A note on ductility for probabilistic seismic design of structures

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SUMMARY – This paper proposes a stochastic approach to estimate the sectional ductility demand of a member and the overall ductility demand in a structure when it is subjected to earthquake excitations with known statistical characteristics. For this, the excitation and the structural response are assumed to be zero mean, Gaussian processes. This assumption is approximately true for a mildly non-linear system with a few inelastic excursions. The proposed method takes into account the number of excursions of the response into non-linear range as a parameter, besides considering the random nature of the excitation. It has been shown how the concepts developed here can be applied to the single-degree-of-freedom and multi-degree-of-freedom systems.

1. Introduction

In case of the earthquake excited structures, both the input and the response are random in nature. Hence, any measure of ductility defined as the ratio of maximum to yield level response inherently becomes a random variable.

An earthquake excitation is different from a cyclic loading as generally considered in the ductility studies, since it is broad-band in nature and consists of several frequency components with highly random amplitudes and phases. Thus any single realization of the excitation process and the system response to it are, in effect, combinations of different realizations of the amplitude and phase variables for various frequencies. For these reasons, the response to monotonic loading, cyclic loading or even to the cyclic loading with varying amplitude and a single frequency component, cannot in reality simulate the complex nature of response. The response and hence the ductility ratio estimated in these situations may thus involve appreciable errors. Further, the option of having a reasonable statistical estimate of ductility by considering a large number of compatible accelerograms is computationally intensive and it cannot be generalized to other cases of interest.

The first effort to provide a probabilistic basis to the case of a yielding elastoplastic system was by Karnopp and Scharton (1966) who obtained an estimate of average rate of energy dissipation when the system is subjected to Gaussian white noise. Based on their approach, Vanmarcke (1969) and Vanmarcke and Veneziano (1973) obtained the probability distributions and statistical characteristics of plastic drift and ductility factor. Assuming a point crossing process in time, with accumulation of drift from each crossing, they obtained the probability of maximum drift and then ductility factor from the first passage problem in time. However, their approach has not accounted for the statistical dependence between the maximum level, yield level and the number of non-linear crossings.

There have been further attempts to account for the randomness and complex nature of the input as well as that of the output of a structural system with reference to ductility. Murakami and Penzien (1977) carried out non-deterministic response analysis on five single-degree-of-freedom (SDOF) systems using stochastic ground motion models. Ridell et al. (1989) studied the reduction factors for the non-linear behaviour of a SDOF system using a large number of recorded ground motions. Miranda (1992) also reported a statistical study based on several ground motions for different soil conditions to obtain the ductility reduction factors. Nakamura and Takewaki (1989) and Hwang and Jaw (1990) assumed ductility as a random variable and tried to obtain its statistical characteristics. It can be observed that growing emphasis has been laid by different researchers on accounting for the randomness in the response. However, their approaches have lacked gene-

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rality regarding the ground motion and response characteristics. Thus, there remains a need for evolving a more rational and generalized probabilistic description of ductility, and for relating it to the statistical characteristics of the response, particularly in the case of the multi-degree-of-freedom (MDOF) systems.

In this paper, we have used a conditional order statistics description of the response peaks to determine the probability density functions for ductility ratio, for a given number of non-linear excursions. Two separate formulations have been proposed with the condition respectively on the linear and maximum levels of response. The density functions for ductility and the expected ductility estimates for a given number of non-linear excursions have been calculated for both the cases. A SDOF model and a MDOF system are used to illustrate the application of the concepts developed in the formulation.

2. Probabilistic ductility ratio

Ductility is defined as the ratio of maximum to yield deformation. To have a probabilistic estimate of ductility, it is required to have the conditional distributions of maximum or yield deformations. In turn, these conditional distributions respectively are the distributions of the largest peak and a higher order peak in the response process. Recalling the concepts of order statistics, it may be noted that the ordered peaks are mutually dependent, even though the unordered peaks are considered to be independent. This is because they satisfy the relationship, $X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n-1)} \geq X_{(n)}$ where, n = number of peaks, and $X_{(i)}$ = i^{th} order peak variable; $i = 1, 2, 3, \ldots, n$. Thus, for obtaining the conditional distribution of the largest peak, with condition on the yield level, it becomes necessary to find the joint distribution functions of the ordered variables corresponding to the yield and maximum levels. By a transformation of variables, this further leads to the conditional distribution for ductility. Similarly, the mutual dependence of the maximum and the yield levels can be accounted for to obtain the conditional distribution of ductility with condition on the maximum level. Following is the formulation for conditional density of ductility, with condition first on the yield level and then on the maximum level.

i) condition on the yield level

Let X_i, $i = 1, 2, \ldots, n$ be the n number of peaks occuring in the response process, $X(t)$. $X(t)$ is assumed to be Gaussian and it may, for example, denote the roof displacement, rotation or curvature of a section in a structure. These response peaks are assumed to be identically and independently distributed random variables. On being normalized by the root-mean-square (r.m.s.) value of the process, they can be characterized by a probability density function (p.d.f.) given by Cartwright and Longuet-Higgins (1956) following Rice (1944, 1945), and written as

$$p(\eta) = \frac{1}{\sqrt{2\pi}}\left[\epsilon e^{-\eta^2/2\epsilon^2} + (1-\epsilon^2)^{1/2}\eta e^{-\eta^2/2}\int_{-\infty}^{\eta(1-\epsilon^2)^{1/2}/\epsilon} e^{-x^2/2}dx\right] \quad (1)$$

where, η is the normalized peak value, and

$$\epsilon = \left[\frac{m_0 m_4 - m_2^2}{m_0 m_4}\right]^{1/2} \quad (2)$$

is a measure of the distribution of energy in various frequencies of the process. m_n is, in general, the n^{th} moment of the energy spectrum, $E(\omega)$, where ω is the frequency. It is defined by

$$m_n = \int_0^{\infty} \omega^n E(\omega)\mathrm{d}\omega;\ n = 0, 1, 2, \ldots. \quad (3)$$

For $\epsilon = 0$, $p(\eta)$ corresponds to the Rayleigh distribution, and for $\epsilon = 1$, it becomes a normal distribution. Let $P(\eta)$ denote the cumulative distribution function of the response peaks where

$$P(\eta) = \int_{-\infty}^{\eta} p(\eta)\mathrm{d}\eta. \quad (4)$$

Let the ordered formation of the response peaks be denoted by $X_{k:n}$ (or $X_{(k)}$); $k = 1, 2, \ldots, n$, and the number of excursions beyond a specified level be i, in a sample of n response peaks. Thus, to arrive at the p.d.f. of the ductility ratio for a given number of non-linear excursions, the p.d.f. of $X_{(1)}$ (i.e. the largest peak) conditioned on $X_{(i+1)} = b$, the yield level, may be obtained as

$$f(X_{(1)} = a \mid X_{(i+1)}) = b)\ or\ f_{X_{(1)}|X_{(i+1)}=b}(a \mid b) = \frac{f_{(i+1),(1)}(b, a)}{f_{(i+1)}(b)} \quad (5)$$

where, $f_{(i+1)(1)}(b, a)$ is the joint p.d.f. of the $(i + 1)^{th}$ and 1^{st} order peaks, and $f_{(i+1)}(b)$ is the marginal density function of the $(i + 1)^{th}$ order peak. To obtain the joint density function, $f_{(i+1),(1)}(b, a))$ let us consider the real axis and partition it as shown in Fig. 1, with one peak each placed at levels, a and b, and $(i - 1)$ number of peaks placed in between a and b. $(n - i - 1)$ peaks are placed below b while there is no peak above a. Making use of the multinomial combination to get the density function for continuous, independent random variables, the joint density function is obtained as (David (1980))

$$f_{(i+1),(1)}(b, a) = \frac{n!}{(n-i-1)!(i-1)!}[P(b)]^{n-i-1}p(b)$$
$$[P(a) - P(b)]^{i-1}p(a). \quad (6)$$

For the univariate density function of the $(i + 1)^{th}$ peak, one peak is placed at b, i peaks are above b and the rest $(n - i - 1)$ peaks are below b, as shown in Fig. 2. Thus, the density function for $X_{(i+1)}$ can be written as (David (1980))

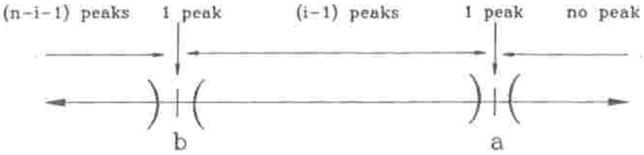

Fig. 1 – Arrangement of peaks for the joint density of the 1st and $(i+1)^{th}$ peaks.

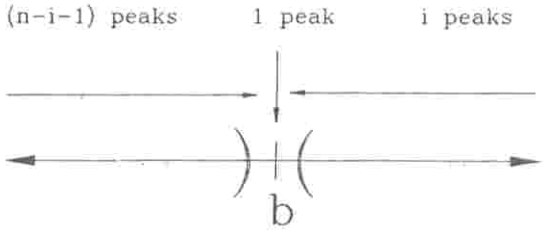

Fig. 2 – Arrangement of peaks for the density function of $(i+1)^{th}$ peak.

$$f_{(i+1)}(b) = \frac{n!}{(n-i-1)!\,i!}[P(b)]^{n-i-1}[1-P(b)]^{i}\,p(b). \qquad (7)$$

On using Eqs. (5), (6) and (7), the conditional density can be written as

$$f_{X_{(1)}|X_{(i+1)}=b}(a \mid b) = i\frac{[P(a)-P(b)]^{i-1}\,p(a)}{[1-P(b)]^{i}};\; b \le a < \infty. \qquad (8)$$

The above expression gives the density function for the largest peak, $X_{(1)}$, on the condition that $X_{(i+1)} = b$. Therefore, the conditional probability density function for the variable, $X_{(1)}/X_{(i+1)}$ to have the value a/b, on the condition that $X_{(i+1)} = b$, would be same as given by Eq. (8). Since the yield level, $X_{(i+1)}$ corresponds to i number of non-linear excursions, the ratio $X_{(1)}/X_{(i+1)}$ gives the ductility for i number of non-linear excursions. Let it be denoted by μ_i. By a variable transformation, $a = \mu b$, the conditional p.d.f. for μ_i becomes

$$f_{\mu_i|X_{(i+1)}=b}(\mu) = \frac{i[P(b\mu)-P(b)]^{i-1}\,p(b\mu)b}{[1-P(b)]^{i}};\; \mu \ge 1. \qquad (9)$$

Accordingly, the expected value of ductility, with i non-linear excursions and yield level, b, is obtained as

$$E(\mu_i) = \int_1^{\infty} \mu f_{\mu_i|X_{(i+1)}=b}(\mu)\,\mathrm{d}\mu = \int_1^{\infty} \frac{\mu i[P(b\mu)-P(b)]^{i-1}\,p(b\mu)b}{[1-P(b)]^{i}}\,\mathrm{d}\mu. \qquad (10)$$

(ii) *condition on the maximum level*

Similarly, as in the formulation for condition on the yield level, we focus our attention on $X_{(1)}$ (largest) and $X_{(i+1)}$ ($(i+1)^{th}$ largest) ordered normalized peak variables and obtain the p.d.f. of the ductility ratio by conditioning on the maximum level for a given number of yield level excursions and total number of peaks.

The conditional p.d.f. of $X_{(i+1)}$ (i.e. the $(i+1)^{th}$ largest peak), given $X_{(1)} = a$, the maximum level, is expressed as

$$f(X_{(i+1)} = b \mid X_{(1)} = a)\; or\; f_{X_{(i+1)}|X_{(1)}=a}(b \mid a) = \frac{f_{(i+1),(1)}(b,a)}{f_{(1)}(a)} \qquad (11)$$

where, $f_{(i+1),(1)}(b, a)$ is same as in Eq. (6), and $f_{(1)}(a)$ is the marginal density function of the 1st order peak, expressed as

$$f_{(1)}(a) = \frac{n!}{(n-1)!}[P(a)]^{n-1}\,p(a). \qquad (12)$$

Eq. (11) thus gives

$$f_{X_{(i+1)}|X_{(1)}=a}(b \mid a) = \frac{(n-1)!}{(n-i-1)!(i-1)!}\,\frac{[P(b)]^{n-i-1}[P(a)-P(b)]^{i-1}\,p(b)}{[P(a)]^{n-1}}. \qquad (13)$$

This expression is the conditional density function for $X_{(i+1)}$ (i.e. the $(i+1)^{th}$ largest peak), given that the largest peak, $X_{(1)} = a$. Now, by a similar approach as followed in Case (i) above, we obtain the expression for the p.d.f. of the ductility ratio, μ_i, with the conditional maximum level, $X_{(1)} = a$ and with the number of non-linear excursions, i as

$$f_{\mu_i|X_{(1)}=a}(\mu) = \frac{(n-1)!}{(n-i-1)!(i-1)!}\frac{a}{\mu^2}\,\frac{[P(a/\mu)]^{n-i-1}[P(a)-P(a/\mu)]^{i-1}\,p(a/\mu)}{[P(a)]^{n-1}}. \qquad (14)$$

Also, the expected value of ductility, $E(\mu_i)$ is obtained as

$$E(\mu_i) = \int_1^{\infty} \frac{(n-1)!}{(n-i-1)!(i-1)!}\frac{a}{\mu}\,\frac{[P(a/\mu)]^{n-i-1}[P(a)-P(a/\mu)]^{i-1}\,p(a/\mu)}{[P(a)]^{n-1}}\,\mathrm{d}\mu. \qquad (15)$$

The order statistics formulation for probabilistic ductility ratio as above, takes into consideration the mutual statistical dependence of the peaks corresponding to the yield and maximum levels. The density functions, therefore, take care of the statistical dependence on the conditional level (yield or maximum), and depend on the number of non-linear excursions. Even intuitively speaking, in assessing the ductility demand when the yield limit is specified, the chances of occurence of the maxima at a particular level are affected by the specified yield level and the number of peaks occuring after it. The present order statistics approach quantifies this dependence in form of the conditional distribution of the ductility ratio.

The conditional density function for ductility as in Eq. (9), with conditioning on the yield level, can be shown to satisfy a Markov dependence. In fact, it could be proved in general that the conditional distribution of any order statistic satisfies a Markov dependence structure. This means that given a sequence of occurence, the present occurence is affected only by the recent past. For illustration, if $X_1, X_2, ..., X_{i-1}, X_i, X_{i+1}, ..., X_{j-1}, X_j$ is the sequential occurence of random variables, then mathematically one can write the Markov property as

$$\text{Prob}(X_j \mid X_i, X_{i-1}, ..., X_2, X_1) = \text{Prob}(X_j \mid X_i). \quad (16)$$

To prove the Markov dependence, let us first consider the joint p.d.f. of the ordered normalized peak variables, $X_{(i+1)}, X_{(i+2)}, ..., X_{(n-1)}, X_{(n)}$. Using the approach as followed earlier of partitioning the real axis, placing the variables suitably and considering the multinomial distribution, this may be expressed as (David (1980))

$$f_{X_{(n)}...X_{(i+1)}}(x_n,...,b) = \frac{n!}{i!}p(x_n)...p(b)[1-P(b)]^i. \quad (17)$$

Similarly, the joint p.d.f. of the ordered normalized peak variables, $X_{(n)}, X_{(n-1)}, ..., X_{(i+2)}, X_{(i+1)}, X_{(1)}$ becomes

$$f_{X_{(n)}...X_{(n-1)}...X_{(i+1)}X_{(1)}}(x_n, x_{n-1}, ..., b, a) =$$

$$= \frac{n!}{(i-1)!}p(x_n)...p(b)[P(a)-P(b)]^{i-1}p(a). \quad 18$$

Further, the conditional p.d.f. of $X_{(1)}$, given $X_{(n)} = x_n$, $X_{(n-1)} = x_{n-1}, ..., X_{(i+1)} = b$ can be expressed as

$$f(a \mid x_n, ..., b) = \frac{f_{X_{(n)}X_{(n-1)}...X_{(i+1)}X_{(1)}}(x_n, x_{n-1}, ..., b, a)}{f_{X_{(n)}X_{(n-1)}...X_{(i+1)}}(x_n, x_{n-1}, ..., b)}. \quad (19)$$

Using Eqs. (17), (18) and (19), and by a variable transformation, $a = \mu b$, as done earlier, the conditional density for ductility ratio, μ is written as

$$f(\mu \mid x_n, ..., b) = \frac{ib[P(b\mu)-P(b)]^{i-1}p(\mu b)}{[1-P(b)]^i}. \quad (20)$$

This expression for $f(\mu|x_n, ..., b)$ is seen to tally exactly with Eq. (9) for $f_{\mu|b}(\mu)$. Thus, Markov property is seen to be satisfied as we can write

$$f(\mu|x_n, ..., b) = f(\mu|b). \quad (21)$$

In the case of conditioning on the maximum level, this property is seen trivially since no peak occurs above the largest maxima, and hence, the dependence has to be on the maximum level only.

The Markov dependence structure as above is consistent with the fact that in the formulation, the interdependence between the yield and the maximum levels has been captured while ignoring the pre-yielding information. This is also in agreement with the physical situation. For ductility, we are only interested to know the range of nonlinear zone, once the yield level is specified. Thus, it is only the probability behaviour of the yield level and the posteriori probability structure of peaks above the yield level that leads to the characterization of the maximum level and thus that of the ductility.

3. Determination of RMS value

The response peaks as considered in the above formulation are normalized with respect to the r.m.s. value of the process. To determine this, let a stationary response process, $f(t)$, be represented by

$$f(t) = \sum_n A_n \cos(\omega_n t + \phi_n) \quad (22)$$

where, ω_n are the circular frequencies, ϕ_n are the random phases with uniform distribution in the interval, $[0,2\pi]$, and A_n are the random amplitudes. A_n are related to the energy spectrum, $E(\omega)$ of the function, $f(t)$ by the following relation

$$\sum_{\omega_n=\omega}^{\omega+d\omega} \frac{1}{2}A_n^2 = E(\omega)d\omega. \quad (23)$$

The r.m.s. value of the function, $f(t)$ is then given by

$$f_{rms} = m_0^{1/2} \quad (24)$$

where m_0 is defined according to Eq. (3).

The input process in the present work is assumed to be Gaussian. Therefore, the output would also be Gaussian for the linear systems. Even for most structural systems with mild non-linearities e.g., those with a few inelastic excursions, the response could be assumed to be Gaussian without significant errors. In case of a linear SDOF oscillator with natural frequency, ω_n and damping ratio, ζ, and subjected to an input ground motion assumed stationary and with (one-sided) power spectral density function (PSDF), $S_0(\omega)$, the r.m.s. value for the relative displacement response, $x(t)$ can be obtained as (Newland (1984)),

$$x_{rms} = \left[\int_0^\infty \frac{S_0(\omega)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}d\omega\right]^{1/2}. \quad (25)$$

The corrections may be applied in this to account for the nonstationarity in response e.g., by using the response spectrum approach suggested by Gupta and Trifunac (1987), and Gupta and Trifunac (1990, 1991). Also, by a simple adjustment in the damping and natural frequency, we can estimate the r.m.s. value of the response of a non-linear system (Iwan and Gates (1979)).

The above SDOF idealization may be suitable, for example, for the single story buildings and for studying the overall structure ductility of the MDOF systems. In those cases, however, where this idealization is not appropriate, one may use the mode shape vectors to obtain the r.m.s. value for the desired response function. To illustrate this for the case of sectional curvature ductility demand, let a N degree-of-freedom framed building subjected to ground motion, $\ddot{u}_g(t)$ at its base, be considered as an equivalent linear system by using the stochastic linearization techniques (Atalik and Utku (1976)). The equations of motion for this system may be written as

$$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\}=\{P\}\ddot{u}_g(t) \qquad (26)$$

where, $\{x\}$ is the response vector of floor displacements and joint rotations, $[m]$ is the structure mass matrix, $[k]$ is the structure stiffness matrix, $[c]$ is the structure damping matrix, and $\{P\}$ is the influence vector for the various degrees of freedom. Let the ground motion, $\ddot{u}_g(t)$ be characterized by the (one-sided) PSDF, $S_0(\omega)$. Considering the linear expansion of various degrees of freedom in the terms of the normal coordinates, these equations of motion can be decoupled and solved to give the transfer functions for any displacement component with ground acceleration as the input. For example, transfer function for the c^{th} and d^{th} displacement components, say X_c and X_d, can be written as

$$H_c(\omega)=\sum_{j=1}^{N}\frac{C_jF_j}{(\omega_j^2-\omega^2)+2i\zeta_j\omega_j\omega}, \qquad (27)$$

$$H_d(\omega)=\sum_{j=1}^{N}\frac{D_jF_j}{(\omega_j^2-\omega^2)+2i\zeta_j\omega_j\omega},$$

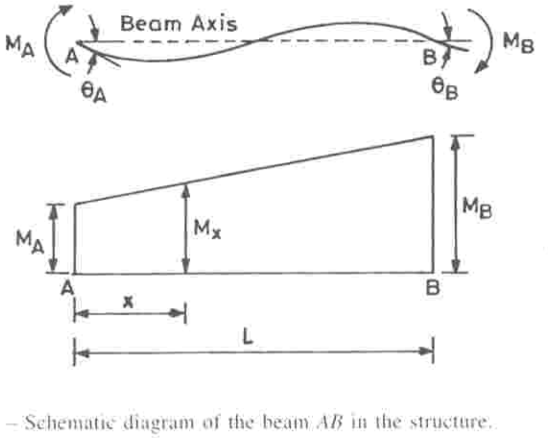

Fig. 3 – Schematic diagram of the beam AB in the structure.

where, C_j and D_j respectively are the modal eigenvector coefficients for the j^{th} mode corresponding to the c^{th} and d^{th} displacement components respectively, F_j is the modal participation factor in the j^{th} mode, and ω_j and ζ_j respectively are the frequency and damping ratio in the j^{th} mode. Thus, the cross-spectral density of X_c and X_d is given by

$$S_{X_cX_d}(\omega)=\sum_{r,s=1}^{N}\frac{1}{2}C_rD_sF_rF_s$$

$$\frac{S_0(\omega)}{[(\omega_r^2-\omega^2)+2i\zeta_r\omega_r\omega][(\omega_s^2-\omega^2)+2i\zeta_s\omega_s\omega]}. \qquad (28)$$

Here, $S_{X_cX_d}(\omega)$ is the two-sided spectral density function defined from $-\infty$ to ∞. Thus, one can obtain the covariance function, $\mathrm{Cov}(X_c, X_d)$ as

$$\mathrm{Cov}(X_c, X_d) = \sum_{r,s=1}^{N}C_rD_sF_rF_s$$

$$\int_0^{\infty}\frac{S_0(\omega)[(\omega_r^2-\omega^2)(\omega_s^2-\omega^2)+4\zeta_r\zeta_s\omega_r\omega_s\omega^2]}{[(\omega_r^2-\omega^2)+(2\zeta_r\omega_r\omega)^2][(\omega_s^2-\omega^2)^2+(2\zeta_s\omega_s\omega)^2]}d\omega, \qquad (29)$$

and the variance for any displacement, say X_c, as

$$\sigma_X^2 = \sum_{r,s=1}^{N}C_rC_sF_rF_s$$

$$\int_0^{\infty}\frac{[(\omega_r^2-\omega^2)(\omega_s^2-\omega^2)+4\zeta_r\zeta_s\omega_r\omega_s\omega^2]S_0(\omega)}{[(\omega_r^2-\omega^2)^2+(2\zeta_r\omega_r\omega)^2][(\omega_s^2-\omega^2)^2+(2\zeta_s\omega_s\omega)^2]}d\omega. \qquad (30)$$

With the knowledge of r.m.s. values of the joint rotations, in particular, it becomes possible to find the r.m.s. values of the section curvatures in an approximate manner as shown below. This can in turn be used in the computation of the section ductility.

Let the r.m.s. values of joint rotation be $\theta_{A_{rms}}$ and $\theta_{B_{rms}}$, and their covariance, $\mathrm{Cov}(\theta_A, \theta_B)$ (as computed from Eqs. (29) and (30)), for the ends A and B of the member AB of the structure as shown in Fig. 3. Applying the slope-deflection equations, the end moments at A and B in the absence of the member loads are obtained as

$$M_A=\frac{2EI}{l}[2\theta_A+\theta_B], \qquad (31)$$

$$M_B=\frac{2EI}{l}[2\theta_B+\theta_A]. \qquad (32)$$

The moment, M_x at any intermediate section of AB is then given by

$$M_x=\frac{x}{l}M_B+\left(1-\frac{x}{l}\right)M_A. \qquad (33)$$

Thus, the curvature, ϕ_x at this section becomes

$$\phi_x = \frac{2}{l^2}[(l+x)\theta_B + (2l-x)\theta_A]. \quad (34)$$

Since ϕ_x is a zero mean process, its r.m.s. value, ϕ_{xrms} becomes

$$\phi_{xrms} = \frac{2}{l^2}[(l+x)^2\theta^2_{Brms} + (2l-x)^2\theta^2_{Arms} +$$
$$+2(l+x)(2l-x)\mathrm{Cov}(\theta_A, \theta_B)]^{1/2}. \quad (35)$$

4. Application to structural systems

(i) Numerical Results

The expected values and the p.d.f. of the ductility ratio with conditions respectively on the linear and maximum levels have been computed using the equations derived in the formulation (i.e. Eqs. (9), (10), (14) and (15)). Different combinations of the governing parameters i.e. the conditioning yield level, b or the conditioning maximum level, a, number of non-linear excursions, i, number of peaks, n in the process, and parameter, ϵ (see Eq. (2)) have been considered.

Figs. 4 and 5 show the variation of expected ductility, $E(\mu)$, with ϵ, for different number, i, of nonlinear excursions, the former for the conditioning on the maximum level, $a = 3$, and the latter for the conditioning on the yield level, $b = 0.75$. The total number of peaks is taken as $n = 40$ in the first case. It is seen that except for the range $\epsilon > 0.7$, the expected ductility remains almost invariant of ϵ. Same behavior is also observed in the case of the p.d.f. of the ductility ratio. This suggests that the distribution of energy in the response with the usual range, $\epsilon < 0.4$, is likely to have negligible effect on any probabilistic measure of ductility, for a given number of nonlinear excursions, (normalized) maximum or yield level and the total number of peaks.

Fig. 6 shows the variation of expected ductility, $E(\mu)$ with the total number of peaks, n for different values of non-linear excursions, with ϵ held constant at 0.4 and (normalized) maximum level at 5.0. It is seen that the expected ductility decreases with the increase in number of peaks for all the four curves (corresponding to i = 2, 4, 6, 8). This is so because an increase in the number of peaks is associated with more close packing of the peaks and thus with the yield level coming closer to the specified maximum level for the number of excursions remaining unchanged. Further, for any given number of peaks, the expected ductility is, as expected, found to increase with the increase in number of excursions. Variation of $E(\mu)$ with the maximum level, a for different values of nonlinear excursions has been shown in Fig. 7, with $\epsilon = 0.4$ and $n = 60$. It is seen that the expected ductility increases almost linearly with the increase in the maximum level, a, the rate of increase being higher for the greater number of excursions. This is again as expected because for the fixed value of n, increase in maximum level, a is associated with increased spacing between the peak amplitudes, and this leads to greater difference between the maximum level and yield level for a given number of excursions, and also to greater effect of number of excursions at any maximum level, a.

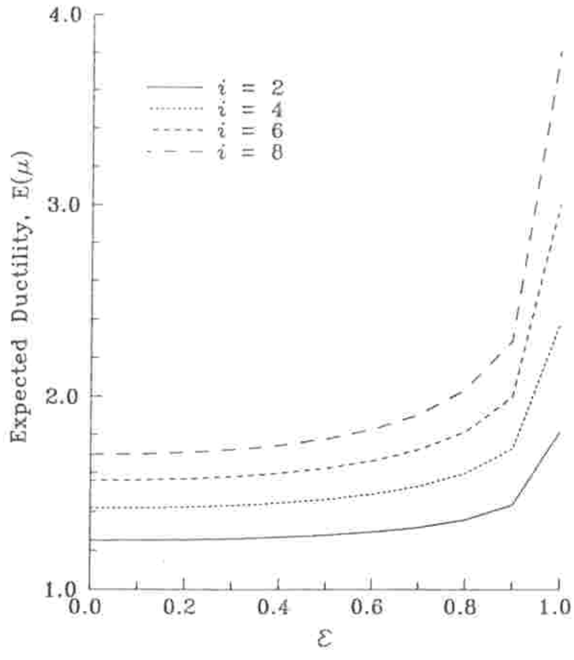

Fig. 4 – Variation in expected ductility with ϵ for $a = 3.0$ and $n = 40$.

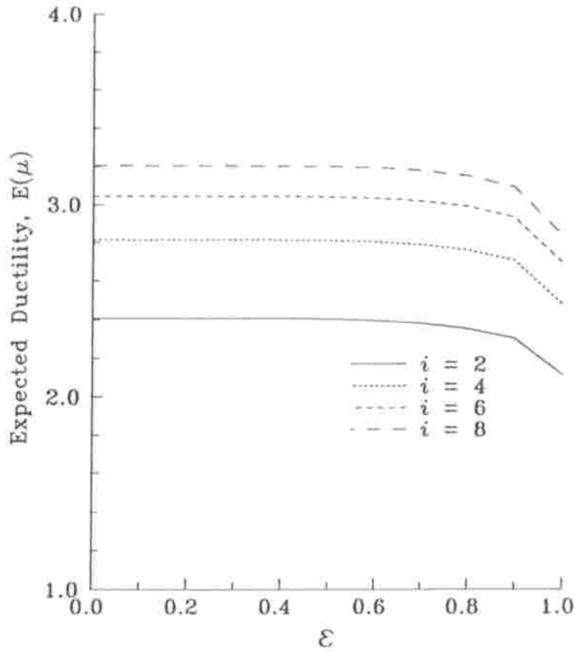

Fig. 5 – Variation in expected ductility with ϵ for $b = 0.75$.

Figs. 8, 9 and 10 show the probability density functions of the ductility ratio, for illustrating the effects of total number of peaks, (normalized) maximum level and the number of non-linear excursions respectively. For the effect of total number of peaks, a and i have been respectively taken as 3 and 2. For the effect of

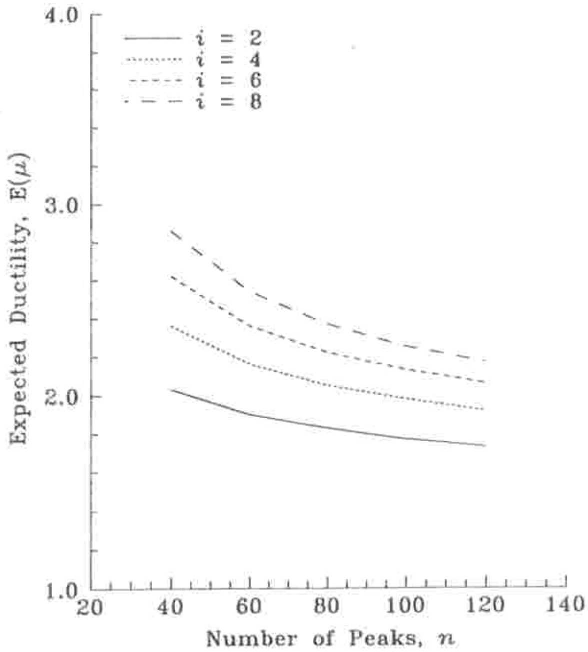

Fig. 6 – Variation in expected ductility with n for a = 5.0 and ϵ = 0.4.

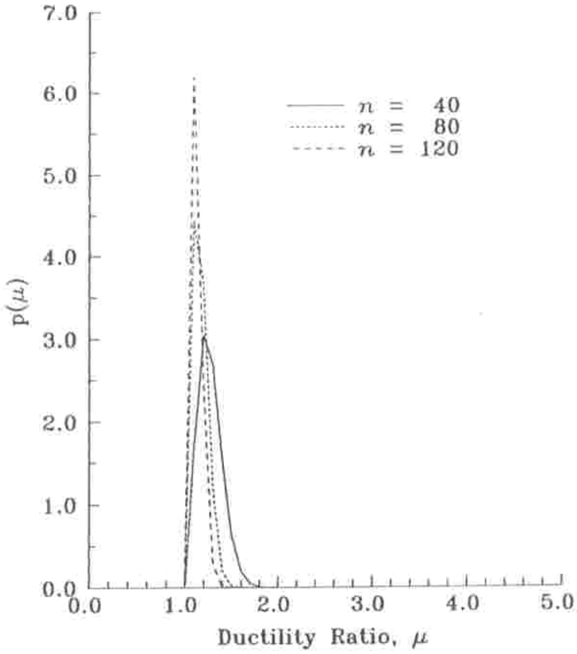

Fig. 8 – Variation in p.d.f. for ductility with n for a = 3.0, i = 2 and ϵ = 0.4.

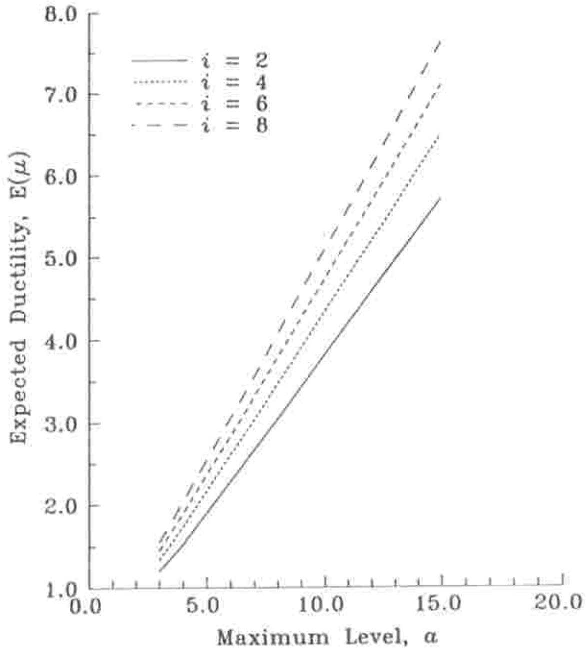

Fig. 7 – Variation in expected ductility with a for n = 60 and ϵ = 0.4.

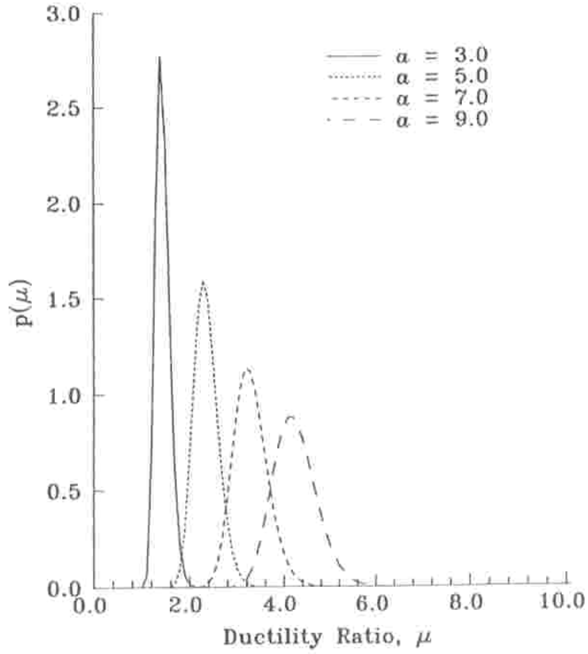

Fig. 9 – Variation in p.d.f. for ductility with a for n = 40, i = 4 and ϵ = 0.4.

maximum level, n = 40 and i = 4, and for the effect of nonlinear excursions, a = 5 and n = 60 have been taken. ϵ has been taken as 0.4 for all the three figures. The curves in Fig. 8 indicate that there is lesser dispersion in the case of greater number of total peaks, and then, the expected value is a good indicator of the ductility ratio with different levels of confidence. Similar observations are also obtained from Fig. 9. Here, the density function becomes more sharp peaked with the decrease in the maximum level. The effect of the number of nonlinear excursions (see Fig. 10) is however different as the greater number of excursions is not associated with the increased or decreased dispersion here. Actually, with a and n remaining fixed, the peak spacing becomes independent of the number of excursions while when a is varied with n remaining fixed or n is varied with a remaining fixed, this spacing also gets changed leading to the different levels of dispersion.

For the case of conditioning on the yield level, the parameter of total number of peaks, n is absent as seen from the formulation. The effects of the other parameters, b and i, on $E(\mu)$ have been shown in Fig. 11 where four curves corresponding to the different number of nonlinear excursions describe the dependence of $E(\mu)$ on the yield level, b. It is seen that for any given number of excursions, expected ductility decreases with the increase in the yield level. For the yield level below 0.5, this change in expected ductility is very sharp while for the yield levels above 1.0, the expected ductility converges asymptotically to the value of one irrespective of the number of excursions. Such a variation can be well approximated by the expression, $E(\mu) = 1 + k/b$ where, k is a constant depending on the number of excursions. It increases with the number of excursions. Physically, this expression may be explained by the fact that the difference in the maximum and yield levels for any given number of excursions is likely to be independent of the yield level. It follows from Fig. 11 that the sections can be most efficient if they are designed for the linear response between 0.5 and 1.0 times the r.m.s. value. In that case, the ductility demand would be around 2 to 4.

Figs. 12 and 13 show the plots for the density function of ductility, with conditioning on the yield level, respectively for different number of nonlinear excursions and for different yield levels. As observed also in the case of conditioning on the maximum level, the higher yield levels are associated with lesser degree of uncertainty with the expected value while there is considerably larger dispersion in the values of ductility at lower yield levels. On the other hand, the curves for different number of excursions show almost similar dispersion characteristics.

(ii) Applications

Let us first consider a SDOF system. Let d and X_{rms} respectively be the maximum allowable and r.m.s. values of the displacement of the mass of this system. Therefore, the normalized maximum displacement is given by

$$X_{(1)} = \frac{d}{X_{rms}}. \tag{36}$$

If n number of non-linear excursions are allowed, then from the curves of expected ductility with conditioning on the maximum level (see Fig. 7), the expected ductility ratio can be obtained (say, $E(\mu) = \overline{\mu}$). Thus, we have linear maximum displacement

$$d_l = \frac{d}{\overline{\mu}}, \tag{37}$$

and maximum elastic design force in the structure as

$$F_D = \frac{kd}{R(\overline{\mu})\Omega}. \tag{38}$$

Here, k is the lateral stiffness of the supporting system, $R(\overline{\mu})$ is the ductility reduction factor estimated from $\overline{\mu}$ based on equivalent displacement or energy considerations, and Ω is the overstrength available in the structure. Ω depends on the degree of redundancy in the structure and on the inherent safety cushion provided in the design methodology.

These concepts can also be applied to the MDOF and continuous systems without any loss of generality. For example, in a multistoried structure, maximum roof

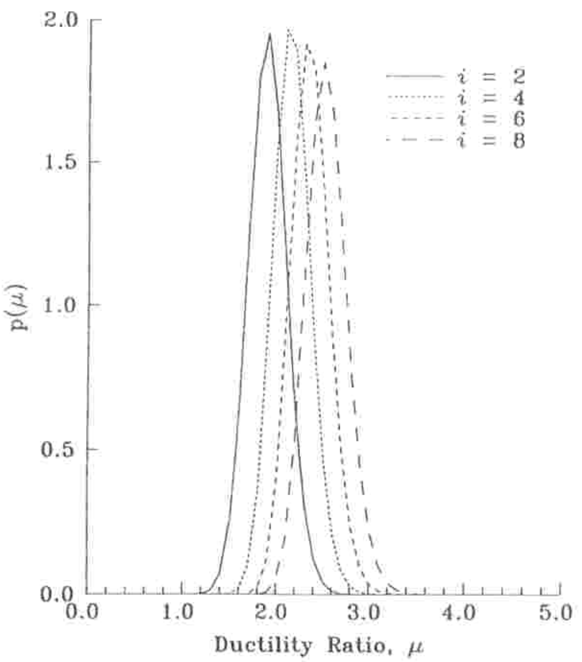

Fig. 10 – Variation in p.d.f. for ductility with i for $n = 60$, $a = 5.0$ and $\epsilon = 0.4$.

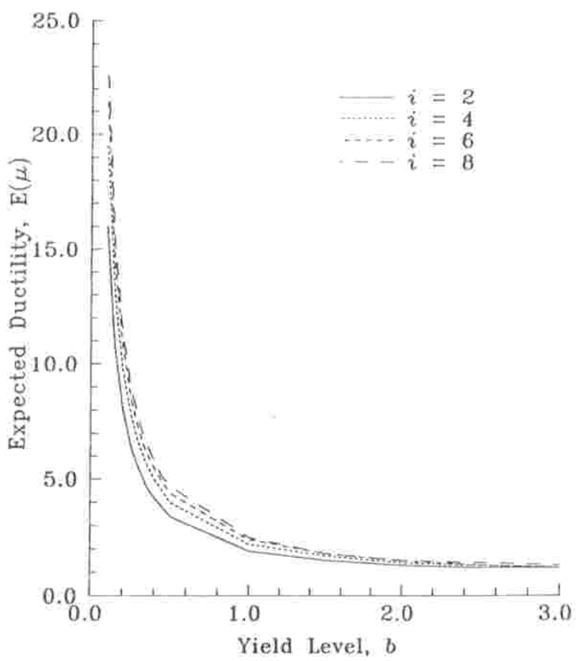

Fig. 11 – Variation in expected ductility with b for $\epsilon = 0.4$.

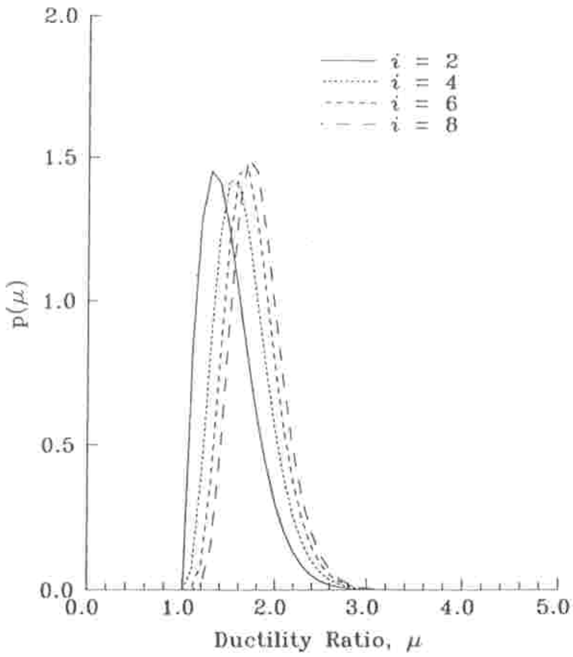

Fig. 12 – Variation in p.d.f. for ductility with i for $b = 1.5$ and $\epsilon = 0.4$.

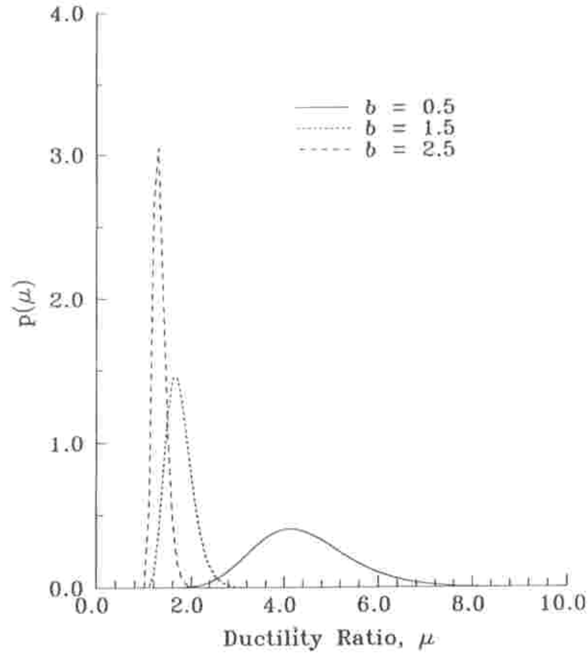

Fig. 13 – Variation in p.d.f. for ductility with b for $i = 6$ and $\epsilon = 0.4$.

displacement and maximum interstory drift considerations can lead to the maximum allowable roof displacement and maximum displacements at the other floor levels. The r.m.s. values of the floor displacements may now be estimated from the ground motion characteristics and transfer functions, and the normalized maximum floor displacements be computed. Further, with the knowledge of the number of peaks and desired excursions at each level, the expected ductility demand for the structure and the story ductility demand can be obtained, e.g., from the curves in Fig. 7 with condition on the maximum level and with the displacement and drift conditions being critically satisfied. Once, the structure is designed for the elastic design forces, the yield curvatures at the critical sections can be estimated by carrying out a static analysis. Now, with the knowledge of the r.m.s. values of section curvatures from the input characteristics and transfer functions (see Eq. (35)), and by using, for example, Fig. 11 for the expected ductility ratio with conditioning on the linear level, the curvature ductility demand can be estimated, for a given number of non-linear excursions. To calculate the expected ductility demand corresponding to a number of nonlinear excursions a designed system has to resist, we must use the curves with the conditioning on the yield level.

It is to be noted that the curves for expected ductility are obtained for specific values of the bandwidth parameter, ϵ, the total number of peaks, n in the process, and the number of nonlinear excursions, i. The first two parameters can be determined as functions of the moments of the energy spectrum, $E(\omega)$ of the process (see Gupta and Trifunac (1988), for example). The estimation of number of non-linear excursions is crucial. It is physically associated with the damage to be allowed in a structure. A single non-linear excursion may be sufficient for the architectural damage whereas the structural damage in beams and girders may depend on how ductile they are to sustain greater number of excursions. The columns form the most critical part in a structure and therefore they should have greater insurance against damage. Stiffer columns may be provided for this purpose, thus raising the linear design level and decreasing the number of excursions. But, a rational estimation of allowable excursions under different cases must be obtained from experimentation and then by correlating the damage with the number of excursions. Computational simulation by introducing suitable damage model may be an alternative. In that case, the estimation of the number of nonlinear excursions is done by the knowledge of the damage to be allowed, and with the help of a damage analysis along with the ductility study which is beyond the scope of this paper. Moreover, it is clear that the provision of ductility is not enough to ensure safety as the damage which strongly depends on the number of nonlinear excursions has also to be kept within an acceptable limit.

5. Conclusions

A probabilistic approach has been proposed for obtaining the ductility ratio with condition on the yield or maximum level using the order statistics formulation. The expressions for the density functions and the expected values have been derived for both the cases, and numerical results have been obtained for various values of the different parameters. It has been shown how these results can be applied to the SDOF and MDOF systems.

It has been found that the density functions and the expected values of ductility ratio do not depend on ϵ (which is a measure of the band-width of the response process) for the practical range of its values. Response characteristics would nevertheless affect the ductility estimates via the response r.m.s. value.

It has been noticed that 0.5-1.0 times the r.m.s. value of response is the optimal range for deciding on the design yield levels from the ductility considerations.

The results obtained from this study may be very useful in the design of ductile structures where the number of inelastic excursions play a significant role making it essential to account for the statistical dependence between the maximum and the yield levels.

References

Atalik, T.S. and Utku, S., Stochastic linearization of multi-degree-of-freedom non-linear systems, *Earthq. Eng. Struct. Dyn.*, **4**, 411-420 (1976).

Cartwright, D.E. and Longuet-Higgins, M.S., The statistical distribution of maxima of a random function, *Proc. Roy. Soc. London*, **A 237**, 212-232 (1956).

David, H.A., Order statistics, *John Wiley*, New York (1980).

Gupta, I.D. and Trifunac, M.D., Order statistics of peaks in earthquake response of multi-degree-of-freedom systems, *Earthq. Eng. Eng. Vib.*, **7(4)**, 15-50 (1987).

Gupta, I.D. and Trifunac, M.D., Order statistics of peaks in earthquake response, *J. Eng. Mech.*, **114(10)**, 1605-1627 (1988).

Gupta, V.K. and Trifunac, M.D., Response of multistoried buildings to ground translation and rocking during earthquakes, *J. Prob. Eng. Mech.*, **5(3)**, 138-145 (1990).

Gupta, V.K. and Trifunac, M.D., Seismic response of multistoried buildings including the effects of soil-structure interaction, *Soil Dyn. Earthq. Eng.*, **10(8)**, 414-422 (1991).

Hwang, H.H.M. and Jaw, J.W., Probabilistic damage analysis of structures, *J. Struct. Div. (ASCE)*, **116(7)**, 1992-2007 (1990).

Iwan, W.D. and Gates, N.C., The effective period and damping of a class of hysteretic structures, *Earthq. Eng. Struct. Dyn.*, **7**, 199-211 (1979).

Karnopp, D. and Scharton, T.D., Plastic deformation in random vibration, *J. Acoust. Soc. Am.*, **39**, 1154-1161 (1966).

Miranda, E., Non-linear response spectra for earthquake resistant design, *Proc. Tenth Wld Conf. Earthq. Eng.*, Madrid, Spain, **X**, 5835-5840 (1992).

Murakami, M. and Penzien, J., Non-linear response spectra for probabilistic seismic design of reinforced concrete structures, *Proc. Sixth Wld Conf. Earthq. Eng.*, New Delhi, India, **II**, 1046-1051 (1977).

Nakamura, T. and Takewaki, I., Ductility design via optimum design of nonlinear elastic frames, *J. Struct. Div. (ASCE)*, **115(3)**, 608-625 (1989).

Newland, D.E., An introduction to random vibrations and spectral analysis, *Longman*, New York (1984).

Rice, S.O., Mathematical analysis of random noise, *Bell Syst. Tech. J.*, **23**, 282 332 (1944).

Rice, S.O., Mathematical analysis of random noise, *Bell Syst. Tech. J.*, **24**, 46-156 (1945).

Ridell, R., P. Hidalgo, and Cruz, E., Response modification factors for earthquake resistant design of short period buildings, *Earthq. Spectra*, **5(3)**, 571-590 (1989).

Vanmarcke, E.H., First passage and other failure criteria in narrow-band random vibration: a discrete state approach, *Dept. of Civil Engg.*, M.I.T., Report R69-68 (1969).

Vanmarcke, E.H. and Veneziano, D., Probabilistic seismic response of simple inelastic systems, *Proc. Fifth Wld Conf. Earthq. Eng.*, Rome, Italy (1973).