EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS
Earthquake Engng Struct. Dyn. 2009; **38**:941–949
Published online 26 November 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/eqe.877

SHORT COMMUNICATION

A new approximation for spectral velocity ordinates at short periods

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SUMMARY

Response spectrum methods in earthquake-resistant design sometimes require information on the spectral velocity (SV) for a given single-degree-of-freedom oscillator and specified seismic hazard. SV has been conventionally approximated as pseudo spectral velocity (PSV) in the case of lightly damped structures that are not too flexible. This study shows that the PSV approximation may lead to large overestimation errors when the structure is stiffer to the ground motion and the ground motion is a long-period motion. It is also shown that a new approximation requiring the use of peak ground acceleration of the motion may significantly reduce these errors. Copyright © 2008 John Wiley & Sons, Ltd.

Received 14 May 2008; Revised 15 September 2008; Accepted 23 October 2008

KEY WORDS: response spectrum method; spectral velocity; pseudo spectral velocity

INTRODUCTION

The concept of response spectrum in earthquake engineering has marked one of the most useful developments for practicing engineers responsible for ensuring seismic safety of structural systems. It has afforded an extremely compact representation of the effects that a ground motion will have on a variety of structures. After its initial development by Biot [1–3], this concept has become popular through the widespread use of spectral displacement (SD), pseudo spectral velocity (PSV) and pseudo spectral acceleration (PSA) curves [4]. It is possible to represent these curves via a single tripartite logarithmic plot due to the relationship between the SD, PSV and PSA curves. An SD curve represents an accurate variation of the maximum displacement of a base-excited single-degree-of-freedom (SDOF) oscillator mass (relative to the ground) with period of the oscillator. On the other hand, PSV and PSA curves represent variations of the maximum relative velocity

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and maximum absolute acceleration responses of the oscillator mass, known as spectral velocity (SV) and spectral acceleration (SA), respectively, in an approximate form [5]. While PSA is considered to be an excellent approximation of SA in the case of lightly damped systems (those become identical for the undamped systems), PSV is considered to be a good approximation of SV, particularly when the system is lightly damped and is not very flexible (see, e.g. Reference [6]).

With the fast and inexpensive computation of response spectra from an accelerogram becoming possible, one can directly use SV and SA curves instead of approximating those by the PSV and PSA curves. However, it is not always possible to have an accelerogram available to the practicing engineer for the specified seismic hazard at the site of interest. Therefore, it may still be desirable to approximate the SV curves, possibly with the help of PSA curves, as those are usually available, wherever such curves are required. This may be the situation, for example, when we need to combine the modal maxima through a modal combination rule for the peak structural response [7, 8].

In this paper, a new approximation of SV is developed by using the theory of random vibrations, and it is shown that the hitherto-used PSV approximation may not be reasonable when the ground motion has strong low-frequency content relative to the frequency of the structural system. With the help of three ground motions having widely different energy characteristics, it is shown that the proposed approximation works much better than the PSV approximation in such a situation.

FORMULATION

The equation of motion for a base-excited SDOF oscillator with natural frequency ω_n and viscous damping ratio ζ_n can be written as

$$\ddot{x}(t)+2\zeta_n\omega_n\dot{x}(t)+\omega_n^2 x(t)=-\ddot{z}(t) \tag{1}$$

where $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are the relative displacement, velocity and acceleration, respectively, of the oscillator mass with respect to the base, and $\ddot{z}(t)$ is the base acceleration. The transfer function relating the absolute acceleration response $(=\ddot{x}(t)+\ddot{z}(t))$ of the oscillator mass to the input ground acceleration may be expressed as

$$H_a(\omega)=1-\omega^2 H(\omega) \tag{2}$$

where

$$H(\omega)=\frac{-1}{\omega_n^2-\omega^2+2i\zeta_n\omega_n\omega} \tag{3}$$

(with $i=\sqrt{-1}$) is the transfer function relating the relative displacement of the oscillator mass to the input base excitation. On assuming stationarity in the excitation and the response, the power spectral density function (PSDF) of the absolute acceleration response may be expressed as [9]

$$\begin{aligned} S_a(\omega) &= S_{\ddot{z}}(\omega)|H_a(\omega)|^2 \\ &= S_{\ddot{z}}(\omega)[1+\omega^4|H(\omega)|^2-2\omega^2\,\mathrm{Re}(H(\omega))] \end{aligned} \tag{4}$$

where $S_{\ddot{z}}(\omega)$ is the PSDF of the input base excitation $\ddot{z}(t)$.

DOI: 10.1002/eqe

Taking the square root of the area under $S_a(\omega)$, the root-mean-square (r.m.s.) value of the absolute acceleration response process may be expressed as

$$a_{\mathrm{rms}} = [\ddot{z}_{\mathrm{rms}}^2 + 2\omega_n^2 \dot{x}_{\mathrm{rms}}^2 - \ddot{x}_{\mathrm{rms}}^2]^{1/2} \tag{5}$$

where

$$\ddot{z}_{\mathrm{rms}} = \left[\int_0^\infty S_{\ddot{z}}(\omega)\,\mathrm{d}\omega\right]^{1/2} \tag{6}$$

is the r.m.s. value of the base acceleration process,

$$\dot{x}_{\mathrm{rms}} = \left[\int_0^\infty |\omega H(\omega)|^2 S_{\ddot{z}}(\omega)\,\mathrm{d}\omega\right]^{1/2} \tag{7}$$

is the r.m.s. value of the relative displacement response process and

$$\ddot{x}_{\mathrm{rms}} = \left[\int_0^\infty |\omega^2 H(\omega)|^2 S_{\ddot{z}}(\omega)\,\mathrm{d}\omega\right]^{1/2} \tag{8}$$

is the r.m.s. value of the relative acceleration response process of the oscillator mass.

Assuming that the r.m.s. values of the absolute acceleration, relative acceleration and relative velocity responses of the oscillator mass, and base acceleration are related to their respective largest peak amplitudes via same peak factors and that the effects of nonstationarity on the r.m.s. values, as obtained in Equations (5)–(8), are identical [10], the largest peak amplitude of the absolute acceleration is approximated as

$$\mathrm{SA} = [\mathrm{PGA}^2 + 2\omega_n^2 \mathrm{SV}^2 - \mathrm{RSA}^2]^{1/2} \tag{9}$$

In Equation (9), PGA (peak ground acceleration) is the largest peak amplitude of the ground acceleration, SV is the largest peak amplitude of the relative velocity response, RSA (relative spectral acceleration) is the largest peak amplitude of the relative acceleration response, and SA is the largest peak amplitude of the absolute acceleration response of the oscillator mass.

Using Equation (9), SV may be expressed as

$$\mathrm{SV} = \frac{1}{\omega_n\sqrt{2}}[\mathrm{SA}^2 + \mathrm{RSA}^2 - \mathrm{PGA}^2]^{1/2} \tag{10}$$

It is a very good approximation to consider PSA in place of SA for low damping ratios, and to consider PRSA (pseudo relative spectral acceleration) in place of RSA. PRSA is expressed as [11]

$$\mathrm{PRSA} = \begin{cases} \sqrt{\mathrm{PSA}^2 - \mathrm{PGA}^2}, & T_n \leqslant T_c \\ \sqrt{\mathrm{PSA}^2 + \mathrm{PGA}^2}, & T_n > T_c \end{cases} \tag{11}$$

DOI: 10.1002/eqe

where T_c is the mean period of the base acceleration corresponding to the centre of gravity of its Fourier spectrum. SV may thus be approximated as

$$\mathrm{SV}=\begin{cases}\dfrac{1}{\omega_n}\sqrt{\mathrm{PSA}^2-\mathrm{PGA}^2}, & T_n<T_c\\[2mm] \dfrac{1}{\omega_n}\mathrm{PSA}, & T_n\geqslant T_c\end{cases} \tag{12}$$

It has been customary to approximate SV by PSV ($=\omega_n$SD), which is equal to PSA$/\omega_n$, but it follows from Equation (12) that such an approximation may be valid only for the periods longer than the mean period of the base acceleration. For short periods, this approximation may result in an overestimation.

NUMERICAL ILLUSTRATION

In order to illustrate the proposed approximation, three earthquake ground motions with the details given in Table I are considered. The first two motions (Nos. 1 and 2) are recorded motions while the third motion has been synthetically generated for a Mexico City site during the 1985 Michoacan earthquake (see Reference [12] for details). The mean period T_c and PGA values for these motions are given in Table I. The Fourier spectra of these motions (as normalized to the unit maximum value) are shown in Figure 1. It may be observed that the lower cutoff period T_{min} (i.e. the period below which there is insignificant energy in the motion) for the three motions covers a wide range of values. Whereas it is less than 0.04 s for the Imperial Valley motion, it is about 0.11 s for the Borrego Mountain motion and 0.55 s for the Michoacan motion.

The SV values and their approximations (as in Equation (12)) are examined for ($4\times 83=$) 332 oscillators, having damping ratios as 0, 2, 5 and 10% and 83 natural periods equispaced on logarithmic scale between 0.04 and 8.5 s, in the case of the three example ground motions. Calculations of the SV and PSA values are carried out as for the Volume 3 data supplied by the Strong Motion Group at the University of Southern California [13, 14].

The goodness of an approximation is estimated by computing the percentage errors in the approximated SV values at various periods (of the 83 oscillators considered) within a specific range and then by finding the r.m.s. error over those periods. The approximations considered are the PSV approximation and the proposed approximation (see Equation (12)) for the periods less

Table I. Details of the example ground motions.

Record No.	Earthquake	Site	Component	PGA (g)	T_c (s)
1	Imperial Valley Earthquake, 1940	El Centro Site, Imperial Valley Irrigation District, CA, U.S.A.	S00E	0.348	0.17
2	Borrego Mountain Earthquake, 1968	Engineering Building, Santa Ana, Orange County, CA, U.S.A.	S04E	0.013	0.38
3	Michoacan Earthquake, 1985	Mexico City, Mexico	Synthetic	0.101	0.96

DOI: 10.1002/eqe

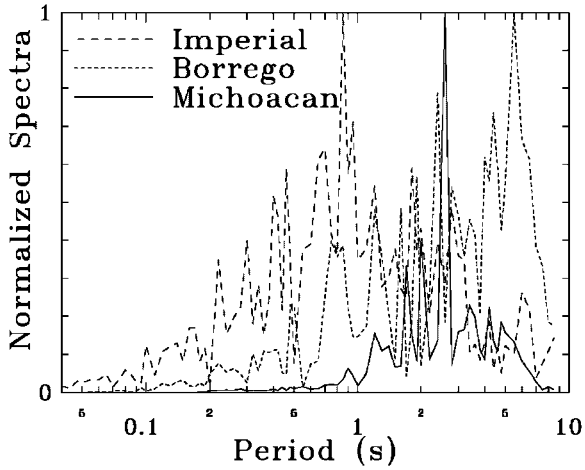

Figure 1. Normalized Fourier spectra of the example ground motions.

Table II. Comparison of R.M.S. error in different approximations of SV values.

Approximation	PSV		Equation (12)		PSV or Equation (12)
	$T_n<T_c$	$T_{\min}\leqslant T_n<T_c$	$T_n<T_c$	$T_{\min}\leqslant T_n<T_c$	$T_n\geqslant T_c$
(1)	(2)	(3)	(4)	(5)	(6)
Example motion	Record No. 1 (Imperial Valley), $T_c=0.17\,\mathrm{s}$				
0% damping	13.36	13.36	6.28	6.28	13.21
2% damping	34.13	34.13	15.33	15.33	18.59
5% damping	46.16	46.16	23.04	23.04	22.64
10% damping	58.11	58.11	30.18	30.18	28.41
Example motion	Record No. 2 (Borrego Mountain), $T_c=0.38\,\mathrm{s}$				
0% damping	208.70	12.24	39.72	8.33	5.37
2% damping	459.80	51.35	53.82	19.01	8.02
5% damping	497.03	82.59	55.27	24.40	9.63
10% damping	527.86	118.23	47.96	24.84	15.31
Example motion	Record No. 3 (Michoacan), $T_c=0.96\,\mathrm{s}$				
0% damping	1481.24	14.00	68.63	8.69	19.39
2% damping	1557.54	55.23	50.92	23.88	21.54
5% damping	1560.75	68.50	52.79	23.37	25.03
10% damping	1569.29	84.75	54.08	21.21	28.14

than T_c. In the proposed approximation, if the PSA value comes out to be less than the PGA value, the approximated SV value is taken to be zero. The r.m.s. error for these cases is compared in Table II (see the second and fourth columns) for all four cases of damping ratios and three example motions. Also compared is the r.m.s. error for the PSV approximation for the periods greater than or equal to T_c for each of these cases (see the sixth column in Table II). Figures 2(a)–(c) show comparisons of the exact 5%-damping SV curves with their PSV and proposed approximations

DOI: 10.1002/eqe

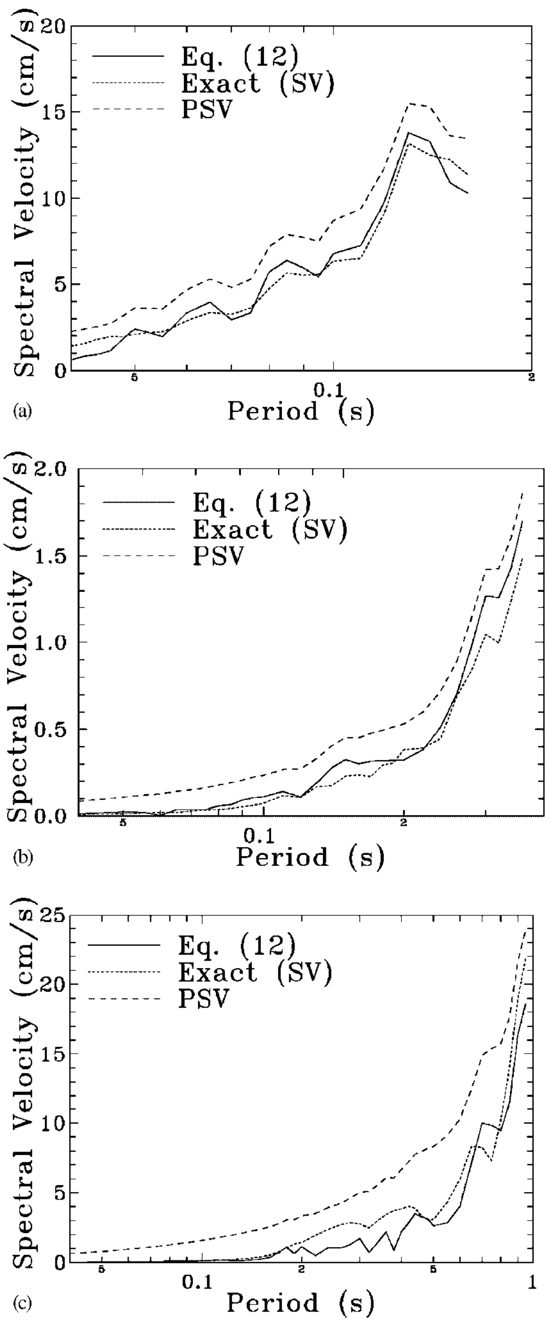

Figure 2. Comparison of the exact and approximate spectral velocity curves for $T_n < T_c$ in the case of the: (a) Imperial Valley motion; (b) Borrego Mountain motion; and (c) Michoacan motion.

DOI: 10.1002/eqe

for the periods $T_n < T_c$ in the cases of Imperial Valley, Borrego Mountain and Michoacan motions, respectively.

It may be observed from Table II that error in an approximation usually increases with damping. This is consistent with the known trends of discrepancies between the SV and PSV values and the fact that PSA values are better estimates of the SA values at lower damping ratios. Further, for any given damping, the PSV approximation for $T_n < T_c$ appears to become poorer with increasing mean period of the ground motion. This happens because for a given oscillator, with ground motion becoming more and more flexible with respect to the oscillator, waves in the ground motion undergo greater deamplification for the velocity response (than for the displacement response). The resulting overestimation in the case of PSV approximation is smaller for the stiffer oscillators. This overestimation however becomes critical at very high values of T_c/T_n because the velocity response tends to zero values for such cases (see Figures 2(a)–(c)). It is thus clearly implied that the error with the PSV approximation should not be neglected for the long-period ground motions, even when the oscillators are not very stiff. To illustrate this further, a case of $T_{\min} \leqslant T < T_c$ is also considered wherein only those periods less than the mean period are considered, which are not shorter than the lower cutoff period $T_{\min}$. The results for the PSV and proposed approximations with this constraint are recomputed and shown in the third and fifth columns of Table II, respectively. The values of $T_{\min}$ are considered to be <0.04 s in the case of the Imperial Valley motion, 0.11 s in the case of the Borrego Mountain motion and 0.55 s in the case of the Michoacan motion. On comparing the second and third columns, it may be noted that the r.m.s. error for the PSV approximation comes down significantly, as the oscillators with very high values of T_c/T_n are not considered in the computation of the error. It is still higher than the error for the $T_n \geqslant T_c$ case, and therefore one needs to use the PSV approximation (for $T_n < T_c$) only for those periods that get closer to T_c, as the low-frequency content in the ground motion becomes stronger. In the case of a high-frequency motion, e.g. the Imperial Valley motion, this would work reasonably well even when T_c/T_n is high.

It may be observed from the results in the sixth column of Table II that r.m.s. errors for the $T_n \geqslant T_c$ case increase with damping and are minimum for the Borrego Mountain motion. Approximating SV by PSV at the oscillator periods longer than the mean period of the ground motion is largely governed by the natural period and damping of the oscillator. While both displacement and velocity responses are not affected significantly by the energy at much shorter periods (in the ground motion) and are affected almost equally by the energy at resonant periods, the displacement response is affected a little less by the energy in the shorter periods and this difference is greater for the highly damped and/or very stiff oscillators. In the case of the Imperial Valley motion with a shorter mean period, the oscillators in the $T_n \geqslant T_c$ range are stiffer than those in the case of the Borrego Mountain and, therefore, the r.m.s. errors are greater in the case of the Imperial Valley motion. However, these errors do not become further smaller for the Michoacan motion (with even longer mean period), because the motion is narrow-banded with low spectral width [15]. Owing to this, there is hardly any energy (in the ground motion) in the periods longer than the natural period of the oscillator. It may be mentioned here that for a motion with large spectral width or with energy over a wide band of periods, the contribution of the periods greater than the oscillator period will be greater to the displacement response (than to the velocity response) and this would nullify the smaller response of the periods shorter than the oscillator period. In any case, the r.m.s. errors in the PSV approximation (for $T_n \geqslant T_c$) appear to remain below 30%.

On comparing the results in the fourth column with those in the second column in Table II, it is seen that the proposed approximation reduces the r.m.s. error (over $T < T_c$) in all the cases

DOI: 10.1002/eqe

considered. As discussed above, there is a greater deamplification in the ground motion waves for the velocity response. The resulting overestimation of SV can be shown to be proportional to the PGA of the motion, and the PGA correction term in Equation (12) basically eliminates this overestimation. It may be noted that the reduction in the r.m.s. error is more significant in the motions with longer mean periods. This observation also holds in the case of $T_{\min} \leqslant T < T_c$ (compare the results in the third and fifth columns in Table II), where a comparison is made only for those periods that have some energy in the ground motion. On comparing the results in the fifth and sixth columns, it is seen that the r.m.s. errors with the proposed approximation in the case of $T_{\min} \leqslant T < T_c$ are of the same order as those for the $T_n \geqslant T_c$ case. Thus, the proposed approximation is associated with a more consistent level of errors over the entire range of significant periods, particularly in the case of long-period ground motions.

It may be concluded from the above that if a ground motion has the mean period close to that for the Imperial Valley motion, the typical periods of structural systems would be sufficiently high to permit the PSV approximation of SV. On the other hand, if the ground motion has the dominance of long-period waves as in the case of Michoacan motion, majority of structural systems would be stiffer to the ground motion and, therefore, the proposed approximation should be useful in such cases. One needs to have an estimate of the mean period for using this approximation, albeit crudely, and then undamped PSV spectrum may be possibly used (instead of Fourier spectrum), since undamped SV spectrum provides upper bound to the Fourier spectrum [16]. It may however be mentioned that the results in this study are based on a specific suite of oscillators with periods between 0.04 and 8.5 s. In addition, this study does not consider the effects of initial conditions, which may govern the response in the cases of short-duration ground motions and/or long-period structures [17].

CONCLUSIONS

A new approximation has been proposed for SV values for oscillators with periods shorter than the mean period of the ground motion. This approximation requires the knowledge of PSA and PGA for the ground motion considered, as against the conventional approximation that required only the knowledge of PSA. Knowledge of the mean period of the ground motion is also required. However, it will be sufficient if one has only a crude idea of this parameter. By assuming that the initial conditions do not govern the maximum response and by considering 83 oscillators with periods between 0.04 and 8.5 s, it has been shown that the proposed approximation may be useful for applications in the case of very stiff structures or when the ground motion is dominated by long-period waves. The conventional PSV approximation may overestimate SV values in such cases.

ACKNOWLEDGEMENTS

The author is grateful to Professor M. D. Trifunac of the University of Southern California, U.S.A. and to Dr I. D. Gupta of Central Water & Power Research Station, India for critical reading of the manuscript and for offering many useful suggestions.

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DOI: 10.1002/eqe