Probabilistic Ductility Demand including Peak Dependence

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SUMMARY – A probabilistic approach has been followed to estimate the expected ductility demand in a mildly non-linear system subjected to a stationary, zero-mean, Gaussian excitation. The proposed approach considers the total number of response peaks and number of non-linear excursions beyond the response yield level as parameters in formulating the density function and expected value of the ductility ratio. Since the peaks occuring in a particular response are mutually dependent, simulated joint density function of these peaks has been used in the formulation. Numerical results on expected ductility demand indicate that the usually made assumption of peak independence may be conservative for assessing the ductility demand in a structure.

1. Introduction

The earthquakes-resistant design philosophy requires the structures to be sufficiently ductile to withstand severe ground shaking with acceptable damage while avoiding a collapse. Thus, assessment of ductility demand in the structures becomes an essential requirement of the aseismic design. Uncertainties in the ground motion however make the probabilistic prediction of the response and hence of the ductility demand a necessity. Several probabilistic studies on energy dissipation, ductility factors, and force reduction factors in elasto-plastik systems have been carried out and a discussion on these is available in Basu and Gupta (1995a). However, some of these studies were based on time-history analyses using a large number of recorded ground motions to obtain the statistical estimates of ductility and response reduction factors. Since a preliminary design assessment may not warrant these computation-intensive analyses, there remained a need for developing a proper and rational probabilistic description of ductility and for relating it to the statistical relationship between the maximum level, yield level, and the number of non-linear crossing while those may have important design implications with regard to the damage of structures [see Basu and Gupta (1995b) for details].

Recently, Basu and Gupta (1995a) have used the order statistics approach to study the probabilistic ductility ratio from a different perspective by considering the effect of the number of the excursions on ductility demand. Their formulation is based on the assumption of statistical independence between the response peaks. This paper generalizes their approach to formulate the distribution of probabilistic ductility ratio while including the effects of peak dependence in a responce process. This dependence has been taken into account by modeling the joint density function of these peaks through digital asimulation as suggested by Basu et al. (1996). Two formulations, one based on the condition on the yield level and the other on the condition on the maximum level, have been presented here. A parametric study has been carried out to study the dependence of expected ductility demand on the governing parameters, and these results have been compared with those by Basu and Gupta (1995a) to see how reasonable is the assumption of peak dependence for estimating the expected ductility demand in a structure.

2. Probabilistic Ductility Demand

Ductility is defined as the ratio of maximum to yield response of a given system. The maximum and yield levels in a response may, in turn, be considered to correspond to the largest peak and a higher order peak (with lesser amplitude) respectively. Thus, a study of

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ductility demand would require the formulation of ductility ratio with condition on either of these two levels.

Let us consider a zero mean, stationary, Gaussian response process, $X(t)$. Unless the system is highly non-linear, an equivalent linear oscillator can be obtained whose response, $X(t)$ is close to the response of the original non-linear oscillator in the meansquare sense [see, for example, Atalik and Utku (1976), and Iwan and Gates (1979)]. $X(t)$ may denote response functions like roof displacement, joint rotation, or curvature of a section in a structure. Let X_i, $i = 1, 2, ..., n$ to be the n number of peaks occuring in $X(t)$. On being normalized by the root-mean-square (r.m.s.) value of the process, those can be characterized by the probability density function (p.d.f.) given by Cartwright and Longuet-Higgins (1956) as

$$p(\eta) = \frac{1}{\sqrt{2\pi}}\left[\varepsilon e^{-\eta^2/2\varepsilon^2} + +(1-\varepsilon^2)^{1/2}\eta e^{-\eta^2/2}\int_{-\infty}^{\eta(1-\varepsilon^2)^{1/2}/e} e^{-x^2/2}\mathrm{d}x\right], \quad (1)$$

where, η is the normalized peak value, and

$$\varepsilon = \left[\frac{m_0 m_4 - m_2^2}{m_0 m_4}\right]^{1/2}. \quad (2)$$

Here, m_0, m_2 and m_4 respectively denote the zeroth, second and fourth moments of the energy spectrum of $X(t)$. Further, the expected number of peaks, n may be estimated from $(T/2\pi)\sqrt{m_4/m_2}$ where, T is the duration of the process, $X(t)$.

Basu et al. (1996) have proposed a scheme to approximately evaluate the joint density function of these statistically dependent peaks by simulation. They have considered $n + 1$ independent random variables, Y_0, Y_1, ..., Y_n each exponentially distributed with parameter, β. By adding the first variable, Y_0 to the remaining n variables, n dependent variables, X_1, X_2, ..., X_n are created. The joint density of Y_0, and X_1, X_2, ..., X_n, is obtained by writing the joint density of X_1, X_2, ..., X_n, i.e. $p(x_1, ..., x_n)$, is obtained by integrating out the effect of Y_0 in its possible range, R_y, of variation. If X_1, X_2, ..., X_n are ordered in decreasing order of amplitudes as $X_{(1)}$, $X_{(2)}$, ..., $X_{(n)}$, the joint density of the ordered peaks follows from this as [Basu et al. (1996)].

$$p(x_{(1)}, ..., x_{(n)}) = \int_{R_y} \frac{n!}{\beta^{(n+1)}} e^{-\sum_{j=1}^{n} x_{(j)}/\beta} e^{(n-1)y_0/\beta}\mathrm{d}y_0. \quad (3)$$

It has been found through digital simulation that a value of $\beta = 0.55$ given reasonably good approximation of the joint density in the response peaks for most structural systems [see Base et al. (1996) for details].

Using the expression in Eq. (3), it is possible to derive the conditional order statistics, with condition first on the yield level and then on the maximum level. Let the number of excursions of a specified response level be i out of a sample of n peaks. Now, to arrive at the p.d.f. of the ductility, we first obtain the joint density function of the 1st and the $(i + 1)$th order peaks using Eq. (3) as

$$p(X_{(1)} = x_{(1)}, X_{(i+1)} = x_{(i+1)}) = = \frac{n!}{\beta^{n+1}}\int_{x_{(3)}}^{x_{(1)}}\int_{x_{(4)}}^{x_{(1)}}\cdots\int_{x_{(i+1)}}^{x_{(1)}}\int_{x_{(i+3)}}^{x_{(i+1)}}\cdots\int_{x_{(n)}}^{x_{(i+1)}}\int_{y_0}^{x_{(i+1)}}\int_{b}^{x_{(i+1)}} e^{-\sum_{j=1}^{n} x_{(j)}/\beta}\cdot e^{(n-1)y_0/\beta}dx_{(2)}dx_{(3)}\ldots dx_{(i)}dx_{(i+2)}\ldots dx_{(n)}dy_0. \quad (4)$$

On successive integration, this becomes

$$p(x_{(1)}, x_{(i+1)}) = \frac{n!}{(n-i-1)!\,(i-1)!}\frac{1}{\beta^3}\int_0^{x_{(i+1)}}[e^{-x_{(i+1)}/\beta} - -e^{-x_{(1)}/\beta}]^{(i-1)}[e^{-y_0/\beta} - e^{-x_{(i+1)}/\beta}]^{(n-i-1)}\cdot \cdot e^{(n-1)y_0/\beta}e^{-x_{(1)}/\beta}e^{-x_{(i+1)}/\beta}dy_0. \quad (5)$$

Similarly, from Eq. (3), the univariate density function of the $(i + 1)$th order peak can be obtained as [see Base et al. (1996)],

$$p(x_{(i+1)}) = \frac{n!}{(n-i-1)!\,i!}\frac{1}{\beta^2}\int_0^{x_{(i+1)}} e^{-ix_{(i+1)}/\beta}(e^{-y_0/\beta} - -e^{-x_{(i+1)}/\beta})^{n-i-1}e^{(n-1)y_0/\beta}dy_0. \quad (6)$$

Using Eq. (5) and (6), we get the conditional order statistics of the largest peak, on the condition that the $(i + 1)$th peak is known, as

$$p(X_{(1)} = x_{(1)} \mid X_{(i+1)} = x_{(i+1)}) = = \frac{i}{\beta}\frac{[e^{-x_{(i+1)}/\beta} - e^{-x_{(1)}/\beta}]^{(i-1)}}{e^{-ix_{(i+1)}/\beta}}e^{-x_{(1)}/\beta}. \quad (7)$$

Thus, the expected value of ductility, $\mu(= X_{(1)}/X_{(i+1)})$ with i number of non-linear excursions of the yield level, b is given by taking $x_{(i+1)} = b$ in Eq. (7) as

$$E(\mu \mid X_{(i+1)} = b) = \int_b^{\infty}\frac{i}{\beta}\frac{x_{(1)}}{b}\frac{[e^{-b/\beta} - e^{-x_{(1)}/\beta}]^{(i-1)}}{e^{-ib/\beta}}e^{-x_{(1)}/\beta}\mathrm{d}x_{(1)}. \quad (8)$$

For the second case with the condition on the maximum level, say a, we first obtain the density function of the largest peak by taking $i = 0$ in Eq. (6). Then the conditional density function of the $(i + 1)$th order peak is obtained by using Eq. (5) as

$$p(X_{(i+1)} = x_{(i+1)} \mid X_{(1)} = a) = \left.\frac{p(x_{(1)}, x_{(i+1)})}{p(x_{(1)})}\right|_{x_{(1)}=a}$$

$$= \frac{(n-1)!}{(n-i-1)!\,(i-1)!}[e^{-x_{(i+1)}/\beta} - e^{-a/\beta}]^{(i-1)}e^{-x_{(i+1)}/\beta}$$

$$\cdot \frac{\int_0^{x_{(i+1)}} [e^{-y_0/\beta} - e^{-x_{(i+1)}/\beta}]^{n-i-1} e^{(n-1)y_0/\beta} \mathrm{d}y_0}{\beta \int_0^{a} [e^{-y_0/\beta} - e^{-a/\beta}]^{(n-1)} e^{(n-1)y_0/\beta} \mathrm{d}y_0} . \quad (9)$$

Thus, the expected ductility, μ for i excursions of the yield level, on the condition that the maximum level is a, is given by

$$E\left(\mu = \frac{a}{x_{(i+1)}}\right) = \int_0^{a} \frac{a}{x_{(i+1)}} \cdot p(X_{(i+1)} = x_{(i+1)} \mid X_{(1)} = a)\mathrm{d}x_{(i+1)} . \quad (10)$$

3. Numerical Results

For the numerical results on parametric variation, the expected values of ductility ratio with conditioning respectively on the yield and maximum levels have been calculated using Eq. (8) and (10). The curves have been plotted for the different combinations of parameters such as the yield level, b, maximum level, a, total number of peaks, n, and the number of non-linear excursions, i, by taking $\varepsilon = 0.4$. These results have been compared with the parallel results obtained by Basu and Gupta (1995a) for the statistically independent peaks.

Figure 1 shows the variation in the expected ductility ratio, $E(\mu)$, with the total number of peaks, n, for the number of non-linear excursions, $i = 2$ and 8. The conditional maximum level, a is taken as 5.0. It is seen that the expected ductility demand decreases with the increase in the total number of peaks for a given number of non-linear excursions as in the case of the indipendence assumption. Ductility demand also increases with the increase in the number of excursions in the same way for a given total number of peaks. However, the results from the proposed formulation are slightly lower than the 'independence' results. The variation in expected ductility demand with the maximum level, a has been shown in Fig. 2 for $n = 60$, and $i = 2$, 8. Though the trends here are alike in the curves for both the cases of dependence and indipendence, the linear increase in ductility demand with the maximum

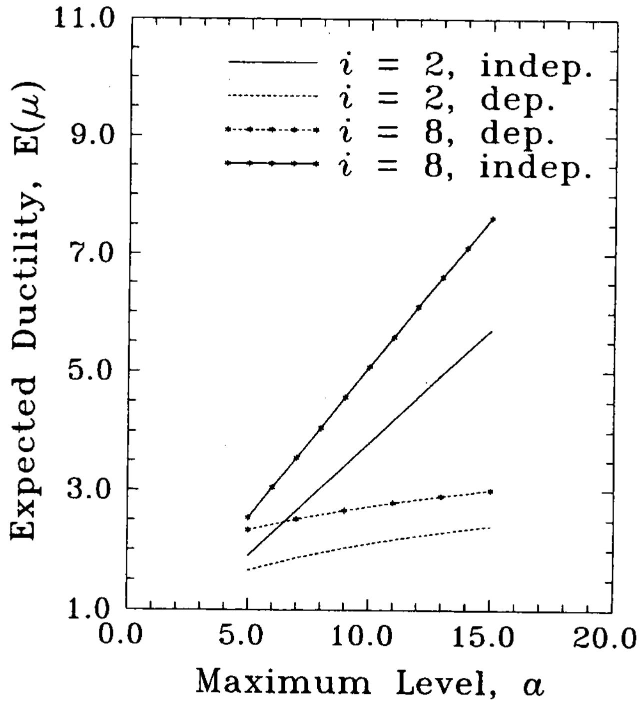

Fig. 1 – Variation in Expected Ductility with n for a = 5.0.

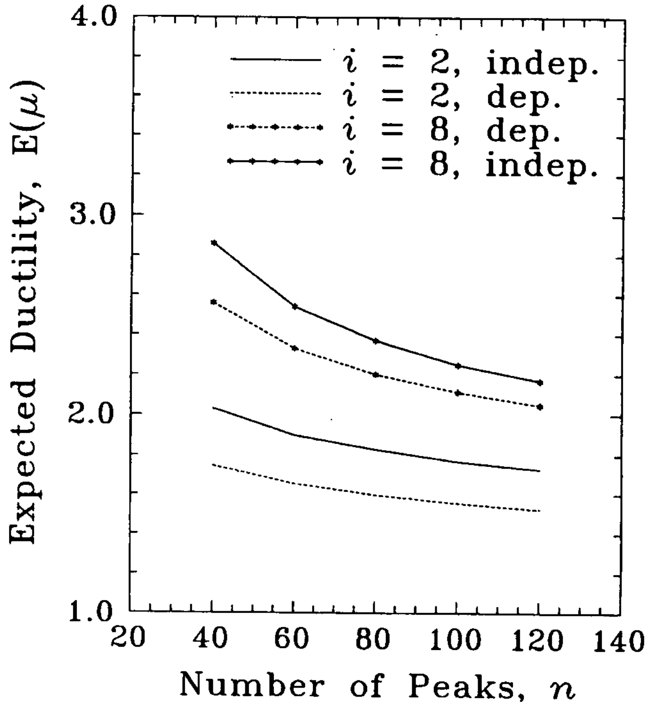

Fig. 2 – Variation in Expected Ductility with a for n = 60.

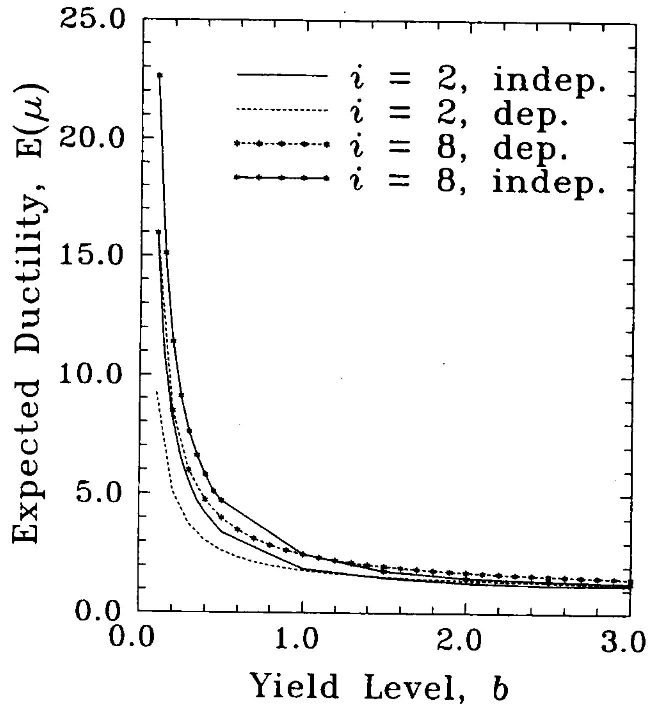

Fig. 3 – Variation in Expected Ductility with b.

level for the 'dependence' case has a much flatter slope. Around $a = 5.0$, the results from the two formulations match but beyond this, the 'dependence' results fall far blow the 'indipendence' results on ductility demand.

In Figure 3, the parameters, i and b have been varied to study the effects on ductility demand when the conditioning is on the yield level, b. Here also, the assumption of independence appears to make no difference so far as the trends of the curves are concerned. The variation in the expected ductility demand may be approximated by the expression, $E(\mu) = 1 + k/b$ where, k is a constant depending on the number of excursions. Thus, the difference in the maximum and yield levels for any given number of excursions is likely to be independent of the yield level. The optimal region for the yield design level is also observed to be 0.5 – 1.0 times the r.m.s. value for both the approaches. However, in Figure 3, the 'dependence' results are lower in value than the 'indipendence' results. These differences are more prominent for the lower values of yield level (say, for $b \leq 0.5$), which is consistent with the greater discrepancies observed at the higher values of the maximum level in Figure 2. Thus, it is seen that whether we have the higher values of maximum level or the lower values of yield level, the two approaches give very different estimates of expected ductility demand. Further, accounting for the dependence corresponds to the lowering in the expected ductility demand estimates, and thus the maximum and yield levels cannot be too distant from each other even at the high maximum or low yield levels with the considered dependence between the unordered peaks.

The above numerical results can be applied to the design of SDOF and multi-degree-of-freedom (MDOF) systems as explained by Basu and Gupta (1995a).

4. Conclusions

The numerical results of the proposed formulation have shown that the dependence of peaks leads to the reduced ductility demand estimates, particularly at the higher values of maximum response level and at the lower values of yield response level. This reduction is significant for the case with the conditioning on the maximum level. No effect of this dependence is however observed in the trends of expected ductility ratio variation for a wide range of parameters. Thus, 0.5 – 1.0 times the r.m.s. value may again be a suitable range for deciding on the optimal response yield level from ductility ratio considerations as reported by Basu and Gupta (1995a) for the independent peaks.

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