Introduction to Hydrodynamic Stability

V. Shankar

Department of Chemical Engineering
IIT Kanpur
vshankar@iitk.ac.in
home.iitk.ac.in/~vshankar

SADEAFFP-2014, Department of Mathematics, IIT Kanpur

Outline

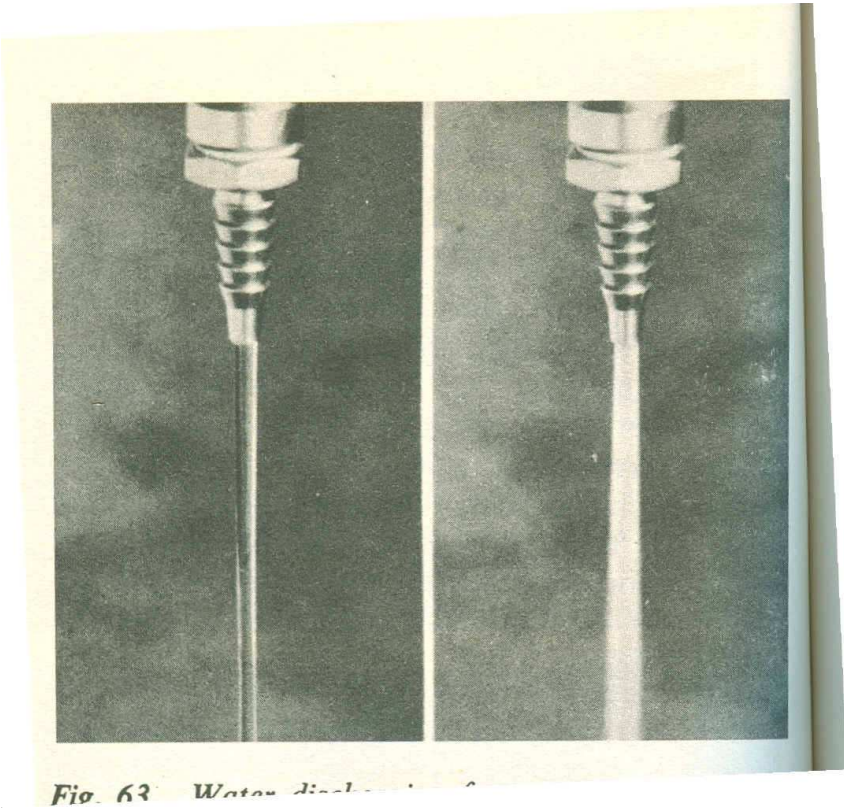
References

1. P. G. Drazin, Introduction to Hydrodynamic Stability, Cambridge (2002).
2. F. Charru, Hydrodynamic Instabilities, Cambridge (2011).
3. P. Schmid and D. Henningson, Stability and Transition in Shear Flows, Springer (2001).
4. P. G. Drazin and W. H. Reid, Hydrodynamic Stability, Cambridge (1981).
5. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford (1961).
6. National Committee for Fluid Mechanics Films, Video on "Instability and Transition", `http://web.mit.edu/hml/ncfmf.html`

Outline

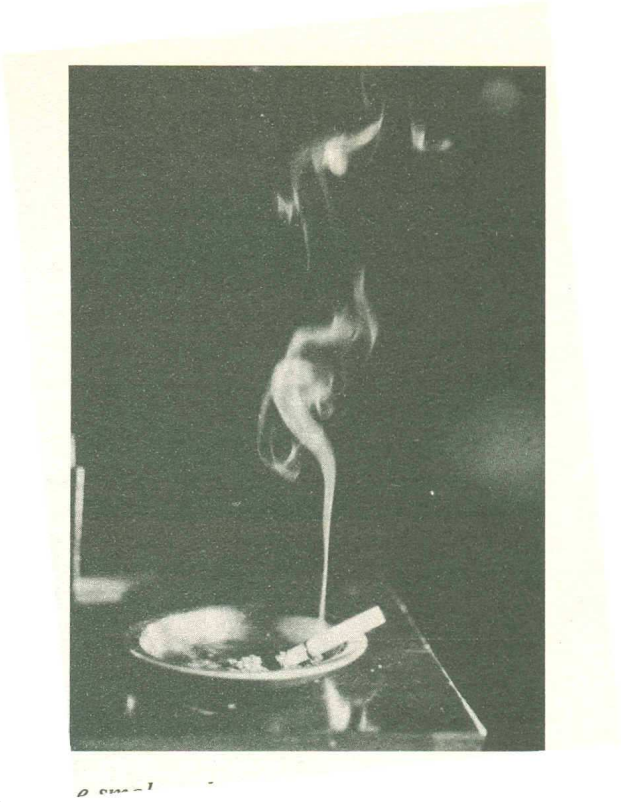
Instabilities are everywhere

- Water flow from a tap.
- Smoke from an incence stick.
- Flow between two concentric cylinders.
- A layer of heated liquid.



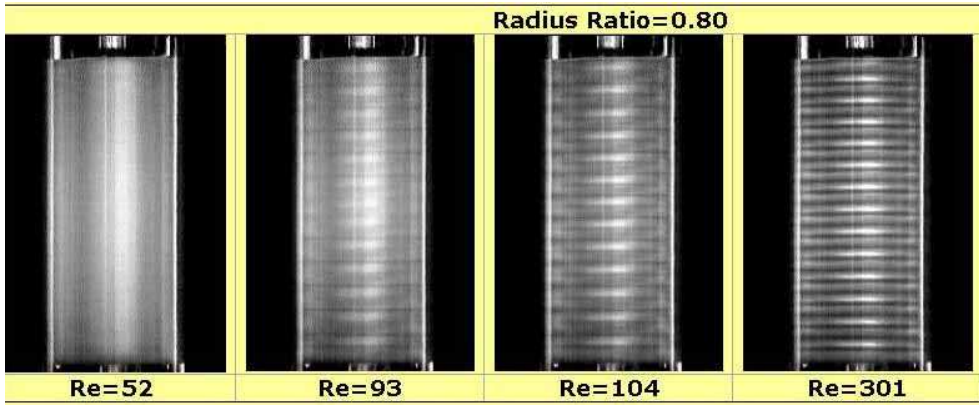
Instabilities are everywhere

- Water flow from a tap.
- Smoke from an incence stick.
- Flow between two concentric cylinders.
- A layer of heated liquid.



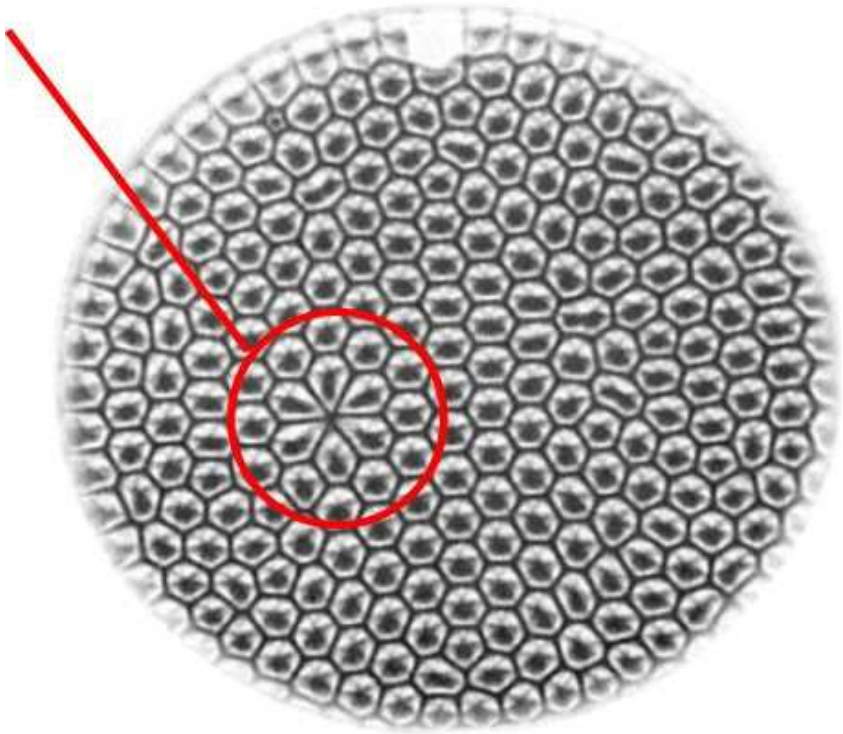
Instabilities are everywhere

- Water flow from a tap.
- Smoke from an incence stick.
- Flow between two concentric cylinders.
- A layer of heated liquid.



Instabilities are everywhere

- Water flow from a tap.
- Smoke from an incence stick.
- Flow between two concentric cylinders.
- A layer of heated liquid.



Hydrodynamic stability: connection with Mathematics

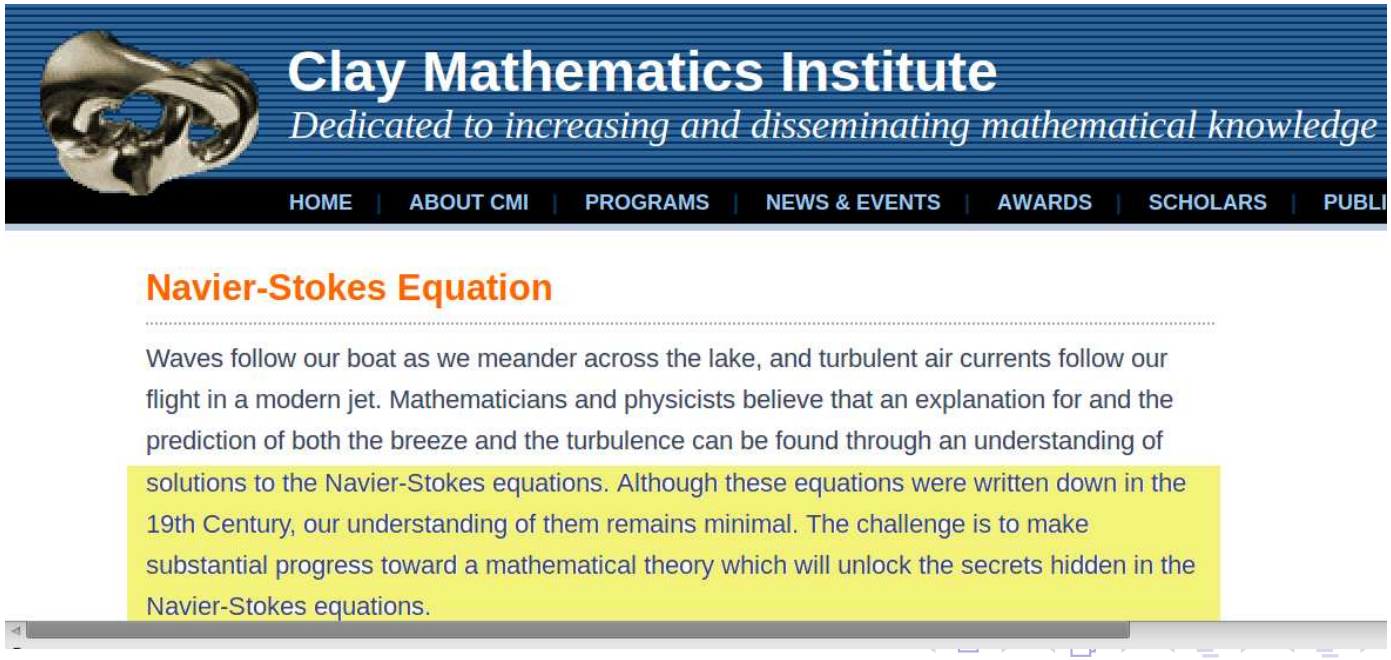
- Pioneers: Applied Mathematicians and Mathematical Physicists in late 1800s and early 1900s.
- E.g. von Helmholtz, Lord Kelvin, Lord Rayleigh, Neils Bohr, A Sommerfeld, W Heisenberg (PhD thesis, 1923), S Chandrasekhar, V I Arnold...
- One of the 7 Millennium Prize problems of Clay Mathematical Institute.

Hydrodynamic stability: connection with Mathematics

- Pioneers: Applied Mathematicians and Mathematical Physicists in late 1800s and early 1900s.
- E.g. von Helmholtz, Lord Kelvin, Lord Rayleigh, Neils Bohr, A Sommerfeld, W Heisenberg (PhD thesis, 1923), S Chandrasekhar, V I Arnold...
- One of the 7 Millennium Prize problems of Clay Mathematical Institute.

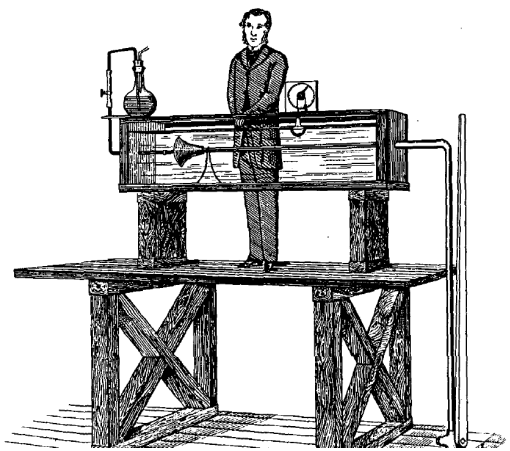
Hydrodynamic stability: connection with Mathematics

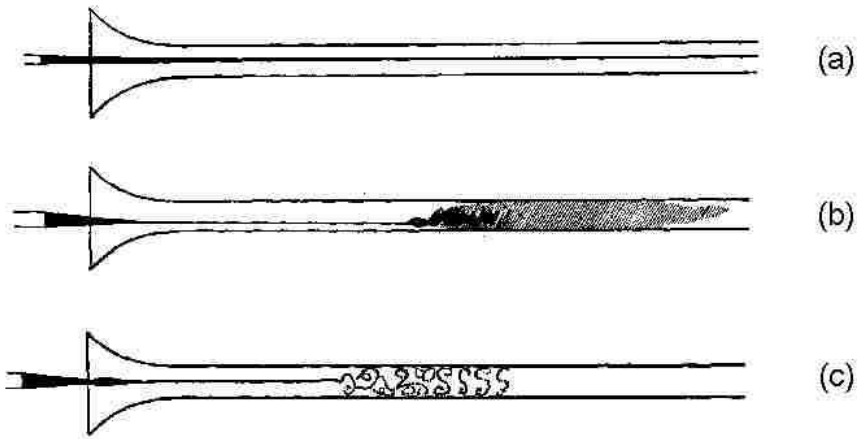
- Pioneers: Applied Mathematicians and Mathematical Physicists in late 1800s and early 1900s.
- E.g. von Helmholtz, Lord Kelvin, Lord Rayleigh, Neils Bohr, A Sommerfeld, W Heisenberg (PhD thesis, 1923), S Chandrasekhar, V I Arnold...
- One of the 7 Millennium Prize problems of Clay Mathematical Institute.



Laminar-turbulent transition

- Osborne Reynolds (1883), "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous.."



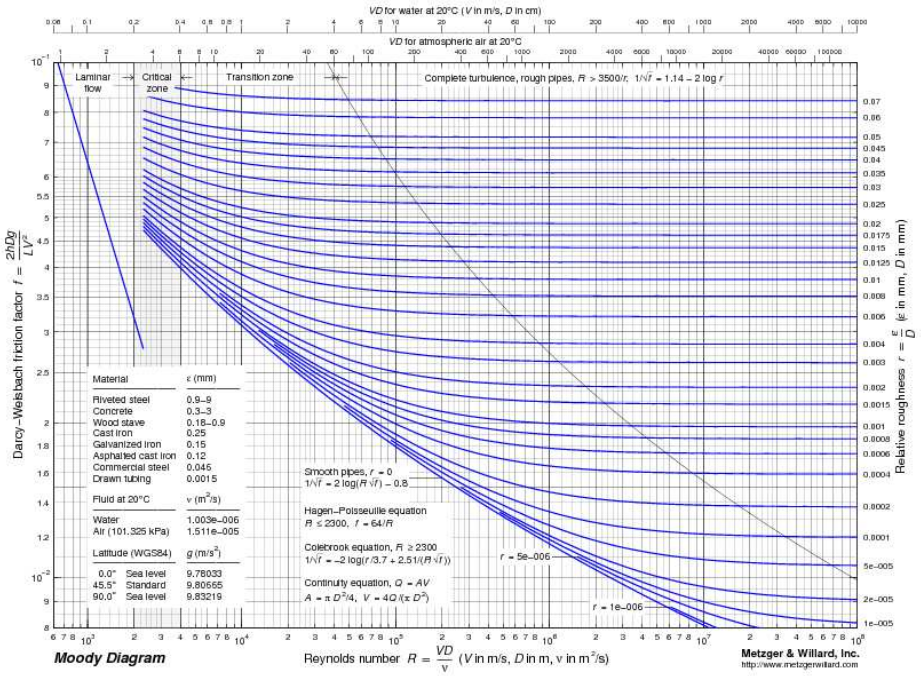


- Discontinuous transition from laminar to a turbulent flow when $Re \equiv \rho V D/\mu > 2000$.

- For rectangular channels, transition at $Re \sim 1200$.

Laminar-turbulent transition

- Osborne Reynolds (1883), "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous.."
- Discontinuous transition from laminar to a turbulent flow when $Re \equiv \rho VD/\mu > 2000$.



- For rectangular channels, transition at $Re \sim 1200$

Laminar-turbulent transition

- Osborne Reynolds (1883), "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous.."
- Discontinuous transition from laminar to a turbulent flow when $Re \equiv \rho V D/\mu > 2000$.

- For rectangular channels, transition at $Re \sim 1200$.

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Practical perspective

- Newtonian fluids: Navier-Stokes equations have all the information about fluid flow.
- Laminar flows: simple solutions to governing equations.
- Examples: plane and pipe Poiseuille flows; plane Couette flow.
- Flow in a pipe: laminar flow unstable at $Re \sim 2000$.
- Instability leads to *turbulence*.
- Turbulent flows: high mixing and drag.
- Laminar flows: low mixing and drag.
- When does a given laminar flow become unstable ?

Laminar flows

- **Navier-Stokes equations**

  $\nabla \cdot \mathbf{v} = 0 \qquad Re[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}] = -\nabla p + \nabla^2 \mathbf{v}$

- Laminar flows: flows with relatively simple kinematics and are usually time-independent.
- Laminar flow solutions satisfy Navier-Stokes equations at any Re.

Landau & Lifshitz

"Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable"

- Need to probe the stability of laminar flows to external disturbances.

Laminar flows

- **Navier-Stokes equations**

  $\nabla \cdot \mathbf{v} = 0 \qquad Re[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}] = -\nabla p + \nabla^2 \mathbf{v}$

- Laminar flows: flows with relatively simple kinematics and are usually time-independent.
- Laminar flow solutions satisfy Navier-Stokes equations at any Re.

Landau & Lifshitz

"Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable"

- Need to probe the stability of laminar flows to external disturbances.

Laminar flows

- **Navier-Stokes equations**

  $\nabla \cdot \mathbf{v} = 0 \qquad Re[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}] = -\nabla p + \nabla^2 \mathbf{v}$

- Laminar flows: flows with relatively simple kinematics and are usually time-independent.
- Laminar flow solutions satisfy Navier-Stokes equations at any Re.

Landau & Lifshitz

"Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable"

- Need to probe the stability of laminar flows to external disturbances.

Laminar flows

- Navier-Stokes equations

  $\nabla \cdot \mathbf{v} = 0 \qquad Re[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}] = -\nabla p + \nabla^2 \mathbf{v}$

- Laminar flows: flows with relatively simple kinematics and are usually time-independent.
- Laminar flow solutions satisfy Navier-Stokes equations at any Re.

Landau & Lifshitz

"Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable"

- Need to probe the stability of laminar flows to external disturbances.

Why do instabilities occur ?

- Real flows subjected to disturbances of various types.
- Disturbances distort the existing force equilibrium.
- Thermal convection & Circular Couette flows.

After instability

- New complex laminar states.
- Direct transition to turbulence.

Why do instabilities occur ?

- Real flows subjected to disturbances of various types.
- Disturbances distort the existing force equilibrium.
- Thermal convection & Circular Couette flows.

After instability

- New complex laminar states.
- Direct transition to turbulence.

Why do instabilities occur ?

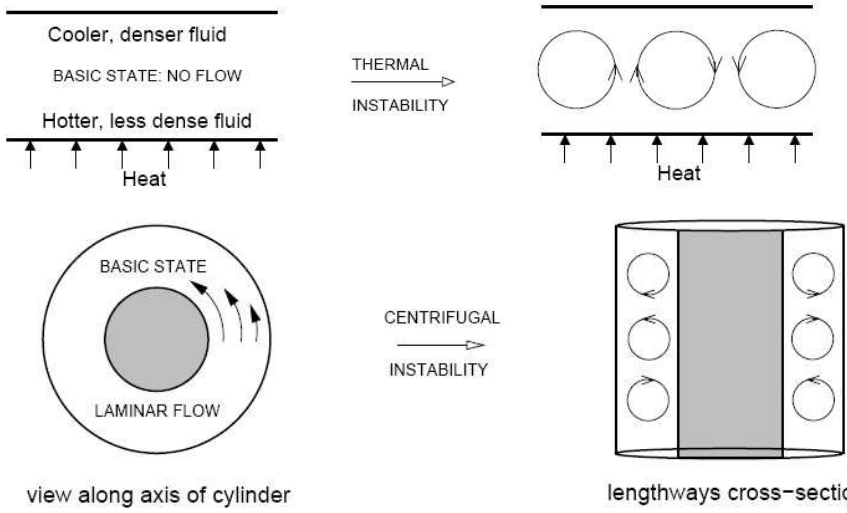
- Real flows subjected to disturbances of various types.
- Disturbances distort the existing force equilibrium.
- Thermal convection & Circular Couette flows.

After instability

- New complex laminar states.
- Direct transition to turbulence.

Why do instabilities occur ?

- Real flows subjected to disturbances of various types.
- Disturbances distort the existing force equilibrium.
- Thermal convection & Circular Couette flows.

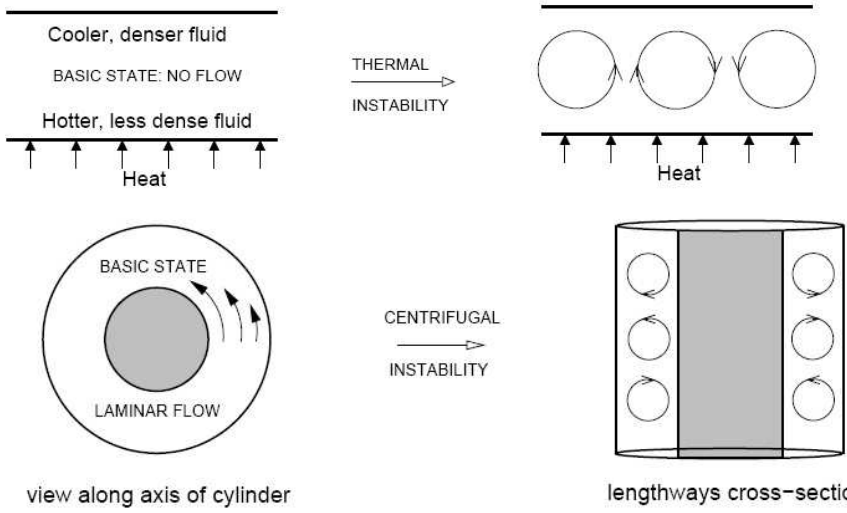


After instability

- New complex laminar states.
- Direct transition to turbulence.

Why do instabilities occur ?

- Real flows subjected to disturbances of various types.
- Disturbances distort the existing force equilibrium.
- Thermal convection & Circular Couette flows.



After instability

- New complex laminar states.
- Direct transition to turbulence.

How to analyze and predict instabilities ?

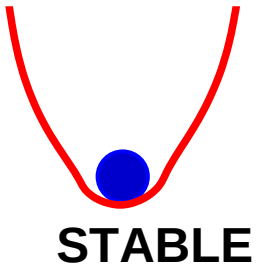
- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

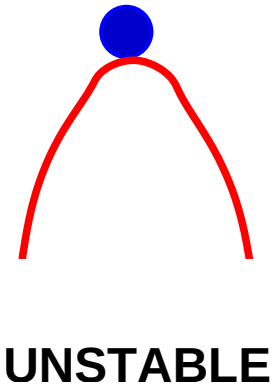
How to analyze and predict instabilities ?

- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

How to analyze and predict instabilities ?

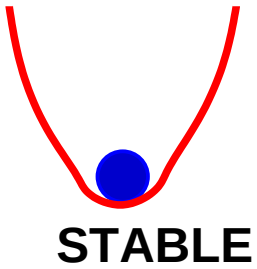
- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

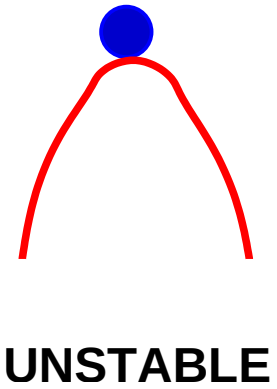




How to analyze and predict instabilities ?

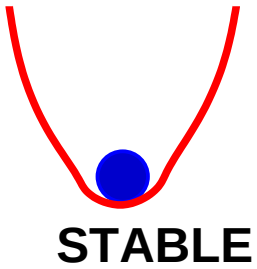
- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

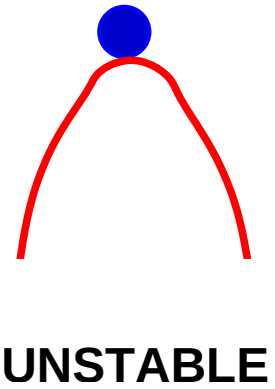




How to analyze and predict instabilities ?

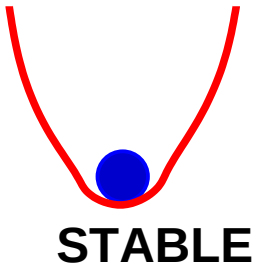
- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

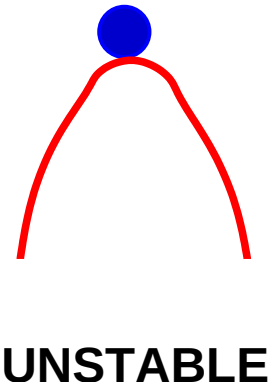




How to analyze and predict instabilities ?

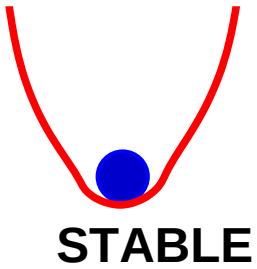
- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.

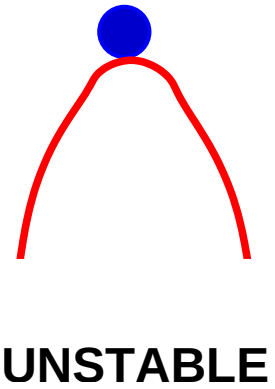




How to analyze and predict instabilities ?

- Experiments, computation and theory.
- What is the effect of an initial disturbance on laminar flow ?
- Do perturbations grow or decay ?
- At what value of Reynolds number ?
- Infinitesimal vs. finite disturbances.
- Infinitesimal disturbances: unavoidable.
- Linear stability theory.





Outline

Specify governing equations

Navier-Stokes equations (Newtonian fluid)

$$\nabla \cdot \mathbf{v} = 0\,,$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}\,.$$

Boundary conditions at rigid surface: no-slip and no-penetration.

Find the base state: $\overline{v}_x(z)$, and $\overline{p}(x)$ (steady, unidirectional).

For complex fluids

Specify appropriate constitutive relation.

Specify governing equations

Navier-Stokes equations (Newtonian fluid)

$$\nabla \cdot \mathbf{v} = 0\,,$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}\,.$$

Boundary conditions at rigid surface: no-slip and no-penetration.

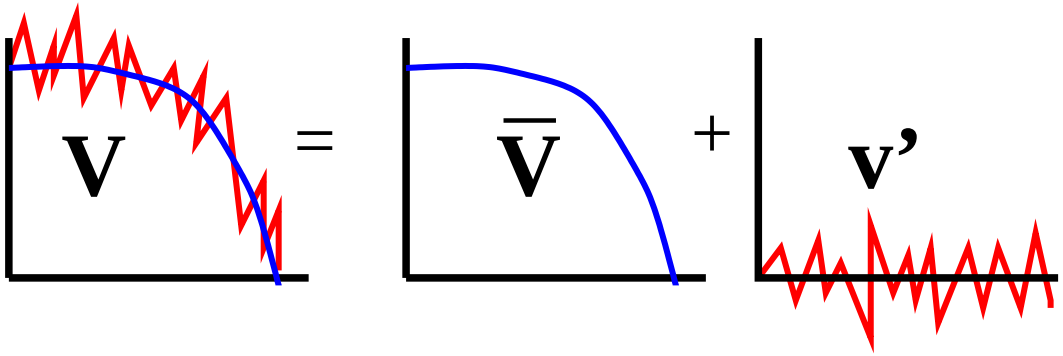
Find the base state: $\overline{v}_x(z)$, and $\overline{p}(x)$ (steady, unidirectional).

For complex fluids

Specify appropriate constitutive relation.

Add a small perturbation

$$\mathbf{v}(\mathbf{x},t) = \overline{\mathbf{v}}_{\mathbf{x}}(\mathbf{x}) + \delta\,\mathbf{v}'(\mathbf{x},t)\,, \quad p(\mathbf{x},t) = \overline{p}(\mathbf{x}) + \delta\,p'(\mathbf{x},t)\,, \quad |\delta| \ll 1.$$

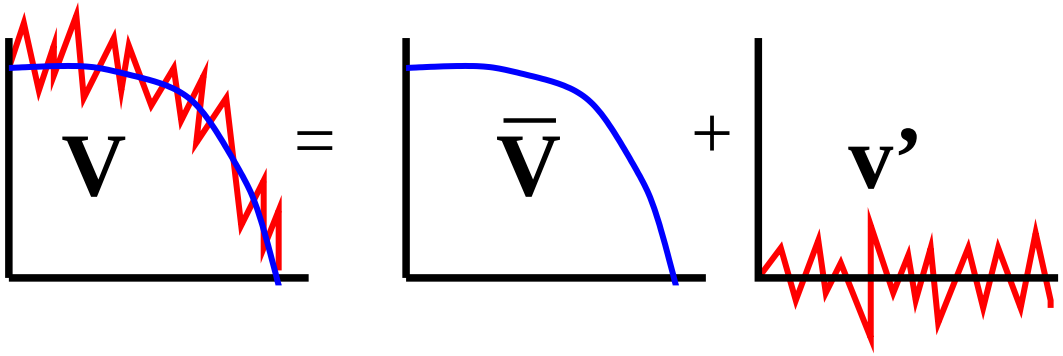


Key questions

- For a given control parameter λ, does $\mathbf{v}'$ grow or decay with time ?
- What is the critical λ for instability ?
- What is the spatial structure at the critical value ?

Add a small perturbation

$$\mathbf{v}(\mathbf{x},t) = \overline{\mathbf{v}}_{\mathbf{x}}(\mathbf{x}) + \delta\,\mathbf{v}'(\mathbf{x},t)\,, \quad p(\mathbf{x},t) = \overline{p}(\mathbf{x}) + \delta\,p'(\mathbf{x},t)\,, \quad |\delta| \ll 1.$$



Key questions

- For a given control parameter λ, does $\mathbf{v}'$ grow or decay with time ?
- What is the critical λ for instability ?
- What is the spatial structure at the critical value ?

Linearize about laminar state

Nonlinear term in Navier-Stokes:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \delta(\overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{v}) + \delta^2 \mathbf{v}' \cdot \nabla \mathbf{v}'$$

- $\overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the trivial laminar-flow contribution.
- At $O(\delta)$, terms linear in the perturbations $\Rightarrow$ Include.
- At $O(\delta^2)$, terms non-linear in the perturbations $\Rightarrow$ Neglected for small perturbations.
- Hence, *linear* stability.
- Can predict only the *onset* of instability.

Linearize about laminar state

Nonlinear term in Navier-Stokes:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \delta(\overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{v}) + \delta^2 \mathbf{v}' \cdot \nabla \mathbf{v}'$$

- $\overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the trivial laminar-flow contribution.
- At $O(\delta)$, terms linear in the perturbations $\Rightarrow$ Include.
- At $O(\delta^2)$, terms non-linear in the perturbations $\Rightarrow$ Neglected for small perturbations.
- Hence, *linear* stability.
- Can predict only the *onset* of instability.

Linearize about laminar state

Nonlinear term in Navier-Stokes:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \delta(\overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{v}) + \delta^2 \mathbf{v}' \cdot \nabla \mathbf{v}'$$

- $\overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the trivial laminar-flow contribution.
- At $O(\delta)$, terms linear in the perturbations $\Rightarrow$ Include.
- At $O(\delta^2)$, terms non-linear in the perturbations $\Rightarrow$ Neglected for small perturbations.
- Hence, *linear* stability.
- Can predict only the *onset* of instability.

Linearize about laminar state

Nonlinear term in Navier-Stokes:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \delta(\overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{v}) + \delta^2 \mathbf{v}' \cdot \nabla \mathbf{v}'$$

- $\overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the trivial laminar-flow contribution.
- At $O(\delta)$, terms linear in the perturbations $\Rightarrow$ Include.
- At $O(\delta^2)$, terms non-linear in the perturbations $\Rightarrow$ Neglected for small perturbations.
- Hence, *linear* stability.
- Can predict only the *onset* of instability.

Linearize about laminar state

Nonlinear term in Navier-Stokes:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \delta(\overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{v}) + \delta^2 \mathbf{v}' \cdot \nabla \mathbf{v}'$$

- $\overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the trivial laminar-flow contribution.
- At $O(\delta)$, terms linear in the perturbations $\Rightarrow$ Include.
- At $O(\delta^2)$, terms non-linear in the perturbations $\Rightarrow$ Neglected for small perturbations.
- Hence, *linear* stability.
- Can predict only the *onset* of instability.

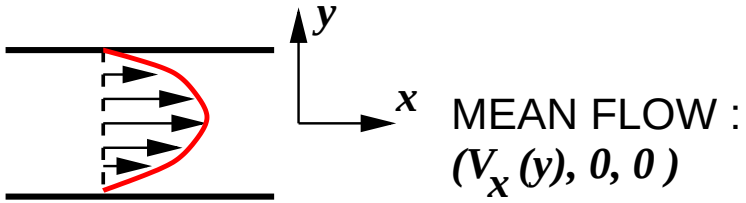
Linearized PDEs: Fourier expansion

Collect terms of $O(\delta)$:

$$\nabla \cdot \mathbf{v}' = 0$$

$$\rho[\partial_t \mathbf{v}' + \overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{\mathbf{v}}] = -\nabla p' + \mu \nabla^2 \mathbf{v}'$$

BCs: $\mathbf{v}' = 0$ at rigid boundaries; Initial condition $\mathbf{v}'(\mathbf{x}, t = 0)$.



Fourier expand the disturbances

$$A(\mathbf{x}, t) = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_z \sum_{n=1}^{\infty} A_n(k, t) F_n(y) \exp[\mathrm{i}(k_x x + k_z z)]$$

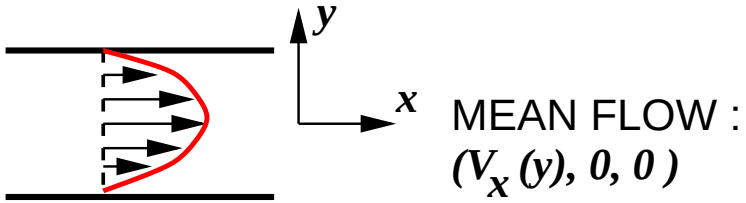
Linearized PDEs: Fourier expansion

Collect terms of $O(\delta)$:

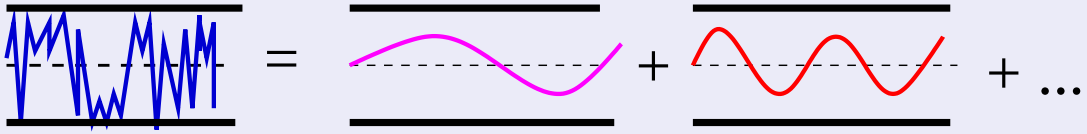
$$\nabla \cdot \mathbf{v}' = 0$$

$$\rho[\partial_t \mathbf{v}' + \overline{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{\mathbf{v}}] = -\nabla p' + \mu \nabla^2 \mathbf{v}'$$

BCs: $\mathbf{v}' = 0$ at rigid boundaries; Initial condition $\mathbf{v}'(\mathbf{x}, t = 0)$.



Fourier expand the disturbances



$$A(\mathbf{x}, t) = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_z \sum_{n=1}^{\infty} A_n(k, t) F_n(y) \exp[\mathrm{i}(k_x x + k_z z)]$$

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Normal modes: Eigenvalue problem

- Linearity $\Rightarrow$ Study the dynamics of all Fourier modes *individually*.
- Time dependence $A_n(k,t) = A_n(k)\exp[s_n t]$
- Complex growth rate $s = s_r + \mathrm{i}s_i$
- $s_r > 0$ Instability; $s_r < 0$ Stability; $s_r = 0$ Neutral stability.
- If *any* Fourier mode grows with time $\Rightarrow$ unstable, exponential growth as $t \to \infty$.
- If *all* Fourier modes decay $\Rightarrow$ stable as $t \to \infty$.
- Need to solve coupled ODEs (for $F(y)$) with an eigenvalue s for various values of control parameter λ.

Outline

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0.$
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

A simple toy example

- Governing equation: $\partial_t f = f - f^2 + \frac{1}{\lambda}\partial_y^2 f$
- Boundary conditions: $f(y=0) = f(y=1) = 0$.
- Base state: $\bar{f} = 0$
- Add perturbation: $f(y,t) = \bar{f} + \delta f'(y,t)$
- Linearize:
  $\partial_t(\bar{f} + \delta f') = \bar{f} - \bar{f}^2 + \frac{1}{\lambda}\partial_y^2 \bar{f} + \delta\left[f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'\right] + O(\delta^2)\cdots$
- At $O(\delta)$: $\partial_t f' = f' - 2\bar{f}f' + \frac{1}{\lambda}\partial_y^2 f'$
- $\partial_t f' = f' + \frac{1}{\lambda}\partial_y^2 f', \quad f'(y=0) = f'(y=1) = 0$
- Normal modes: $f'(y,t) = F(y)\exp[st]$
- $sF = F + \frac{1}{\lambda}d_y^2 F, \quad F(y=0) = F(y=1) = 0$

Solution of toy problem

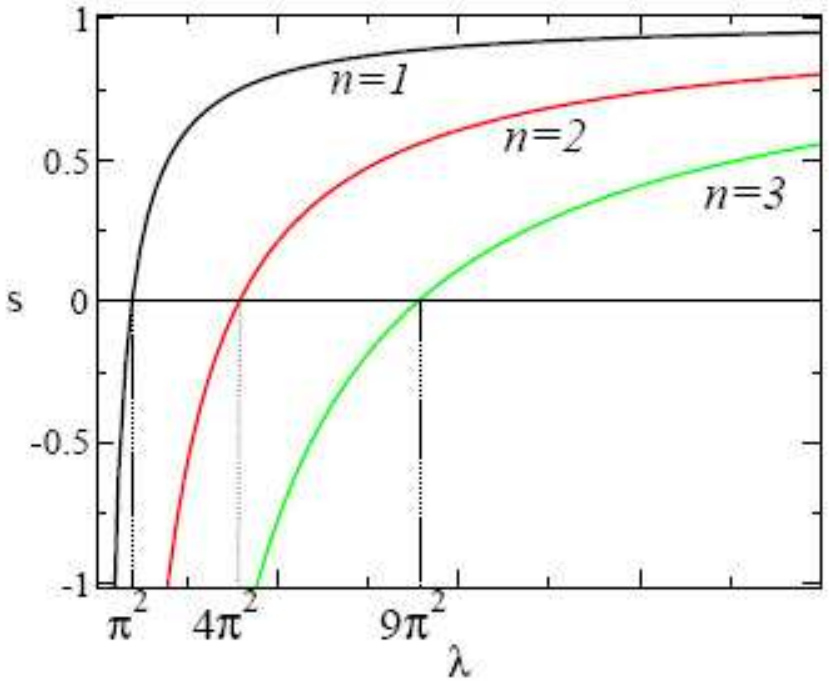
- $F(y) = c\sin[n\pi y] \quad n$ integer.
- $s_n = 1 - \frac{n^2\pi^2}{\lambda}$
- Stable $s < 0$ for $\lambda < n^2\pi^2$
- Neutral $s = 0$ for $\lambda = n^2\pi^2$
- Unstable $s > 0$ for $\lambda > n^2\pi^2$
- Most unstable mode:
- $f'(y,t) = c\sin[\pi y]\exp\left[\left(1 - \frac{\pi^2}{\lambda}\right)t\right]$

Solution of toy problem

- $F(y) = c\sin[n\pi y] \quad n$ integer.
- $s_n = 1 - \frac{n^2\pi^2}{\lambda}$
- Stable $s < 0$ for $\lambda < n^2\pi^2$
- Neutral $s = 0$ for $\lambda = n^2\pi^2$
- Unstable $s > 0$ for $\lambda > n^2\pi^2$
- Most unstable mode:
- $f'(y,t) = c\sin[\pi y]\exp\left[\left(1 - \frac{\pi^2}{\lambda}\right)t\right]$

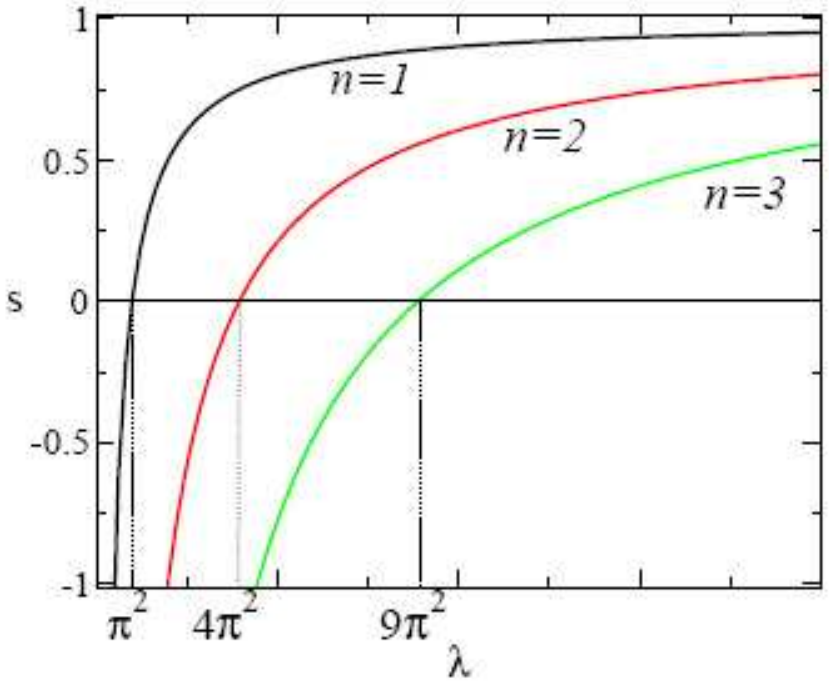
Solution of toy problem

- $F(y) = c\sin[n\pi y] \quad n$ integer.
- $s_n = 1 - \frac{n^2\pi^2}{\lambda}$
- Stable $s < 0$ for $\lambda < n^2\pi^2$
- Neutral $s = 0$ for $\lambda = n^2\pi^2$
- Unstable $s > 0$ for $\lambda > n^2\pi^2$
- Most unstable mode:
- $f'(y,t) = c\sin[\pi y]\exp\left[\left(1 - \frac{\pi^2}{\lambda}\right)t\right]$



Solution of toy problem

- $F(y) = c\sin[n\pi y] \quad n$ integer.
- $s_n = 1 - \frac{n^2\pi^2}{\lambda}$
- Stable $s < 0$ for $\lambda < n^2\pi^2$
- Neutral $s = 0$ for $\lambda = n^2\pi^2$
- Unstable $s > 0$ for $\lambda > n^2\pi^2$
- Most unstable mode:
- $f'(y,t) = c\sin[\pi y]\exp\left[\left(1 - \frac{\pi^2}{\lambda}\right)t\right]$



Outline

Kelvin-Helmholtz instability - I

- Two incompressible, inviscid fluids in horizontal parallel infinte streams of different densities and velocities, one stream above the other.
- $\mathbf{U}(z) = U_2\mathbf{i}$, $\rho(z) = \rho_2$, $P(z) = p_0 - g\rho_2 z$ for $z > 0$.
- $\mathbf{U}(z) = U_1\mathbf{i}$, $\rho(z) = \rho_1$, $P(z) = p_0 - g\rho_1 z$ for $z < 0$.
- Irrotational perturbations: restrictive, but enough for a proof of instability.
- Does not prove stability as the analysis gives no information about rotational disturbances.
- Perturbed interface $z = \zeta(x, y, t)$.
- $\mathbf{u} = \nabla\phi$ where $\phi = \phi_1$ for $z > \zeta$, and $\phi = \phi_2$ for $z < \zeta$.

Kelvin-Helmholtz instability - II

- $\nabla \cdot \mathbf{u} = 0$, $\nabla \times \mathbf{u} = 0 \Rightarrow \mathbf{u} = \nabla \phi$.
- $\nabla^2 \phi_1 = 0$ for $z > \zeta$ and $\nabla^2 \phi_2 = 0$ for $z < \zeta$.
- BC: $\nabla \phi = \mathbf{U}$ as $z = \pm\infty$.
- Kinematic condition at moving interface $z = \zeta$, $k = 1, 2$:
  $\frac{D\zeta}{Dt} = u_z^k = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi_k}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi_k}{\partial y}\frac{\partial \zeta}{\partial y}$
- Normal stress continuity (Bernoulli theorem) at $z = \zeta$:
  $\rho_1 \left[C_1 - \frac{1}{2}(\nabla \phi_1)^2 - \frac{\partial \phi_1}{\partial t} - gz \right] = \rho_2 \left[C_2 - \frac{1}{2}(\nabla \phi_2)^2 - \frac{\partial \phi_2}{\partial t} - gz \right]$
- The base flow also satisfies this condition (at $z = 0$):
  $\rho_1 \left(C_1 - \frac{1}{2} U_1^2 \right) = \rho_2 \left(C_2 - \frac{1}{2} U_2^2 \right)$

Kelvin-Helmholtz instability - Linearization

- $\phi_2 = U_2 x + \phi_2'$ for $z > \zeta$, $\phi_1 = U_1 x + \phi_1'$ for $z < \zeta$. Neglect products of small perturbations ϕ_1', ϕ_2' and ζ.
- How small ? $\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y} \ll 1$ and $g\zeta \ll U_1^2, U_2^2$.
- Taylor-expand $z = \zeta$ about $z = 0$:
  $\phi_k'|_{z=\zeta} = \phi_k'|_{z=0} + \zeta \frac{\partial \phi_k'}{\partial z}|_{z=0} + \cdots$
- $\nabla^2 \phi_2 = 0$, $\nabla^2 \phi_1 = 0$
  BC: $\nabla \phi_k' \to 0$ for $z \to \mp\infty$ for $k = 1, 2$
  $\frac{\partial \phi_k'}{\partial z} = \frac{\partial \zeta}{\partial t} + U_k \frac{\partial \zeta}{\partial x}$ at $z = 0$ for $k = 1, 2$.
  $\rho_1 \left(U_1 \frac{\partial \phi_1'}{\partial x} + \frac{\partial \phi_1'}{\partial t} + g\zeta \right) = \rho_2 \left(U_2 \frac{\partial \phi_2'}{\partial x} + \frac{\partial \phi_2'}{\partial t} + g\zeta \right)$ at $z = 0$.

Kelvin-Helmholtz instability - Normal Modes

- $(\zeta, \phi_1', \phi_2') = (\tilde{\zeta}, \tilde{\phi}_1, \tilde{\phi}_2)\exp[\mathrm{i}(kx + ly) + st]$
- $\nabla^2 \phi_k' \Rightarrow \left[\frac{\mathrm{d}^2\tilde{\phi}_k}{\mathrm{d}z^2} - (k^2 + l^2)\tilde{\phi}_k\right] = 0$
- $\tilde{\phi}_2 = A_2 \exp[-\tilde{k}z] + B_2 \exp[\tilde{k}z]$, $\tilde{k}^2 = (k^2 + l^2)$.
- BC at $y \to \infty$, $u_z^2 = 0 \Rightarrow B_2 = 0$.
- So: $\tilde{\phi}_2 = A_2 \exp[-\tilde{k}]z$ and $\tilde{\phi}_1 = A_1 \exp[\tilde{k}z]$.
- Using the kinematic condition at interface:
  $A_2 = -(s + \mathrm{i}kU_2)\tilde{\zeta}/\tilde{k}, \quad A_1 = (s + \mathrm{i}kU_1)\tilde{\zeta}/\tilde{k}.$
- Using the normal stress condition at interface:
  $\rho_1[\tilde{k}g + (s + \mathrm{i}kU_1)^2] = \rho_2[\tilde{k}g - (s + \mathrm{i}kU_2)^2]$

Kelvin-Helmholtz instability - Growth Rate

- $s = -\mathrm{i}k\frac{\rho_1 U_1+\rho_2 U_2}{\rho_1+\rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2} - \frac{\tilde{k}g(\rho_1-\rho_2)}{\rho_1+\rho_2}\right]^{1/2}$
- Both roots are neutrally stable if $\tilde{k}g(\rho_1^2-\rho_2^2) \geq k^2\rho_1\rho_2(U_1-U_2)^2$. When the equality holds, marginal stability.
- One root is unstable (with $Re[s] > 0$) if $\tilde{k}g(\rho_1^2-\rho_2^2) < k^2\rho_1\rho_2(U_1-U_2)^2$.

Surface Gravity Waves

- $\rho_2 = 0$ and $U_1 = 0, U_2 = 0$. Surface gravity waves on deep water.
- Stable with phase velocity:
  $c = \mathrm{i}s/\tilde{k} = \pm(g/\tilde{k})^{1/2}$
- Oscillatory, stable normal modes.
- Waves – a special case of hydrodynamic stability.

Internal Gravity Waves

- No basic flow: $U_1 = 0, U_2 = 0$.
- $s = \pm[\tilde{k}g(\rho_2 - \rho_1)/(\rho_1 + \rho_2)^2]^{1/2}$
- Instability if $\rho_1 < \rho_2$ (heavey fluid rests above light fluid).
- If $\rho_1 > \rho_2$, there is stability, and there are waves with phase velocity: $c = \pm[g(\rho_1 - \rho_2)/\tilde{k}(\rho_1 + \rho_2)]^{1/2}$.
- The eigenfunctions decay exponentially away from the interface. Motion confined to the vicinity of the interface.
- Observed between layers of fresh and salt water that occur in estuaries.
- Rayleigh-Taylor instability when the whole system has an upward acceleration f. Same result with $g' = f + g$. If $\rho_2 > \rho_1$, then instability occurs only of $g' < 0$.

Instability Due to Shear

- No effect of buoyancy: $\rho_1 = \rho_2$, but $U_1 \neq U_2$.
- $s = -\frac{1}{2}\mathrm{i}k(U_1 + U_2) \pm \frac{1}{2}k(U_1 - U_2)$
- Flow always unstable if $U_1 \neq U_2$. Waves of all wavelengths are unstable.

Outline

Capillary Instability of a Jet

- A cylindrical jet of liquid moving with *uniform* velocity in air (e.g. water jet from a slightly-open tap).
- Surface tension at the liquid-air interface.
- Assume density of outside fluid is zero, and inviscid dynamics for the liquid.
- $\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p \qquad \nabla\cdot\mathbf{u} = 0$
  Pressure inside the jet: $P = P_\infty + \gamma\nabla\cdot\mathbf{n}$ at $r = \zeta(x,\theta,t)$
- Perturbed unit normal to the jet:
  $$\mathbf{n} = \frac{(-\frac{\partial\zeta}{\partial x}, 1, -\frac{\partial\zeta}{\partial r\theta})}{\left[\left(\frac{\partial\zeta}{\partial x}\right)^2 + 1 + \left(\frac{\partial\zeta}{\partial r\theta}\right)^2\right]^{1/2}}$$

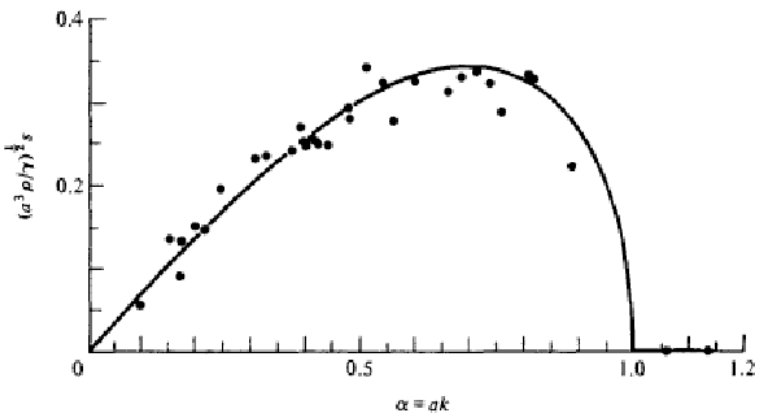
Capillary Instability of a Jet

- Kinematic condition: $u_r = \frac{D\zeta}{Dt}$ at $r = \zeta$.
- Base flow: $\mathbf{U} = 0$, $P = p_\infty + \gamma/a$ for $0 \le r \le a$, as $\nabla \cdot \mathbf{n} = 1/r$ when $\mathbf{n} = \mathbf{e}_r$.
- Disturbances: $\mathbf{u} = \mathbf{U} + \mathbf{u}'$, $p = P + p'$, and $\zeta = a + \zeta'$
- $\rho \dfrac{\partial \mathbf{u}'}{\partial t} = -\nabla p'$ and $\qquad \nabla \cdot \mathbf{u}' = 0.$
- $\nabla \cdot \mathbf{n} = \frac{1}{r} - \dfrac{\partial^2 \zeta'}{\partial x^2} - \frac{1}{r^2} \dfrac{\partial^2 \zeta'}{\partial \theta^2}$
- Normal stress BC at $r = a$: $p' = -\gamma \left(\frac{\zeta'}{a^2} + \dfrac{\partial^2 \zeta'}{\partial x^2} + \frac{1}{a^2} \frac{\zeta'}{\theta^2} \right)$
- $\nabla^2 p' = 0$, where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial r^2} + \frac{1}{r} \dfrac{\partial}{\partial r} + \frac{1}{r^2} \dfrac{\partial^2}{\partial \theta^2}$.
- Normal modes: $(\mathbf{u}', p', \zeta') = (\tilde{\mathbf{u}}(r), \tilde{p}(r), \tilde{\zeta}) \exp[st + \mathrm{i}(kx + n\theta)]$

Capillary Instability of a Jet

- $\frac{\mathrm{d}^2\tilde{p}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}r} - \left(k^2 + \frac{n^2}{r^2}\right)\tilde{p} = 0$
- Linearly independent solutions: modified Bessel functions $I_n(kr)$, $K_n(kr)$; take $n \geq 0$ without loss of generality.
- Physically allowed solution: $\tilde{p}(r) = AI_n(kr)$.
- $\tilde{\mathbf{u}} = -A(\rho s)^{-1}(\mathrm{i}kI_n(kr), kI_n'(kr), \mathrm{i}nr^{-1}I_n(kr))$
- Linearized BCs: $AI_n(\alpha) = -\gamma(1-\alpha^2-n^2)\tilde{\zeta}/a^2$, $-A(a\rho s)^{-1}\alpha I_n'(\alpha) = s\tilde{\zeta}$ where $\alpha = ak$.
- Eigenvalue relation:
- $s^2 = \frac{\gamma}{a^3\rho}\frac{\alpha I_n'(\alpha)}{I_n(\alpha)}(1-\alpha^2-n^2)$
- $\alpha I_n'(\alpha)/I_n(\alpha) > 0$ for all $\alpha \neq 0$. So $s^2 < 0$ if $n \neq 0$.
- $s^2 > 0$ for $-1 < \alpha < 1$ if $n = 0$.

Dispersion relation: theory and experiments



- Jet stable to all nonaxisymmetric disturbances ($n \neq 0$).
- Jet unstable to axisymmetric modes with wavelengths $\lambda = 2\pi/k > 2\pi a$.
- If k_m is the wavenumber at which s is maximum, $k_m = 0.7/a$.
- Jets of all radii are unstable. *No critical parameter that marks the domain of stability*.
- In experiments the liquid jet will break up with wavelength about $2\pi/k_m \approx 9a$

Physical interpretation

- Displacement of the jet radius: $R_{new} = R + \epsilon \cos(kz)$, $k = 2\pi/\lambda$.
- Surface area of the perturbed jet:

$$A = \int_0^\lambda 2\pi R_{new}\, ds$$

$$V = \int_0^\lambda \pi R_{new}^2 dz$$

$$ds = \left[\left(\frac{\mathrm{d}R_{new}}{\mathrm{d}z}\right)^2 + 1\right]^{1/2} dz$$

For small ϵ, $ds = \left[1 + \frac{1}{2}\left(\frac{\mathrm{d}R_{new}}{\mathrm{d}z}\right)^2\right] dz$

$$A = 2\pi R\lambda + \frac{1}{2} R\epsilon^2 k^2 \lambda$$

Physical interpretation

$$V = \int_0^\lambda dz\, \pi[R + \epsilon \cos(kz)]^2$$

$$\frac{V}{\lambda} = \pi R^2 + \frac{1}{2}\pi\epsilon^2$$

Require $V/\lambda = \pi R_0^2$, so $R = R_0 \left[1 - \frac{1}{2}\frac{\epsilon^2}{R_0^2}\right]^{1/2}$

To $O(\epsilon^2)$, $R = R_0 - \frac{1}{4}\frac{\epsilon^2}{R_0^2}$

Then, change in surface area of the jet (per unit wavelength) due to the displacement $R_{new} = R + \epsilon \cos(kz)$ is then

$$\frac{1}{2}\pi\frac{\epsilon^2}{R_0^2}[(2\pi R_0)^2 - \lambda^2)]$$

Physical interpretation

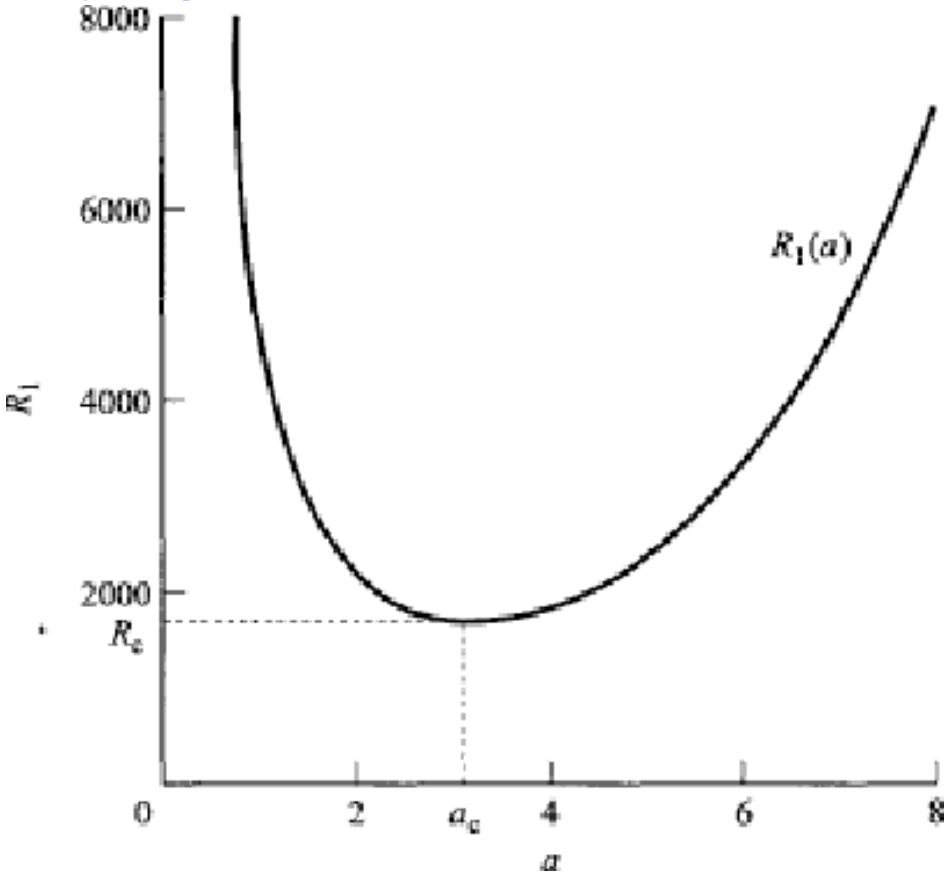
In terms of λ, the change in surface area is

$$\frac{1}{2}\pi\frac{\epsilon^2}{R_0^2\lambda^2}[(2\pi R_0)^2 - \lambda^2]$$

For $\lambda > 2\pi R_0$, the surface area decreases.

Linear stability analysis: success stories

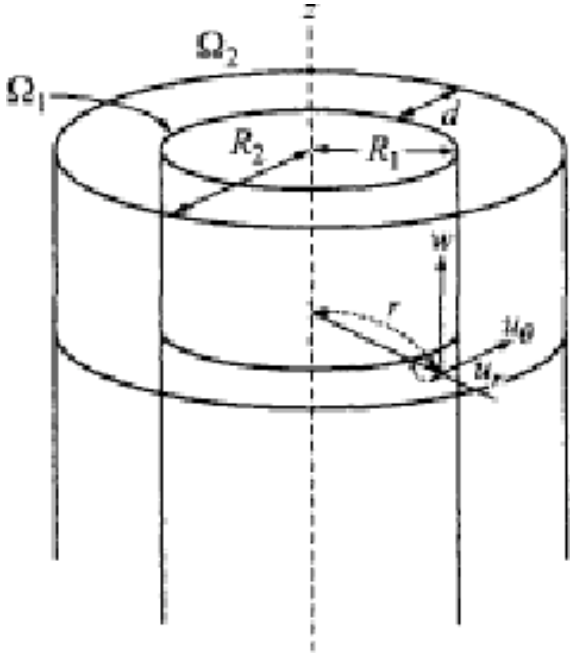
Rayleigh-Benard thermal convection

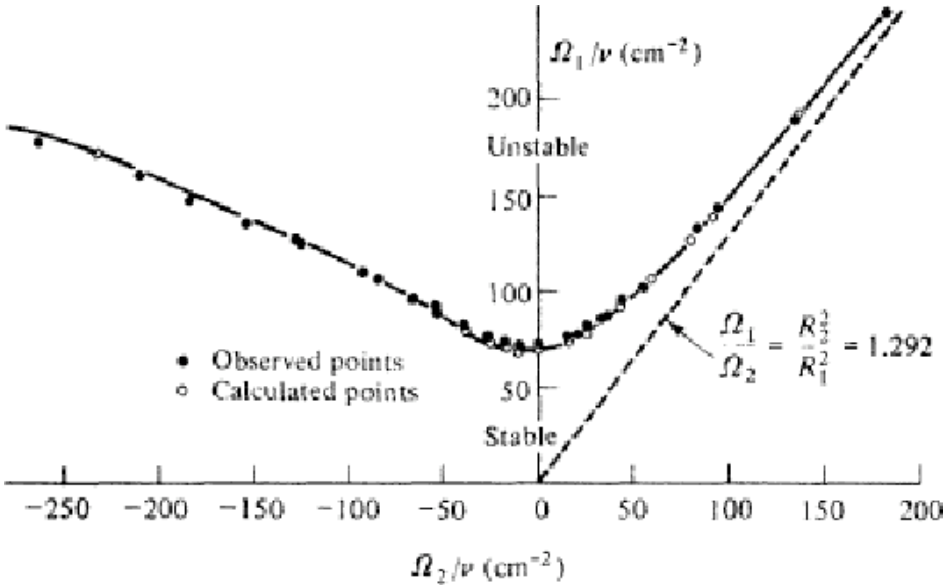


Critical Rayleigh number $Ra_c = 1708$. Experiments: 1705 ± 10.

Linear stability analysis: success stories

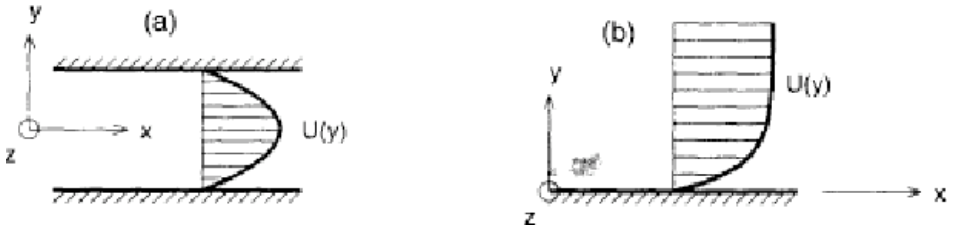
Taylor-Couette Centrifugal Instability





Outline

Stability of Parallel Shear Flows



- Parallel Base Flow: $\mathbf{U} = (U(y), 0, 0)$
- Perturbations: u, v, w

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

Stability of Parallel Shear Flows

- $$\nabla^2 p = -2U' \frac{v}{x}$$
- $\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^4\right] v = 0$
- $\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right]\eta = -U''\frac{\partial v}{\partial z}$
- Normal vorticity: $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$.
- Boundary conditions: $v = v' = \eta = 0$ at solid walls and in the far field.

Orr-Sommerfeld and Squire equations

- $v(x, y, z, t) = \tilde{v}(y) \exp[\mathrm{i}(\alpha x + \beta z - \omega t)]$
- $\eta(x, y, z, t) = \tilde{\eta}(y) \exp[\mathrm{i}(\alpha x + \beta z - \omega t)]$
- $k^2 = (\alpha^2 + \beta^2)$

$$\left[(-\mathrm{i}\omega + \mathrm{i}\alpha U)(\mathcal{D} - k^2) - \mathrm{i}\alpha U'' - \frac{1}{Re}(\mathcal{D}^2 - k^2)^2\right]\tilde{v} = 0$$

$$\left[(-\mathrm{i}\omega + \mathrm{i}\alpha U) - \frac{1}{Re}(\mathcal{D}^2 - k^2)\right]\eta = -\mathrm{i}\beta U'\tilde{v}$$

- BCs: $\tilde{v} = \mathcal{D}\tilde{v} = \tilde{\eta} = 0$ at solid walls and in free stream.

Orr-Sommerfeld and Squire equations

- Temporal problem: α, β real, ω complex.
- Spatial problem: ω real, α, β complex.
- We will consider the temporal problem: $c = \omega/\alpha$ is the complex wavespeed (eigenvalue) of the OS equation, and the associated $\tilde{v}$ are the eigenfunctions.
- OS equation is homogeneous, while the Squire equation is forced by the solutions of the OS equation.
- Two classes of eigenmodes: OS modes and Squire modes.
- OS modes: Find $\tilde{v}_n$ and ω_n by solving OS equation and then find $\tilde{\eta}_n^p$ by solving the inhomogeneous Squire equation.
- Squire modes: $\tilde{v} = 0$, $\tilde{\eta}_m$ ω_m

Squire's transformation and Squire's theorem

- $(U-c)(\mathcal{D}-k^2)\tilde{v} - U''\tilde{v} - \frac{1}{\mathrm{i}\alpha Re}(\mathcal{D}^2-k^2)^2\tilde{v} = 0$
- OS equation with $\beta = 0$ (no variation in the z direction):
  $(U-c)(\mathcal{D}-\alpha_{2D}^2)\tilde{v} - U''\tilde{v} - \frac{1}{\mathrm{i}\alpha_{2D} Re_{2D}}(\mathcal{D}^2-\alpha_{2D}^2)^2\tilde{v} = 0$
- Comparing: the two equations will have identical solutions if the following relations hold: $\alpha_{2D} = k = \sqrt{\alpha^2+\beta^2}$
  $\alpha_{2D} Re_{2D} = \alpha Re$
  $\Rightarrow Re_{2D} = Re\frac{\alpha}{k} < Re$
- To each 3D OS mode, there is a corresponding 2D OS mode at a *lower* Reynolds number.

Squire's transformation and Squire's theorem

- Damped Squire modes: The solutions to the Squire equation are always damped with $c_i < 0$ for all α, β, and Re.
- To prove, multiply Squire equation by $\tilde{\eta}^*$ and integrate from $y = -1$ to $y = 1$ (the fluid domain), and take imaginary part.

$$c_i \int_{-1}^{1} dy\, |\tilde{\eta}|^2 = -\frac{1}{\alpha Re} \int_{-1}^{1} dy\, |\mathcal{D}\tilde{\eta}|^2 + k^2 |\tilde{\eta}|^2 < 0$$

Squire's theorem

Given Re_L as the critical Reynolds number for the onset of linear instability for a given α, β, the Reynolds number Re_c below which no exponential instabilities exisit for any wavenumber satisfies
$Re_c \equiv \min_{\alpha,\beta} Re_L(\alpha, \beta) = \min_{\alpha} Re_L(\alpha, 0)$

Inviscid Analysis

- Rayleigh Equation: Neglect viscous terms in OS equation

$$(U - c)(\mathcal{D}^2 - k^2)\tilde{v} - U''\tilde{v} = 0$$

with $k^2 = \alpha^2 + \beta^2$, and BCs: $\tilde{v} = 0$ at $y = \pm 1$ at solid boundaries.

- We have to forgo the no-slip BC due to the reduced order of the ODE.
- Since the coefficients of the Rayleigh equation are real, any complex eigenvalue will appear in conjugate pairs. If c is an eigenvalue, so is c^*.
- Regular singular point in the complex y plane when $U(y) = c$. The corresponding real part of this location y_c is the "critical layer" where $U(y) = c_r$.

Inviscid Analysis

- Singularity is logarithmic, and Frobenius series can be used to find the solution about y_c.

$$\begin{aligned} \tilde{v}_1(y) &= (y - y_c)P_1(y) \\ \tilde{v}_2(y) &= P_2(y) + \frac{U_c''}{U_c'} \ln(y - y_c) \end{aligned}$$

  Here, P_1 and P_2 are analytic.
- The second solution is multivalued due to the logarithmic term. When $c_i = 0$, the critical layer is on the real axis, $\ln(y - y_c) = \ln|y - y_c| \pm \mathrm{i}\pi$ for $y < y_c$.
- The correct sign of the imaginary part cannot be determined within the inviscid analysis.
- Must do a matched asymptotic expansion of the Rayleigh invisicd solution with the full OS solution about $y = y_c$.

Classical theorems in inviscid stability

- Multiply Rayleigh equation by $\tilde{v}^*$ and integrate from $y = -1$ to $y = 1$. Then integrate by parts:

$$\int_{-1}^{1} dy\, |\mathcal{D}\tilde{v}|^2 + k^2|\tilde{v}|^2 + \int_{-1}^{1} dy\, \frac{U''}{U-c}|\tilde{v}|^2 = 0$$

- Take imaginary part:

$$c_i \int_{-1}^{1} dy\, U'' \frac{|\tilde{v}|^2}{|U-c|^2} = 0$$

- Both $|\tilde{v}|^2$ and $|U-c|^2$ are nonnegative. If c_i is positive, *then* U'' has to change sign in order for the integral to be zero.
- Rayliegh's inflexion point theorem only a necessary condition.

Classical theorems in inviscid stability

Rayleigh's inflexion point theorem

If there exist perturbations with $c_i > 0$, then $U''(y)$ must vanish for some $y \in [-1, 1]$ for instability.

- $$\int_{-1}^{1} dy\, |\mathcal{D}\tilde{v}|^2 + k^2|\tilde{v}|^2 + \int_{-1}^{1} dy\, \frac{U''}{U-c}|\tilde{v}|^2 = 0$$

  Take real part:

  $$\int_{-1}^{1} dy\, \frac{U''(U-c_r)}{|U-c|^2} = -\int_{-1}^{1} dy\, |\mathcal{D}\tilde{v}|^2 + k^2|\tilde{v}|^2$$

  Then add the following expression to the left side of above equation:

  $$(c_r - U_s)\int_{-1}^{1} dy\, U''\frac{|\tilde{v}|^2}{|U-c|^2} = 0$$

  The above expression is zero due to inflexion point theorem

Classical theorems in inviscid stability

- $$\int_{-1}^{1} dy \, \frac{U''(U - Us)}{|U - c|^2} |\tilde{v}|^2 = -\int_{-1}^{1} dy \, |\mathcal{D}\tilde{v}|^2 + k^2 |\tilde{v}|^2$$

  For the integral to be negative, $U''(U - U_s) < 0$ in the flow field.

Fjortoft's Criterion

Given a monotonic mean velocity profile $U(y)$, a necessary condition for instability is that $U''(U - U_s) < 0$, with $U_s = U(y_s)$ as the mean velocity at the inflexion point, i.e., $U''(y_s) = 0$.

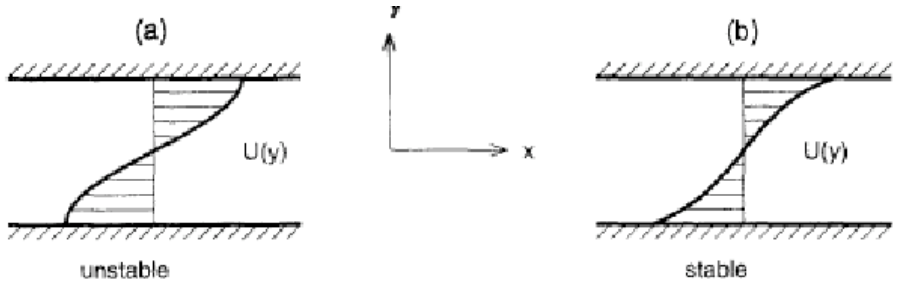


FIGURE 2 3. Demonstration of Fjørtoft's criterion as a necessary condition for instability in inviscid flow (a) unstable according to Fjørtoft, (b) stable according to Fjørtoft

Classical theorems in inviscid stability

Howard's semicircle theorem

The unstable eigenvalues of the Rayleigh equation satisfy

$$\left[c_r - \frac{1}{2}(U_{max} + U_{min})\right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{max} - U_{min})\right]^2$$

- For $c_i \to 0^+$, $U_{min} < c_r < U_{max}$ for marginally stable modes.
- For plane-Poiseuille flow and many internal channel flows, U'' does *not* vanish in the domain of the flow, and so these flows are *stable* in the inviscid limit to 2D infinitesimal perturbations.
- Unbounded jets and free shear layers are unstable in the inviscid limit.

Viscous instability

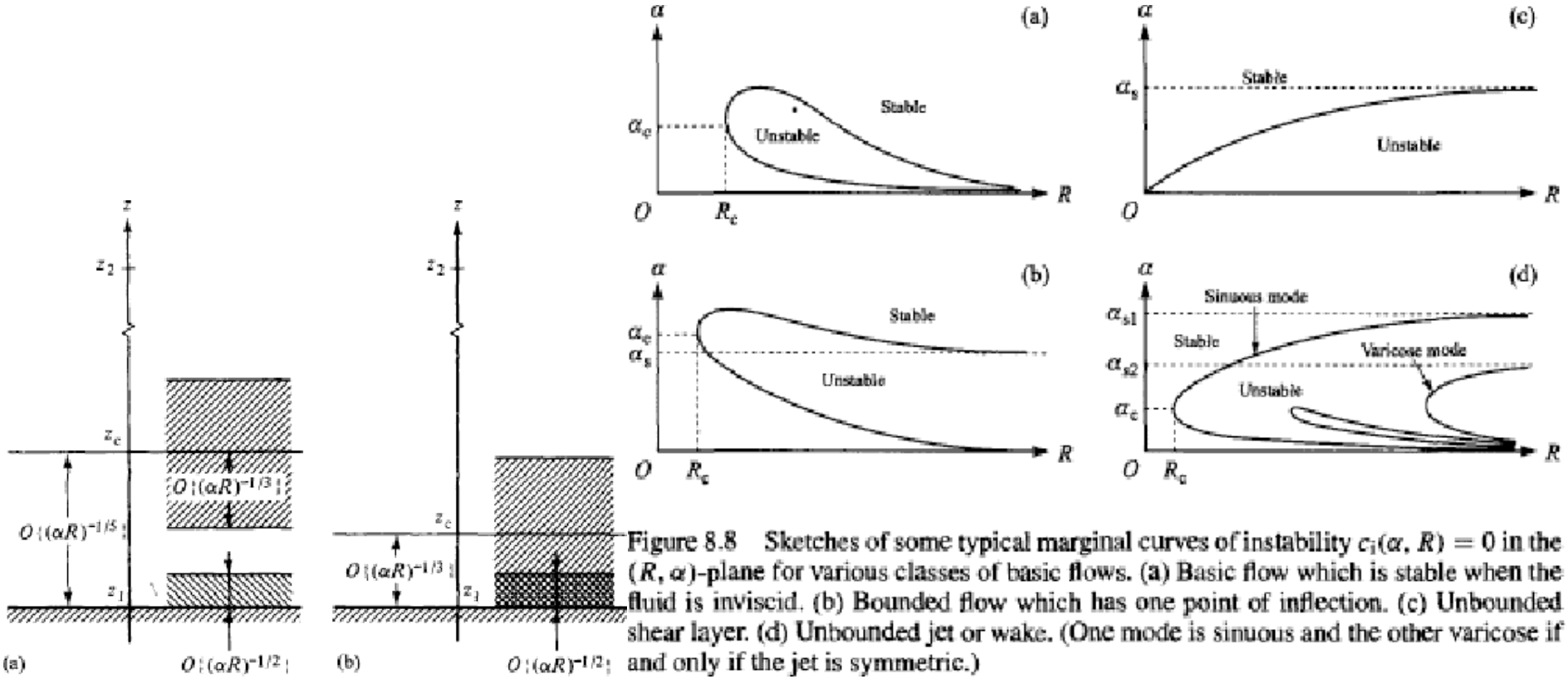


Figure 8.8 Sketches of some typical marginal curves of instability $c_i(\alpha, R) = 0$ in the (R, α)-plane for various classes of basic flows. (a) Basic flow which is stable when the fluid is inviscid. (b) Bounded flow which has one point of inflection. (c) Unbounded shear layer. (d) Unbounded jet or wake. (One mode is sinuous and the other varicose if and only if the jet is symmetric.)

Numerical Eigenspectrum

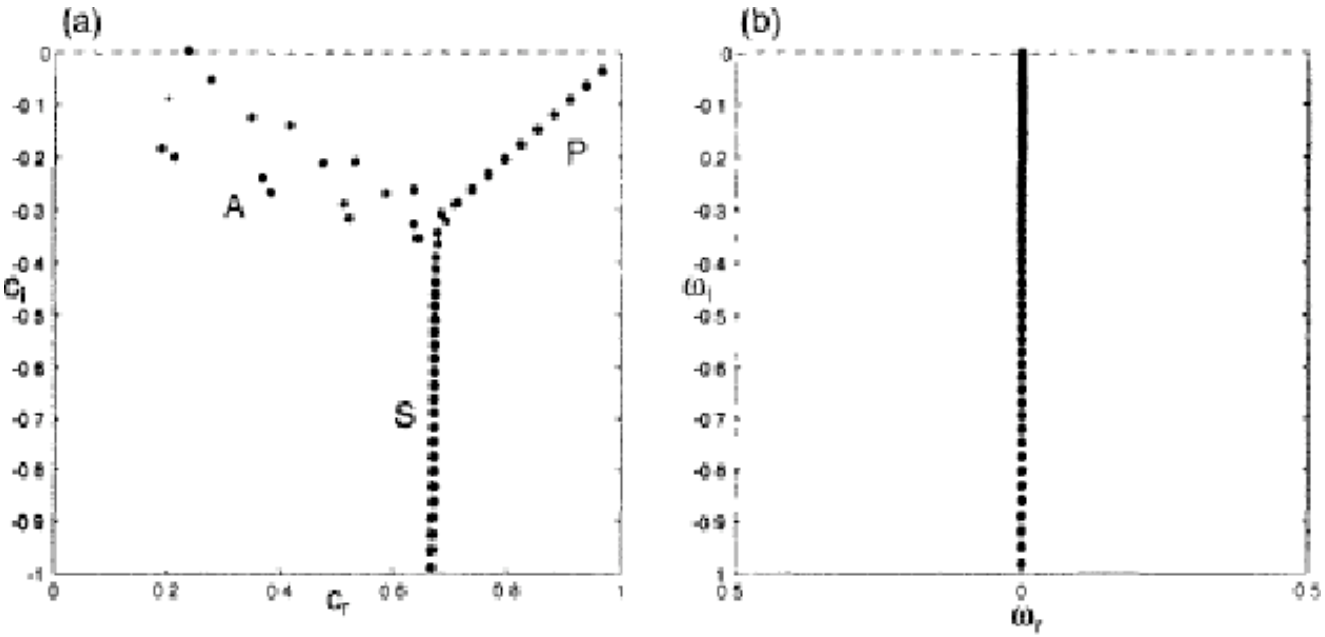
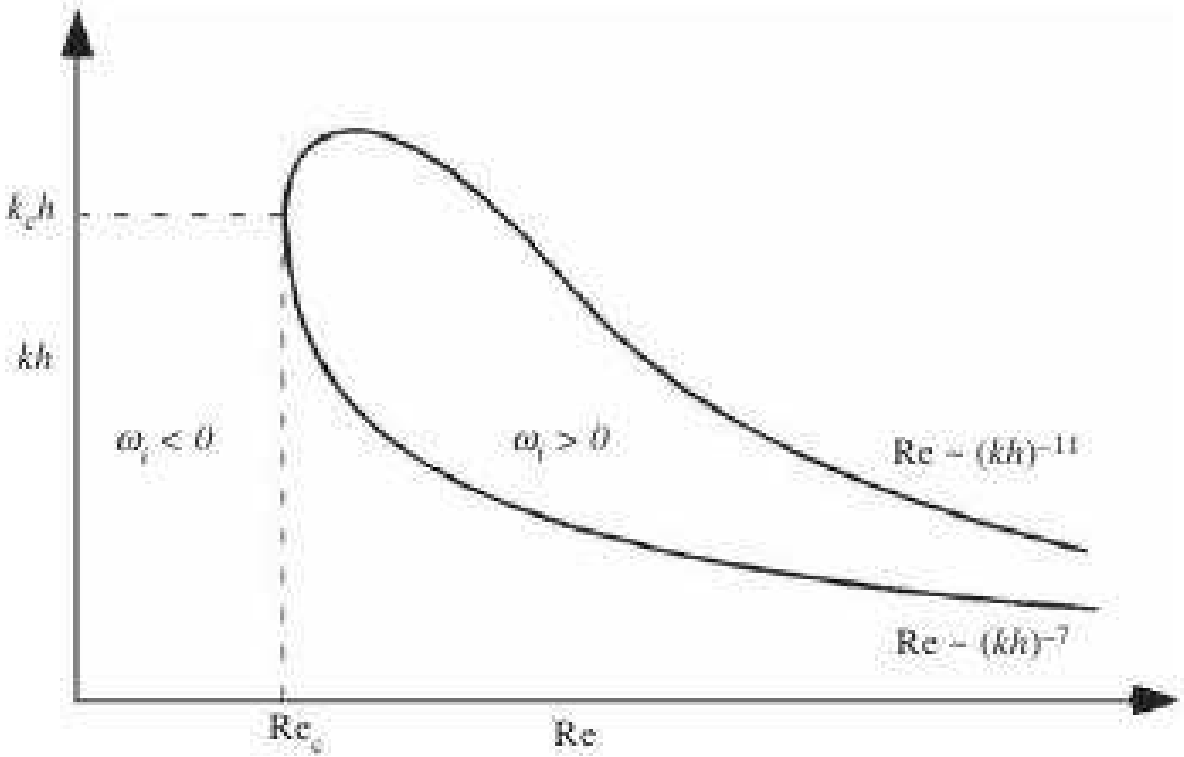


FIGURE 3.1 Orr-Sommerfeld spectrum of plane Poiseuille flow for Re = 10000 (a) wave numbers $\alpha = 1, \beta = 0$. (b) wave numbers $\alpha = 0, \beta = 1$

Numerical Eigenspectrum



Comparison with experiments

- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

Comparison with experiments

- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

Comparison with experiments

- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

Comparison with experiments

- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

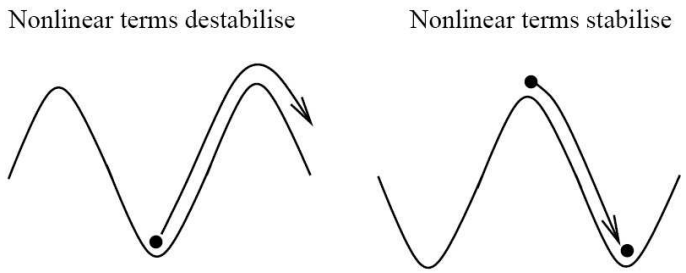
Comparison with experiments

- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

Comparison with experiments

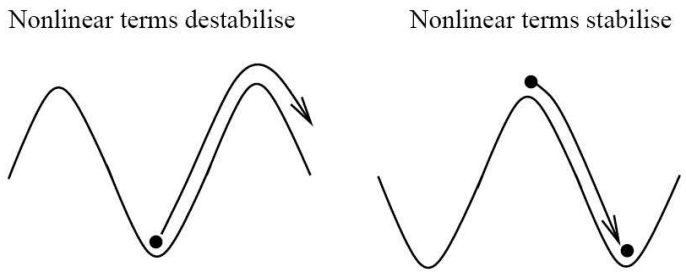
- Plane-Poiseuille flow (numerical solution): $Re_c = 5722.2$
- Experiments: Re_c could be as low as 1000.
- Pipe-Poiseuille flow (asymptotic/numerical solution): $Re_c = \infty$
- Experiments: $Re_c \sim 2000$.
- Plane Couette flow (numerical solution): $Re_c = \infty$.
- Experiments: $Re_c \sim 360$.
- Can a unavoidable disturbance in an experiment be treated as "infinitesimal" ?
- Very careful experiments in pipe flow: Re for transition could be 10^5.
- Still need to explain the usual value of $Re \sim 2000$.

Nonlinear stability



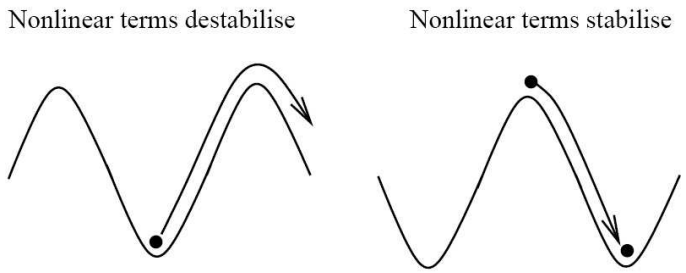
- A sufficiently "large" disturbance could destabilize or stabilize.
- A toy nonlinear ODE: $\frac{dx}{dt} = ax - bx^3$
- Base-state: $x = x_{B1} = 0\,, x = x_{B2} = +\sqrt{a/b}\,, x_{B3} = -\sqrt{a/b}$ for $a/b > 0$.
- Stability: $x = x_{B1} + x'$, $\frac{dx'}{dt} = ax'$, $x'(t) = A\exp[at]$, Unstable for $a > 0$ and Stable for $a < 0$.
- Stability of x_{B2} and x_{B3}: $\frac{dx'}{dt} = ax' - 3bx_B^2 x'$, $x'(t) = A\exp[st]$, $s = -2a$.
- x_{B2} and x_{B3} stable if $a > 0$ and are unstable if $a < 0$.

Nonlinear stability



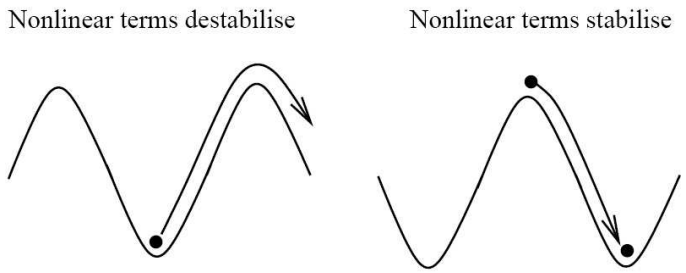
- A sufficiently "large" disturbance could destabilize or stabilize.
- A toy nonlinear ODE: $\frac{dx}{dt} = ax - bx^3$
- Base-state: $x = x_{B1} = 0\,, x = x_{B2} = +\sqrt{a/b}\,, x_{B3} = -\sqrt{a/b}$ for $a/b > 0$.
- Stability: $x = x_{B1} + x'$, $\frac{dx'}{dt} = ax'$, $x'(t) = A\exp[at]$, Unstable for $a > 0$ and Stable for $a < 0$.
- Stability of x_{B2} and x_{B3}: $\frac{dx'}{dt} = ax' - 3bx_B^2 x'$, $x'(t) = A\exp[st]$, $s = -2a$.
- x_{B2} and x_{B3} stable if $a > 0$ and are unstable if $a < 0$.

Nonlinear stability



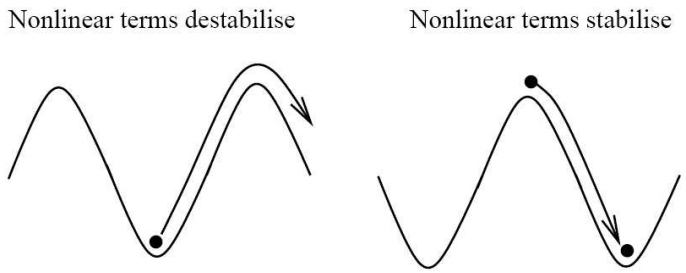
- A sufficiently "large" disturbance could destabilize or stabilize.
- A toy nonlinear ODE: $\frac{dx}{dt} = ax - bx^3$
- Base-state: $x = x_{B1} = 0\,, x = x_{B2} = +\sqrt{a/b}\,, x_{B3} = -\sqrt{a/b}$ for $a/b > 0$.
- Stability: $x = x_{B1} + x'$, $\frac{dx'}{dt} = ax'$, $x'(t) = A\exp[at]$, Unstable for $a > 0$ and Stable for $a < 0$.
- Stability of x_{B2} and x_{B3}: $\frac{dx'}{dt} = ax' - 3bx_B^2 x'$, $x'(t) = A\exp[st]$, $s = -2a$.
- x_{B2} and x_{B3} stable if $a > 0$ and are unstable if $a < 0$.

Nonlinear stability



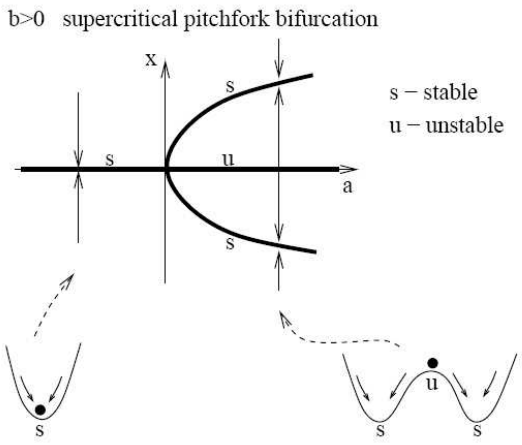
- A sufficiently "large" disturbance could destabilize or stabilize.
- A toy nonlinear ODE: $\frac{dx}{dt} = ax - bx^3$
- Base-state: $x = x_{B1} = 0\,, x = x_{B2} = +\sqrt{a/b}\,, x_{B3} = -\sqrt{a/b}$ for $a/b > 0$.
- Stability: $x = x_{B1} + x'$, $\frac{dx'}{dt} = ax'$, $x'(t) = A\exp[at]$, Unstable for $a > 0$ and Stable for $a < 0$.
- Stability of x_{B2} and x_{B3}: $\frac{dx'}{dt} = ax' - 3bx_B^2 x'$, $x'(t) = A\exp[st]$, $s = -2a$.
- x_{B2} and x_{B3} stable if $a > 0$ and are unstable if $a < 0$.

Nonlinear stability



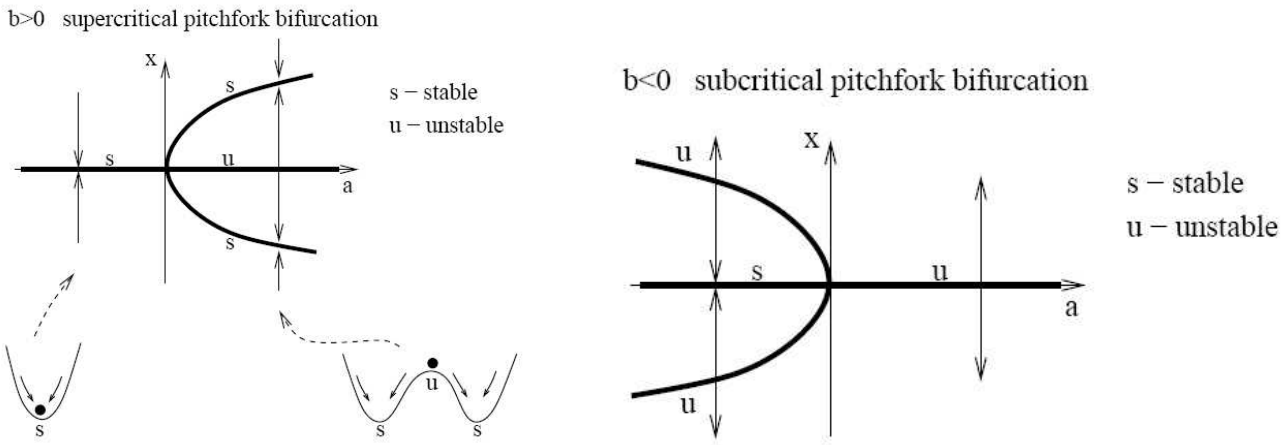
- A sufficiently "large" disturbance could destabilize or stabilize.
- A toy nonlinear ODE: $\frac{dx}{dt} = ax - bx^3$
- Base-state: $x = x_{B1} = 0\,, x = x_{B2} = +\sqrt{a/b}\,, x_{B3} = -\sqrt{a/b}$ for $a/b > 0$.
- Stability: $x = x_{B1} + x'$, $\frac{dx'}{dt} = ax'$, $x'(t) = A\exp[at]$, Unstable for $a > 0$ and Stable for $a < 0$.
- Stability of x_{B2} and x_{B3}: $\frac{dx'}{dt} = ax' - 3bx_B^2 x'$, $x'(t) = A\exp[st]$, $s = -2a$.
- x_{B2} and x_{B3} stable if $a > 0$ and are unstable if $a < 0$.

Supercritical and Subcritical Bifurcation



- Supercritical bifurcation for $b > 0$: Solutions x_{B2} and x_{B3} exist only for $a > 0$; for $a < 0$, $x = 0$ is the only solution.
- For $a > 0$ $x = 0$ becomes unstable and gives rise to two *new* stable solutions x_{B2} or x_{B3}. (Rayleigh–Benard convection).
- Subcritical bifurcation for $b < 0$: For $b < 0$, x_{B2} and x_{B3} exist only for $a < 0$.
- For $a < 0$, x_{B2} and x_{B3} are unstable.
- The $x = 0$ solution could become unstable even for $a < 0$ to a sufficiently large disturbance (Shear flow instabilities).

Supercritical and Subcritical Bifurcation



- Supercritical bifurcation for $b > 0$: Solutions x_{B2} and x_{B3} exist only for $a > 0$; for $a < 0$, $x = 0$ is the only solution.
- For $a > 0$ $x = 0$ becomes unstable and gives rise to two *new* stable solutions x_{B2} or x_{B3}. (Rayleigh–Benard convection).
- Subcritical bifurcation for $b < 0$: For $b < 0$, x_{B2} and x_{B3} exist only for $a < 0$.
- For $a < 0$, x_{B2} and x_{B3} are unstable.
- The $x = 0$ solution could become unstable even for $a < 0$ to a sufficiently large disturbance (Shear flow instabilities).

Weakly nonlinear effects: Landau equation

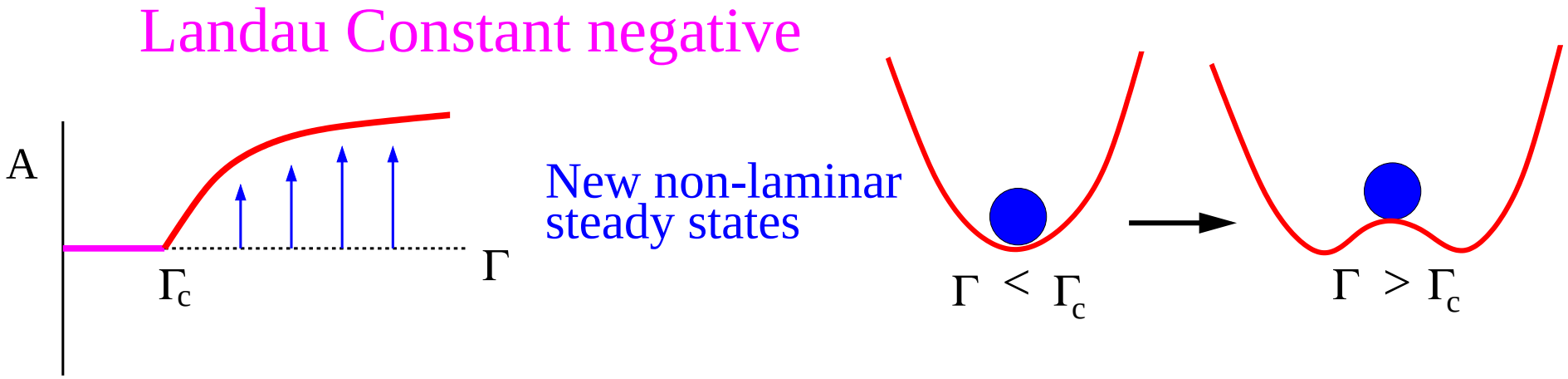
- Linear theory: $\frac{1}{A}\frac{dA}{dt} = s_0 A \Rightarrow A(t) = A(0)\ \exp[s_0 t]$
- When $s_0 > 0 \Rightarrow$ Flow unstable: Only onset predicted.
- Exponential growth: Cannot neglect nonlinearities anymore.
- Nonlinearities can saturate exponential growth, or further accelerate exponential growth.

Weakly nonlinear theory:

Landau Equation: $\frac{1}{A}\frac{dA}{dt} = s_0 + s_1 A^2$;

s_1: Landau constant $\rightarrow$ effect of nonlinearities.

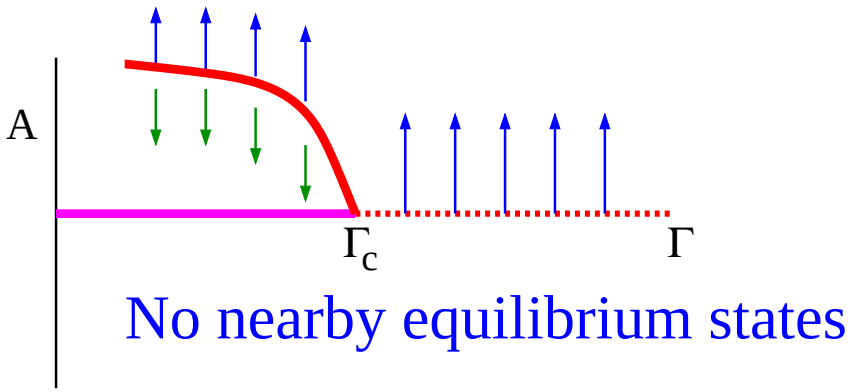
Supercritical Equilibrium

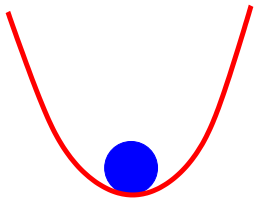


- $\frac{1}{A}\frac{dA}{dt} = \underbrace{s_0}_{+ve} - |s_1|A^2 \Rightarrow A_s^2 = s_0/|s_1|.$
- A linearly unstable mode saturates to a new steady state.
- Rayleigh – Bernard convection cells, Taylor vortices ...

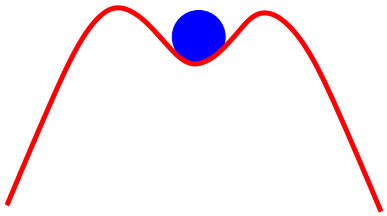
Subcritical Instability

Landau constant positive





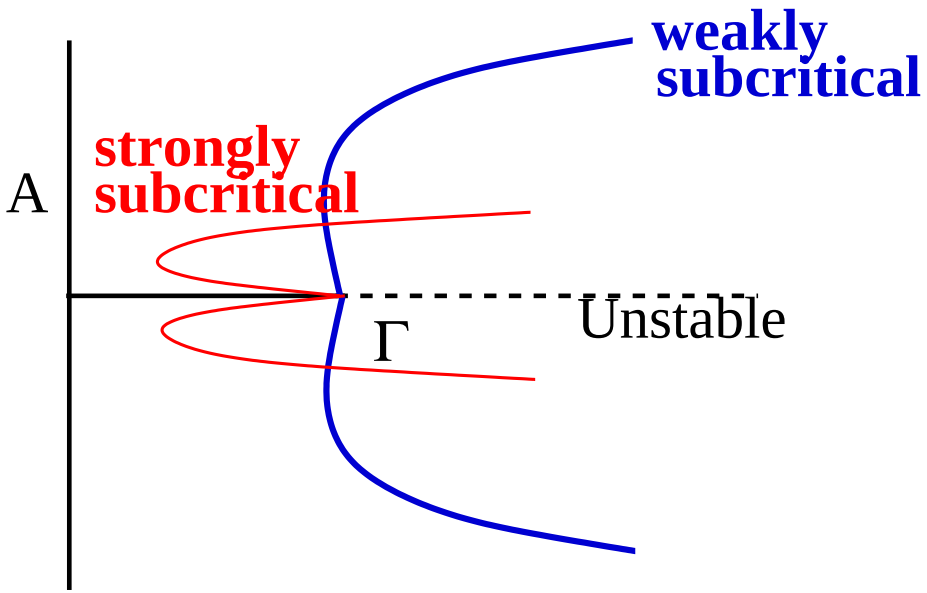
Stable to
Small disturbances



Unstable to
finite disturbances

- $\frac{1}{A}\frac{dA}{dt} = -\underbrace{|s_0|}_{-ve} + s_1 A^2 \Rightarrow$ Instability for $\Gamma < \Gamma_c$ if $A^2_{intial} > |s_0|/s_1$.
- A linearly stable mode becomes unstable to finite amplitude disturbances.
- Plane Poiseuille flow in a rigid channel: 65% reduction in critical Re for $v'_x/V = 0.025$; Strongly subcritical.

Subcritical Instability



- $\frac{1}{A}\frac{dA}{dt} = -\underbrace{|s_0|}_{-ve} + s_1 A^2 \Rightarrow$ Instability for $\Gamma < \Gamma_c$ if $A^2_{intial} > |s_0|/s_1$.
- A linearly stable mode becomes unstable to finite amplitude disturbances.
- Plane Poiseuille flow in a rigid channel: 65% reduction in critical Re for $v'_x/V = 0.025$; Strongly subcritical.

Transient or algebraic growth: A simple example

- Coupled ODEs: $\frac{dv}{dt} = -\frac{1}{Re}v\,, \quad \frac{d\eta}{dt} = v - \frac{2}{Re}\eta$
- Solution: Exponential decay of $v(t)$ and $\eta(t)$

$$v(t) = v_0 \exp[-t/Re]$$

$$\eta(t) = v_0 Re(\exp[-t/Re] - \exp[-2t/Re]) + \eta_0 \exp[-2t/Re]$$

- For small times, series expand:

$$Rev_0(\exp[-t/Re] - \exp[-2t/Re]) = v_0 t - \frac{3v_0}{Re}t^2 + \cdots$$

- Growth $\propto t$ for small times $t < O(Re)$, and exponential decay for long times.
- During this "transient growth" nonlinearities could become important and lead to instabilities.
- "Bypass transition".

Transient or algebraic growth: A simple example

- Coupled ODEs: $\frac{dv}{dt} = -\frac{1}{Re}v\,, \quad \frac{d\eta}{dt} = v - \frac{2}{Re}\eta$
- Solution: Exponential decay of $v(t)$ and $\eta(t)$

$$v(t) = v_0 \exp[-t/Re]$$

$$\eta(t) = v_0 Re(\exp[-t/Re] - \exp[-2t/Re]) + \eta_0 \exp[-2t/Re]$$

- For small times, series expand:

$$Rev_0(\exp[-t/Re] - \exp[-2t/Re]) = v_0 t - \frac{3v_0}{Re}t^2 + \cdots$$

- Growth $\propto t$ for small times $t < O(Re)$, and exponential decay for long times.
- During this "transient growth" nonlinearities could become important and lead to instabilities.
- "Bypass transition".

Transient or algebraic growth: A simple example

- Coupled ODEs: $\frac{dv}{dt} = -\frac{1}{Re}v\,, \quad \frac{d\eta}{dt} = v - \frac{2}{Re}\eta$
- Solution: Exponential decay of $v(t)$ and $\eta(t)$

$$v(t) = v_0 \exp[-t/Re]$$

$$\eta(t) = v_0 Re(\exp[-t/Re] - \exp[-2t/Re]) + \eta_0 \exp[-2t/Re]$$

- For small times, series expand:

$$Rev_0(\exp[-t/Re] - \exp[-2t/Re]) = v_0 t - \frac{3v_0}{Re}t^2 + \cdots$$

- Growth $\propto t$ for small times $t < O(Re)$, and exponential decay for long times.
- During this "transient growth" nonlinearities could become important and lead to instabilities.
- "Bypass transition".

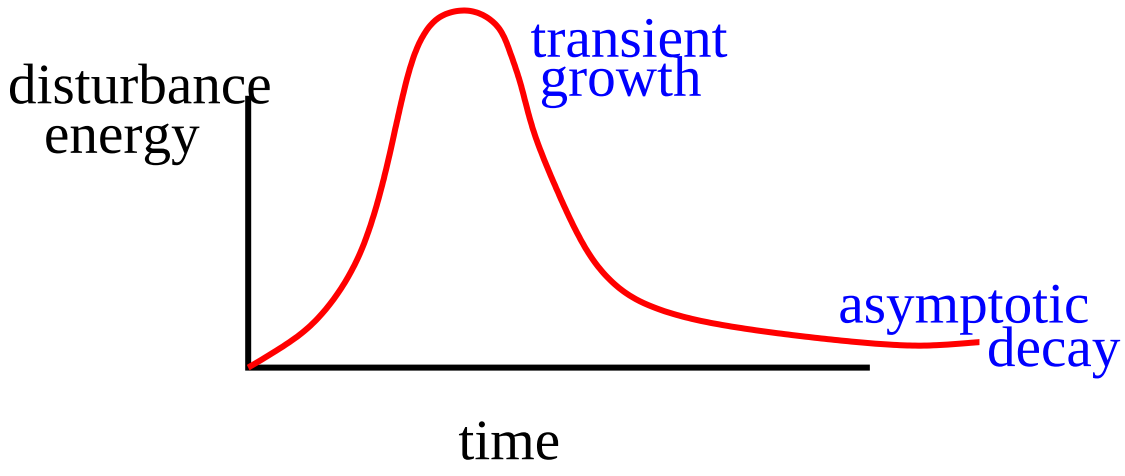
Non-modal (transient) growth

- In rigid tubes, pipe flow always (asymptotically) stable at any Re as per normal mode analysis.
- Plane-Poiseuille flow unstable at $Re_c = 5772$, but experiments show instability at $Re \approx 1200$.
- Underlying differential operators are non-normal, and eigenfunctions are non-orthogonal.
- Possibility of algebraic or transient growth at early times, which eventually decay as $t \to \infty$.

Schmid and Henningson, 2001

Non-modal (transient) growth

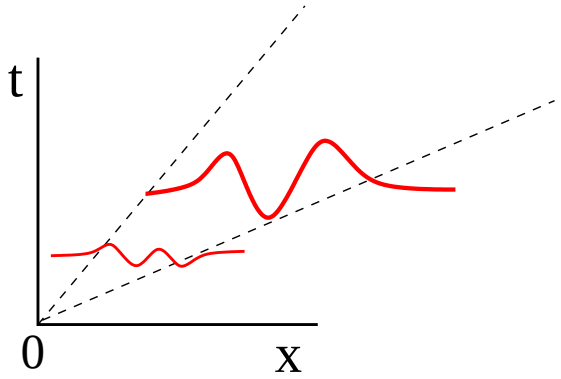
- In rigid tubes, pipe flow always (asymptotically) stable at any Re as per normal mode analysis.
- Plane-Poiseuille flow unstable at $Re_c = 5772$, but experiments show instability at $Re \approx 1200$.
- Underlying differential operators are non-normal, and eigenfunctions are non-orthogonal.
- Possibility of algebraic or transient growth at early times, which eventually decay as $t \to \infty$.



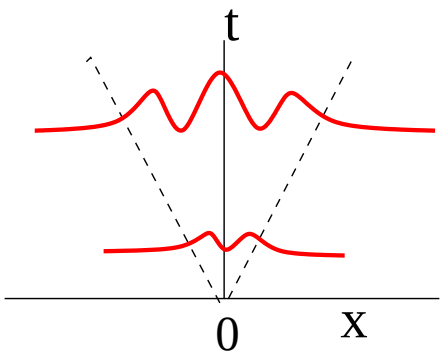
Schmid and Henningson, 2001

Spatio-temporal analysis

- Response to impulse forcing at $x = 0, t = 0$.



Convective



Absolute

- $G(x,t) = \int_L \frac{d\omega}{2\pi}\, I(x,\omega) \exp(-\mathrm{i}\omega t)$

- $I(x,\omega) = \int_F \frac{d\alpha}{2\pi}\, D(k,\omega)^{-1} \exp(\mathrm{i}\alpha x)$

- Absolute instability: Contour deformation in the L-domain until there is a pinching of branches in the F-domain.