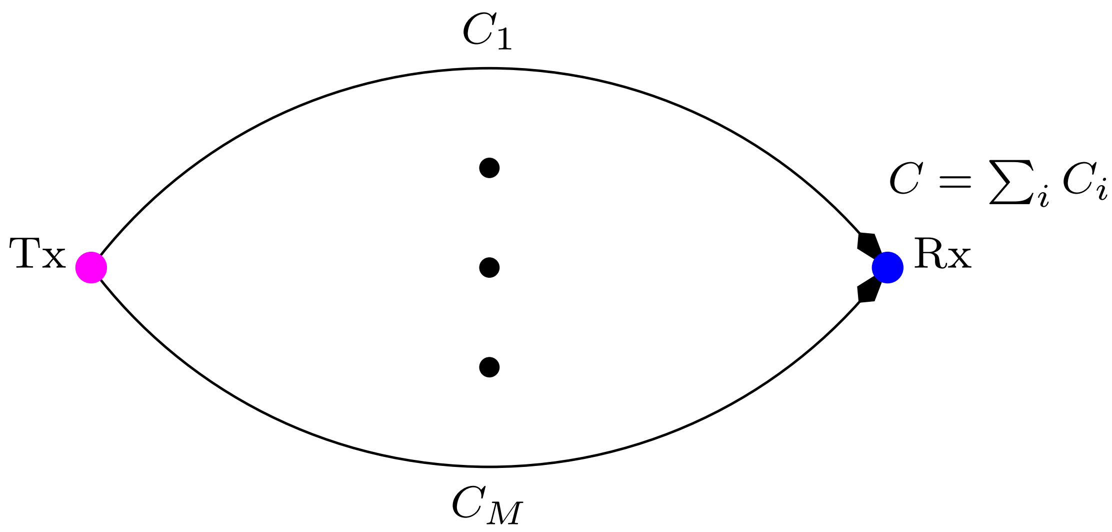

Figure 5: Proposition 1.

Channel Capacity

Proposition 0.2 *Conversely, if C bits per transmission are sent over N complex dimensions, it seems reasonable to assume that each complex dimension receives C/N bits per transmission*

Channel Capacity

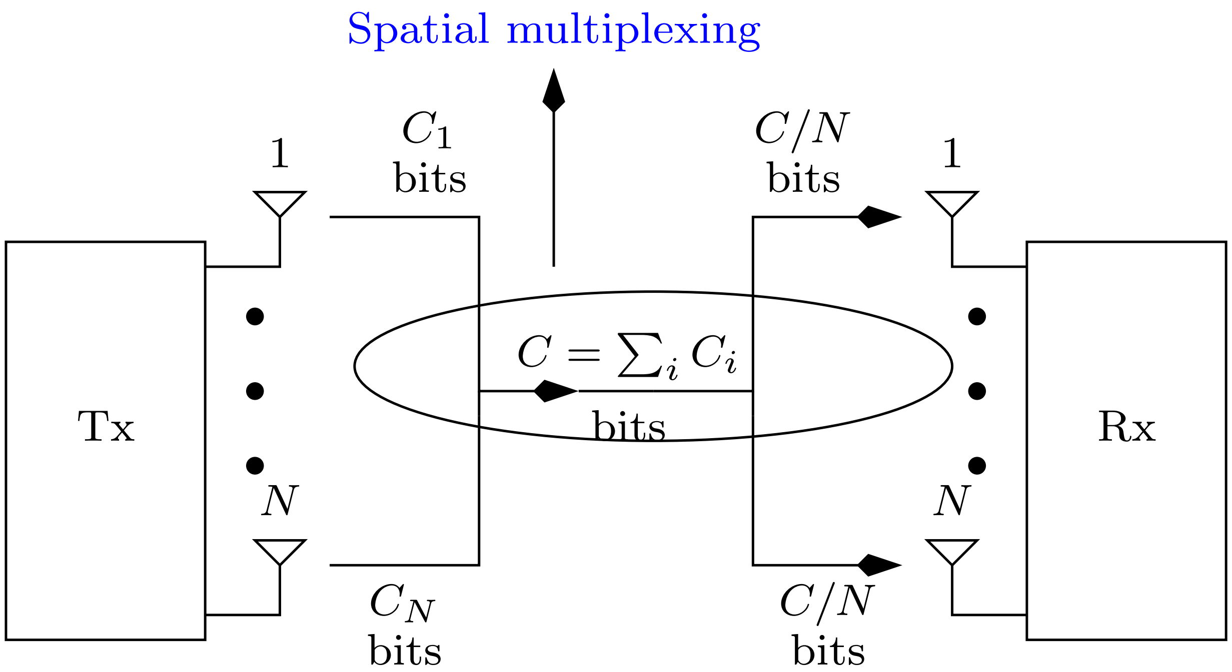

Figure 6: Proposition 2.

Channel Capacity

- The i^{th} element of $\tilde{\mathbf{R}}_k$ in (1) is

$$\tilde{R}_{k,\,i} = \sum_{j=1}^{N} \tilde{H}_{k,\,i,\,j} S_j + \tilde{W}_{k,\,i}. \tag{39}$$

- Now substitute

$$\begin{aligned} \tilde{x}_i &= \sum_{j=1}^{N} \tilde{H}_{k,\,i,\,j} S_j \\ \tilde{w}_i &= \tilde{W}_{k,\,i} \end{aligned} \tag{40}$$

Channel Capacity

- The average SNR at each rx antenna is

$$\text{SNR} = \frac{2NP_{\text{av}}\sigma_H^2}{2\sigma_W^2}. \tag{41}$$

- $C = 1/(2N_{rt})$ bits/transmission at each rx antenna
- The average SNR per bit is

$$\text{SNR}_{\text{av},\,b} = \frac{2NP_{\text{av}}\sigma_H^2 \cdot 2N_{rt}}{2\sigma_W^2} = \frac{\text{SNR}}{C} \tag{42}$$

Channel Capacity

- Capacity relation at each rx antenna is

$$C = \log_2 \left(1 + C\, \text{SNR}_{\text{av},\, b}\right) \qquad \text{bits per transmission} \tag{43}$$

- Re-arranging terms in (43) we get

$$\text{SNR}_{\text{av},\, b} = \frac{2^C - 1}{C}. \tag{44}$$

- As $C \to 0$, $\text{SNR}_{\text{av},\, b} \to \ln(2) \equiv -1.6$ dB.

Table 1: Simulation parameters.

Parameter	Value
L_{d1}	512
L_d	1024
N	1, 16, 512
N_{rt}	1, 2, 4
No of frames simulated	10^5, 10^6
No of turbo decoder iterations	8

Simulation Parameters

- 4-state turbo code with generating matrix given by

$$\mathbf{G}(D) = \begin{bmatrix} 1 & \frac{1+D^2}{1+D+D^2} \end{bmatrix}. \tag{45}$$

- 16-state turbo code with generating matrix given by

$$\mathbf{G}(D) = \begin{bmatrix} 1 & \frac{1+D^2+D^3+D^4}{1+D+D^4} \end{bmatrix}. \tag{46}$$

- Spectral efficiency $N/(2N_{rt})$ bits/transmission

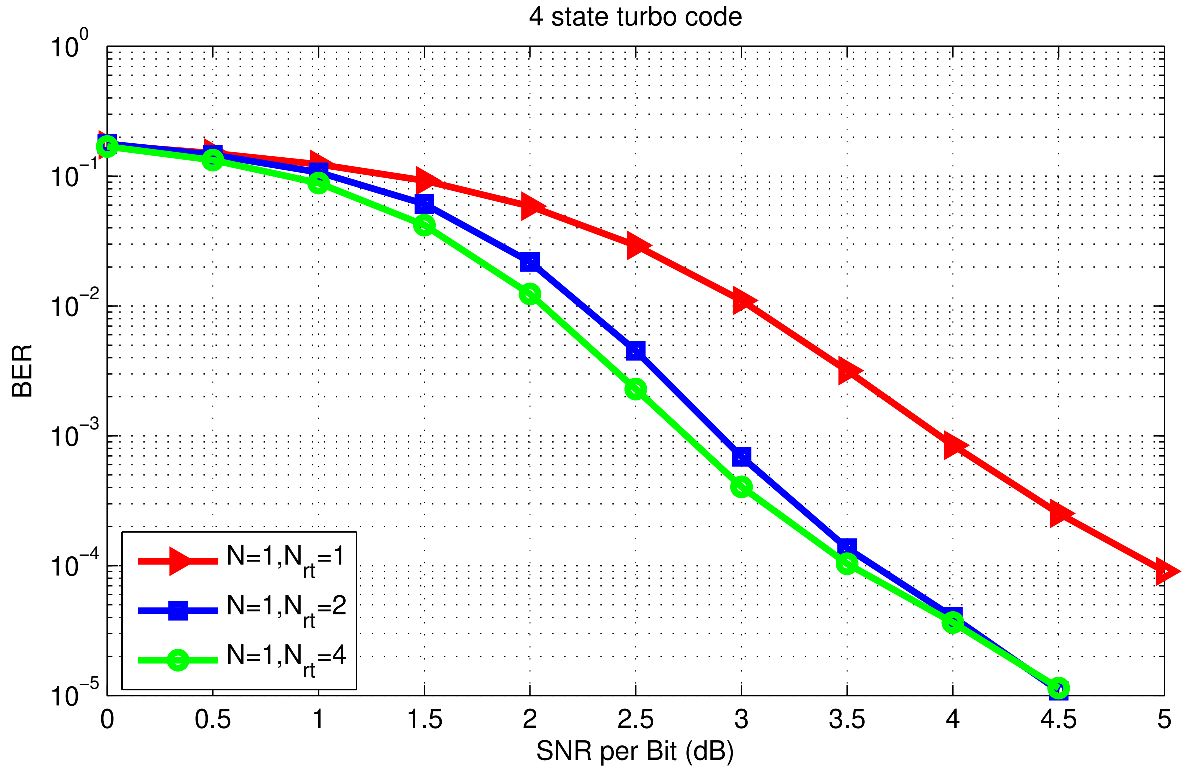

Figure 7: Results for 4-state turbo code, $N = 1$.

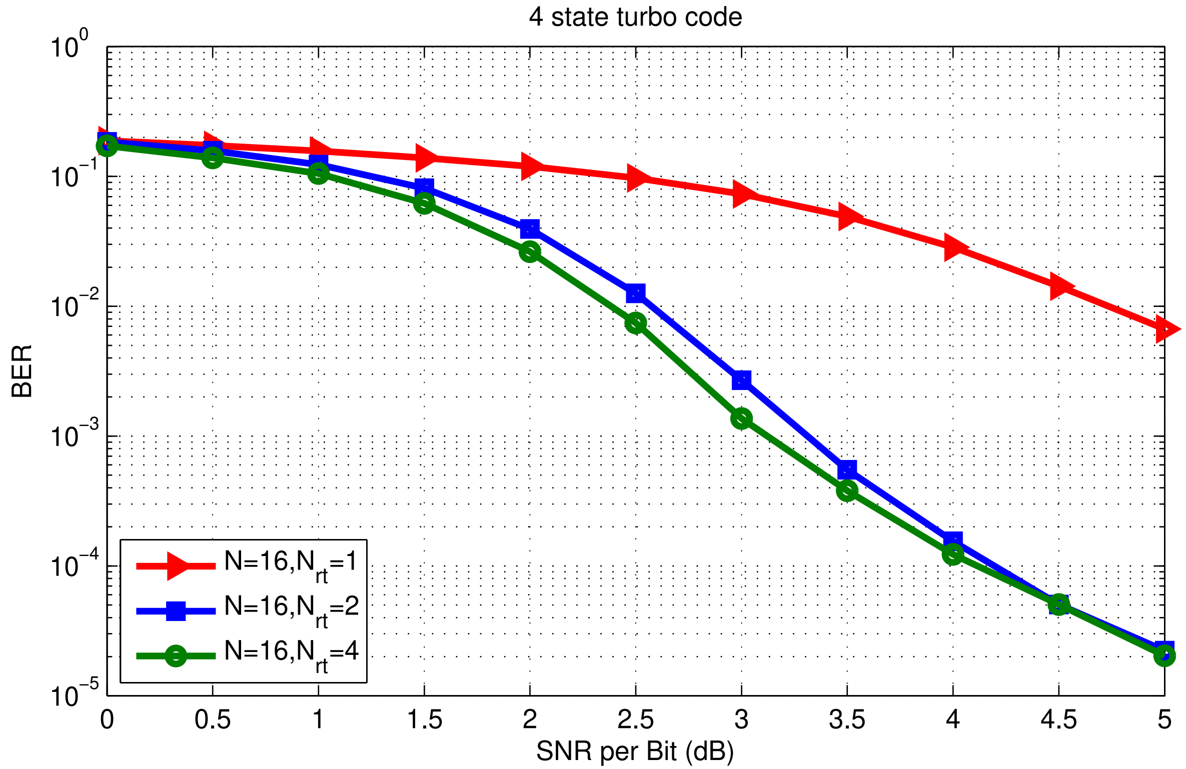

Figure 8: Results for 4-state turbo code, $N = 16$.

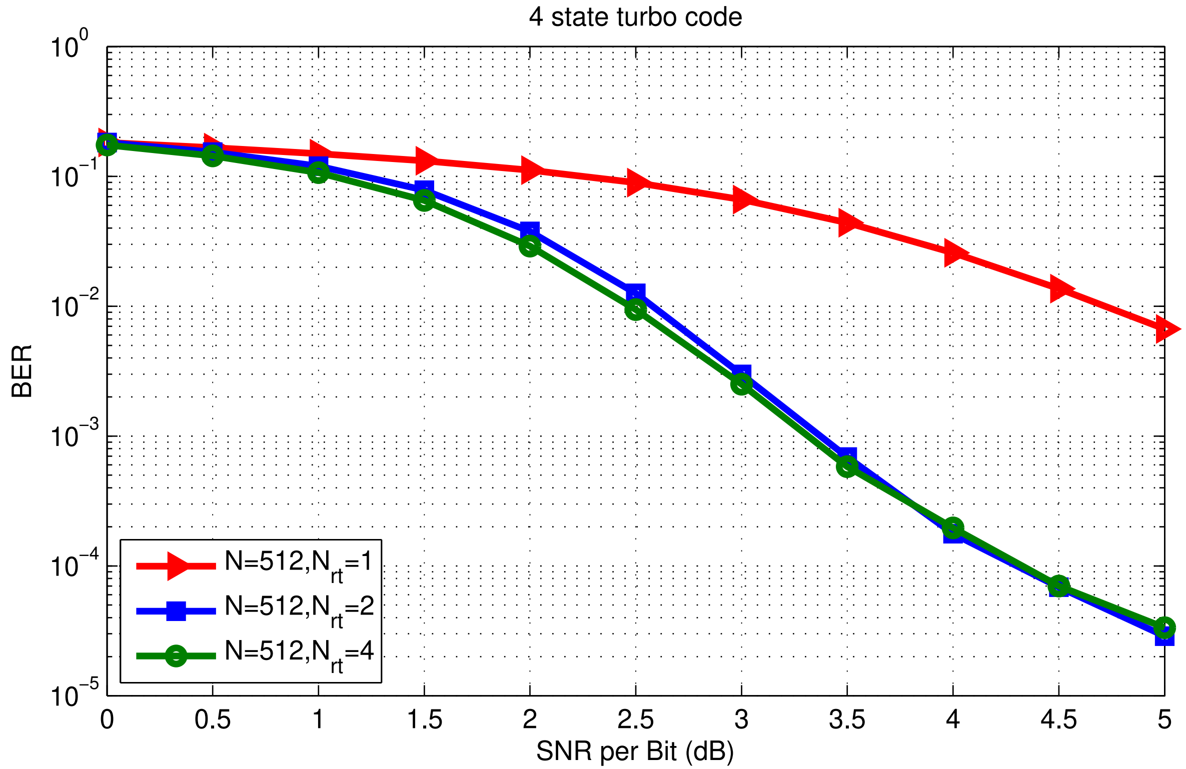

Figure 9: Results for 4-state turbo code, $N = 512$.

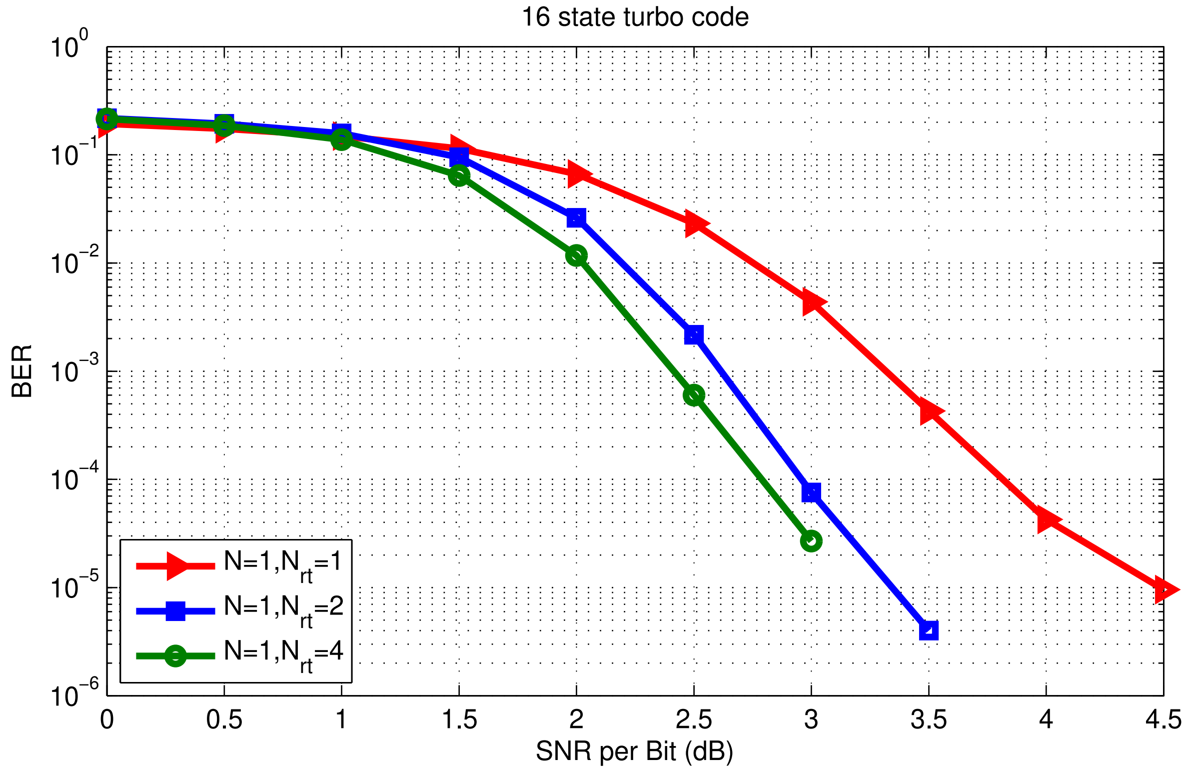

Figure 10: Results for 16-state turbo code, $N = 1$.

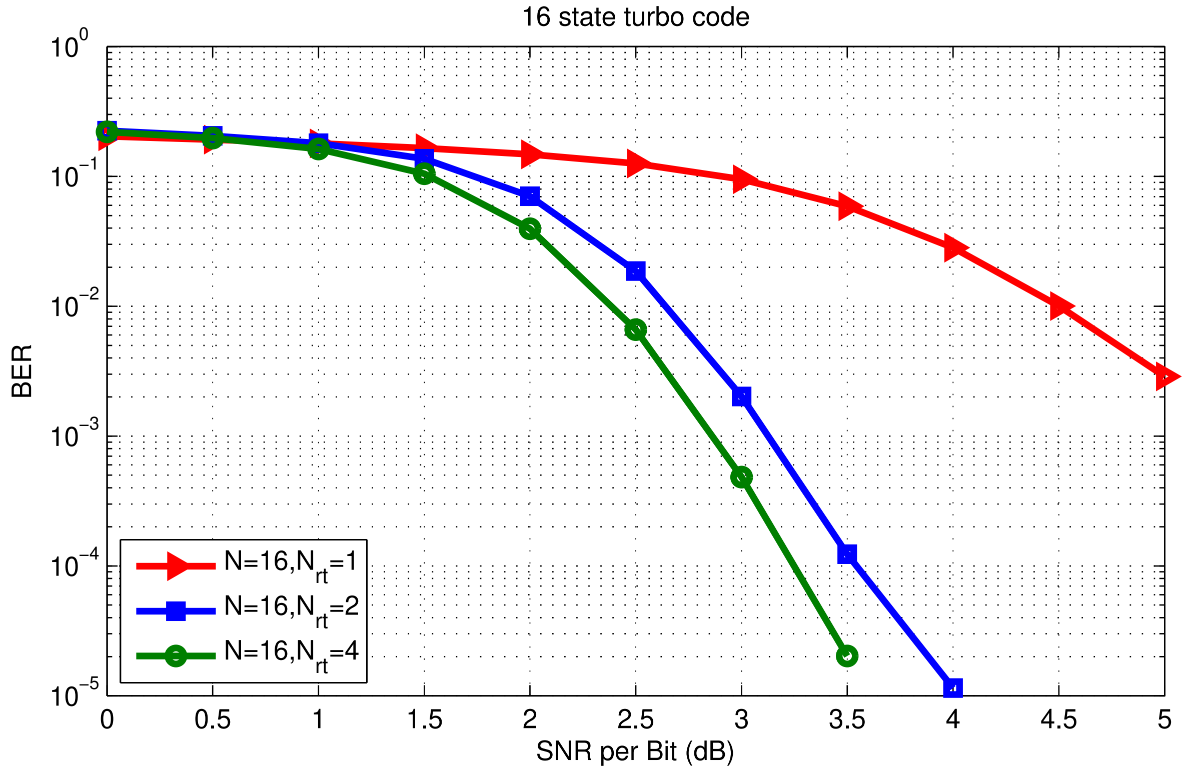

Figure 11: Results for 16-state turbo code, $N = 16$.

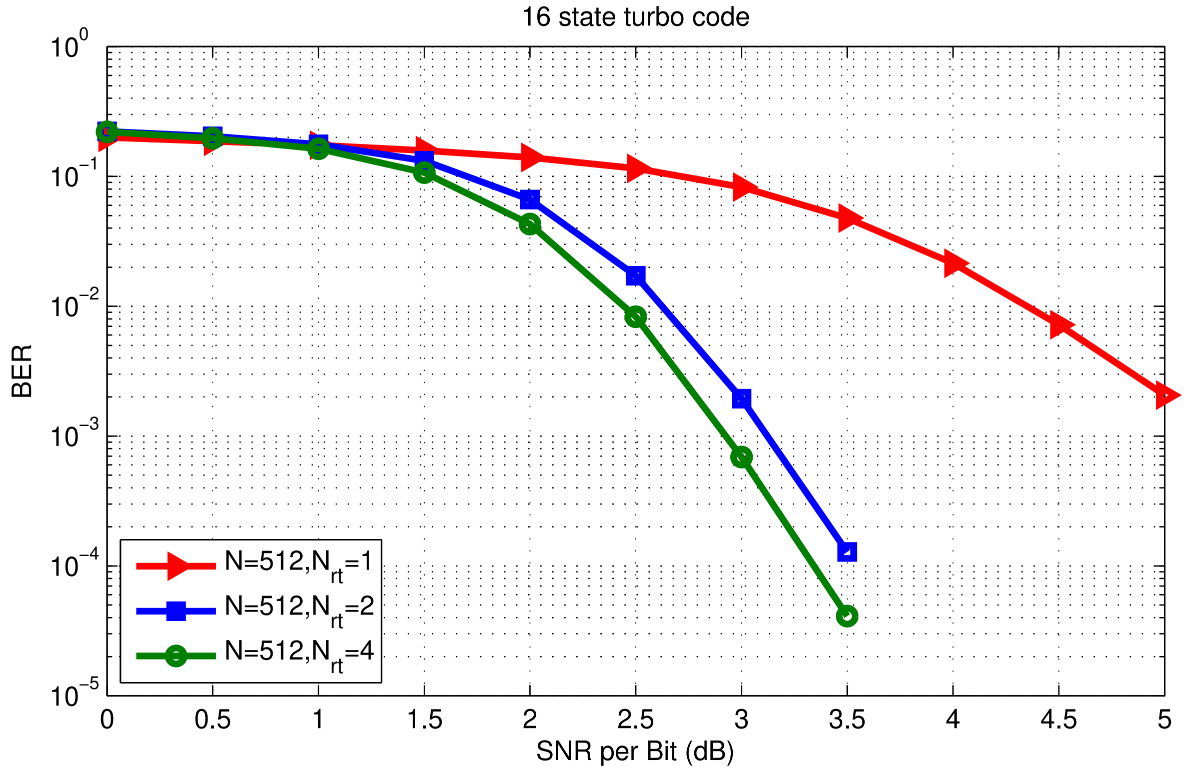

Figure 12: Results for 16-state turbo code, $N = 512$.

Conclusions & Future Work

- A simple method of data detection in massive MIMO proposed
- Future work – to simulate a pratical massive MIMO system