ChE 381: Solution Key to Midsem exam

1. $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2$$

$$y(0) = y'(0) = 0$$

Laplace domain:

$$s^2 Y(s) + 2sY(s) + 2Y(s) = \frac{2}{s}$$

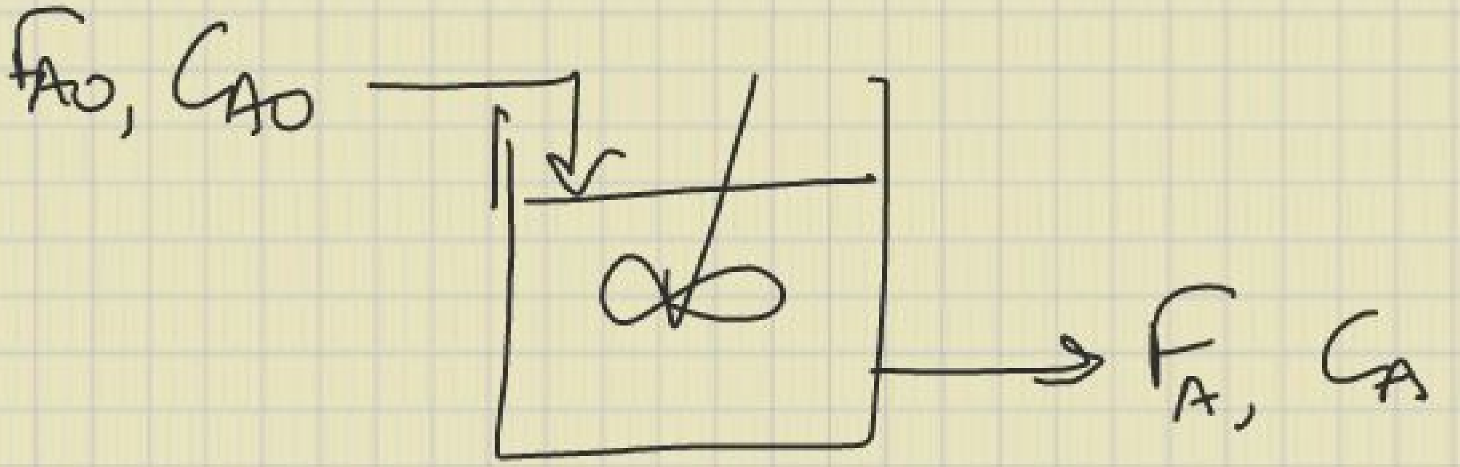
$$Y(s) = \frac{2}{s[s^2+2s+2]}$$

Inverse transform:

$$\boxed{y(t) = 1 + [\cos t + \sin t][\sinh t - \cosh t]}$$

(10 points).

② Isothermal CSTR, constant volume.



$F_{A0} = 0.085\ m^3/min \qquad V = 2.1\ m^3$

$C_{A0} = 0.925\ mol/m^3$

a) $$V \frac{dC_A}{dt} = F_{A0} C_{A0} - F_A C_A - k C_A V$$

$$\frac{dC_A}{dt} + \left(\frac{F_{A0}}{V} + k \right) C_A = \frac{F_{A0}}{V} C_{A0}$$

(4 points).

b) Existing steady-state:

$$C_{A,ini} = \frac{\frac{F_{A0}}{V} C_{A0}}{\left(\frac{F_{A0}}{V} + k \right)}$$

$$\boxed{C_{A,ini} = 0.465237 \ \frac{mol}{m^3}}$$ (4 points)

(c) There is a step-jump in the inlet concn. We have to solve the unsteady ODE with the existing steady state of part b as the initial condition.

$$\frac{dC_{A,new}}{dt} + \left(\frac{F_{A0}}{V} + k\right) C_{A,new} = \frac{F_{A0}}{V} C_{A0,new}$$

2×0.925

with initial condition

$$C_{A,new}(t=10) = 0.465237 \ \frac{mol}{m^3}$$

solving, we obtain

$$\boxed{C_{A,new} = 0.930474 - 0.465 \, e^{-0.0804 t}}$$

(4 points)

d) The ultimate steady-state value of $C_{A,new}$ after the step jump is

$$\boxed{C_{A,new}(t \to \infty) = 0.930474 \ \frac{mol}{m^3}}$$

(4 points)

e) Time constant of the process:

$$\tau = \left(\frac{F_{A0}}{V} + k\right)^{-1}$$

$$= \frac{1}{0.080474} \ \text{min}$$

$$\boxed{\tau = 12.42 \ \text{min}}$$

(4 points).

③ a) $A\frac{dh}{dt} = F_i(t) - F_0$

Constant exit flow rate is fixed by the pump.

b) There exists a steady-state if the inlet flow rate $F_i(t) = F_0$ (i.e $\frac{dh}{dt} = 0$). Since F_0 is a constant, a steady state will exist only if $F_i(t)$ is also = the same constant.

c) Given $F_i(t) = F_0 + Q\sin(\omega t)$

$y = h(t) - \bar{h}$

$$\frac{dy}{dt} = \frac{Q}{A} \sin(\omega t)$$

$$y(t) = \frac{Q}{A\omega} [1 - \cos(\omega t)]$$

$$y(t) = \frac{10}{2.5 \times \omega} [1 - \cos\omega t]$$

$$\boxed{y(t) = \frac{4}{\omega} [1 - \cos\omega t]}$$ (5 points)

d) for $\omega = 0.2$,

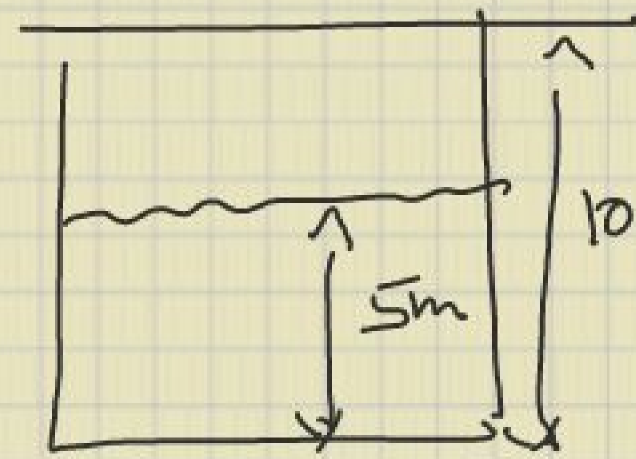
$$y(t) = 20 [1 - \cos(\omega t)]$$

maximum value of deviation when $\cos(\omega t) = -1 \Rightarrow 40$ m

minimum value of deviation when $\cos(\omega t) = 1 \Rightarrow 0$ m.

(5 points)

(c) If $\bar{h} = 5\,m$, and the tank height is 10m



$$h(t) - \cancel{\bar{h}}^{\,5} = \frac{4}{\omega}\left[1 - \cos(\omega t)\right]$$

If the height $h(t)$ should not exceed 10m, then:

$$10 - 5 = \frac{4}{\omega}\underbrace{\left[1 - \cos(\omega t)\right]}$$

maxm. value

$$5 = \frac{4}{\omega} \cdot 2$$

$$\boxed{\begin{aligned} \omega &= \frac{8}{5}\ \frac{rad}{hr} \\ \omega &= 1.6\ \frac{rad}{hr} \end{aligned}}$$

(5 points)

The tank will not run dry, since if $\cos(\omega t) = 1$, $h(t) = \bar{h} = 5\,m$.

(4) $$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}.$$

$\tau_1 > \tau_2$; unit step input.

a) $$Y(s) = \frac{1}{s}\frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$y(t) = K\left[1 + e^{-\frac{t}{\tau_1}}\frac{(\tau_a - \tau_1)}{(\tau_1 - \tau_2)} - e^{-\frac{t}{\tau_2}}\frac{(\tau_a - \tau_2)}{(\tau_1 - \tau_2)}\right]$$

$$\frac{dy}{dt} = K\left[e^{-t/\tau_1}\frac{(\tau_1 - \tau_a)}{\tau_1(\tau_1 - \tau_2)} - e^{-\frac{t}{\tau_2}}\frac{(\tau_2 - \tau_a)}{\tau_2(\tau_1 - \tau_2)}\right]$$

Set $\frac{dy}{dt} = 0$; since $\tau_1 > \tau_2$

$$e^{-\frac{t}{\tau_1}} > e^{-\frac{t}{\tau_2}}$$

So $$\boxed{1 - \frac{\tau_a}{\tau_1} < 1 - \frac{\tau_a}{\tau_2}}$$ (5 points)

b) $$y(t) = K\left[1 + \frac{e^{-\frac{t}{\tau_1}}(\tau_a - \tau_1)}{(\tau_1 - \tau_2)} - \frac{e^{-\frac{t}{\tau_2}}(\tau_a - \tau_2)}{(\tau_1 - \tau_2)}\right]$$

$y(t \to \infty) \to \underline{1}$.

overshoot happens only if

$$\boxed{\tau_a > \tau_1}$$ (5 points)

c) Inverse response happens if

$$\left.\frac{dy}{dt}\right|_{t=0} < 0$$

$$\left.\frac{dy}{dt}\right|_{t=0} = K\left[\frac{(\tau_1 - \tau_a)}{\tau_1(\tau_1 - \tau_2)} - \frac{(\tau_2 - \tau_a)}{\tau_2(\tau_1 - \tau_2)}\right]$$

will be < 0 only if

$$\boxed{\tau_a < 0}$$ (5 points)

(d) $\frac{dy}{dt} = 0$ at $t = t^*$

$$e^{-\frac{t^*}{\tau_1}}\left[1 - \frac{\tau_a}{\tau_1}\right] = e^{-\frac{t^*}{\tau_2}}\left[1 - \frac{\tau_a}{\tau_2}\right]$$

$$\boxed{t^* = \frac{1}{\left[\frac{1}{\tau_2} - \frac{1}{\tau_1}\right]} \ln\left[\frac{1 - \frac{\tau_a}{\tau_2}}{1 - \frac{\tau_a}{\tau_1}}\right]}$$

(5 points).

(5)- $g(s) = g_1(s)\, e^{-\alpha s}$

For a sinusoidal input,

$$Y(s) = g_1(s)\, e^{-\alpha s} \frac{A\omega}{s^2+\omega^2}$$

partial fraction except $e^{-\alpha s}$ term

$$y_1(s) = \sum_{i=1}^{n} \left[\frac{A_i}{(s-r_i)} \right] + \frac{B_1}{(s-j\omega)} + \frac{B_2}{(s+j\omega)}$$

$$y_1(t) = \sum_{i=1}^{n} A_i e^{r_i t} + B_1 e^{j\omega t} + B_2 e^{-j\omega t}$$

Response without time delay.

With delay

$y(t) = 0 \qquad t \leq \alpha$

$$y(t) = \sum A_i e^{r_i(t-\alpha)} + B_1 e^{j\omega(t-\alpha)} + B_2 e^{-j\omega(t-\alpha)} \quad (t > \alpha)$$

For $t \to \infty$ (ultimate periodic response)

$$y(t) = B_1 e^{j\omega(t-\alpha)} + B_2 e^{-j\omega(t-\alpha)}$$

$$= \tilde{B}_1 e^{j\omega t} + \tilde{B}_2 e^{-j\omega t}$$

$$\tilde{B}_1 = B_1 e^{-\alpha j\omega} \qquad \tilde{B}_2 = B_2 e^{\alpha j\omega}$$

Evaluate B_1 and B_2:

$$B_1 = \frac{A\, g_1(j\omega)}{2j} \qquad B_2 = \frac{A\, g_1(-j\omega)}{2j}$$

$$\tilde{B}_1 = \frac{A\, g_1(j\omega)\, e^{-\alpha j\omega}}{2j} \; ; \; B_2 = \frac{A\, g_1(-j\omega)\, e^{\alpha j\omega}}{2j}$$

$$\tilde{B}_1 = \frac{A\, g(j\omega)}{2j} \; ; \; B_2 = \frac{A\, g(-j\omega)}{2j}$$

$$y(t)\Big|_{t\to\infty}$$

$$= \left[\frac{A\, g(j\omega)}{2j}\right] e^{j\omega t} + \left[\frac{A\, g(-j\omega)}{2j}\right] e^{-j\omega t}$$

$$\boxed{\begin{aligned} y(t)\Big|_{t\to\infty} &= A\left[\mathrm{Re}[g(j\omega)]\right] \sin\omega t \\ &\quad + A\left[\mathrm{Im}[g(j\omega)]\right] \cos\omega t \end{aligned}}$$

Thus, the result derived in the lectures hold even for transfer functions with delay.

[25 points].