J. Fluid Mech. (2001), *vol.* 434, *pp.* 337–354. Printed in the United Kingdom

Weakly nonlinear stability of viscous flow past a flexible surface

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(Received 13 January 2000 and in revised form 25 September 2000)

The weakly nonlinear stability of viscous fluid flow past a flexible surface is analysed in the limit of zero Reynolds number. The system consists of a Couette flow of a Newtonian fluid past a viscoelastic medium of non-dimensional thickness H (the ratio of wall thickness to the fluid thickness), and viscosity ratio μ_r (ratio of the viscosities of wall and fluid media). The wall medium is bounded by the fluid at one surface and two different types of boundary conditions are considered at the other surface of the wall medium – for 'grafted' gels zero displacement conditions are applied while for 'adsorbed' gels the displacement normal to the surface is zero but the surface is permitted to move in the lateral direction. The linear stability analysis reveals that for grafted gels the most unstable modes have $\alpha \sim O(1)$, while for adsorbed gels the most unstable modes have $\alpha \rightarrow 0$, where α is the wavenumber of the perturbations. The results from the weakly nonlinear analysis indicate that the nature of the bifurcation at the linear instability is qualitatively very different for grafted and absorbed gels. The bifurcation is always subcritical for the case of flow past grafted gels. It is found, however, that relatively weak but finite-amplitude disturbances do not significantly reduce the critical velocity required to destabilize the flow from the critical velocity predicted by the linear stability theory. For the case of adsorbed gels, it is found that a supercritical equilibrium state could exist in the limit of small wavenumber for a wide range of parameters μ_r and H, while the bifurcation becomes subcritical at larger values of the wavenumber and there is a transition from supercritical to subcritical bifurcation as the wavenumber is increased.

1. Introduction

The dynamics of fluid flow past flexible solid surfaces is qualitatively different from that past rigid surfaces because of the coupling between the fluid and wall dynamics, and the elasticity of the flexible surface could affect the transition from laminar to turbulent flow. Experiments conducted by Krindel & Silberberg (1979) in a gel-walled tube indicate that there is an anomalous drag force at Reynolds numbers (*Re*) as low as about 600, and the authors concluded that this is due to a transition to a turbulent flow at a Reynolds number which is far lower than the critical Reynolds number for the flow through a rigid tube (around 2100). Motivated by these experimental results, there has been renewed interest in the understanding of the stability of fluid flow through flexible channels and tubes (Kumaran, Fredrickson & Pincus 1994; Srivatsan & Kumaran 1997; Kumaran 1995, 1996, 1998*a*, *b*; Shankar & Kumaran 1999). These studies have focused on the linear stability of the fluid flow, where 'small' disturbances are imposed on the laminar flow and their temporal evolution is determined by solving the linearized governing equations. The interesting possibility of the flow being unstable even in the absence of fluid inertia (zero Reynolds number) was reported in Kumaran *et al.* (1994) and Kumaran (1995). The Reynolds number of flows in the biological realm is often very low (typically $\sim 10^{-2}$). Fluid inertia is negligible at such low Reynolds numbers, and the dynamics is determined by a balance between viscous stresses in the fluid and elastic stresses in the flexible medium. Kumaran *et al.* (1994) studied the linear stability of the Couette flow of a Newtonian fluid past a viscoelastic medium of finite thickness in the zero Reynolds number limit. Their analysis revealed that the flow could be unstable when the non-dimensional velocity of the top plate ($V\mu/GR$) exceeds a critical value. Here V is the dimensional velocity of the top plate, μ is the viscosity of the fluid, G is the shear modulus of the gel, and R is the thickness of the fluid. A similar instability was predicted for the case of flow through a flexible tube by Kumaran (1995). The physical mechanism driving this instability is the transfer of energy from mean flow to fluctuations due to the deformation work done by the mean flow at the interface. When this rate of energy transfer exceeds the viscous dissipation of energy, the flow becomes unstable (see Kumaran 1995 for a detailed energy balance argument). This zero Reynolds number instability is qualitatively different from the instability at the interface between two Newtonian viscous fluids (Hooper & Boyd 1983), which requires fluid inertia for destabilization (for a physical argument, see Hinch 1984).

Recently, Kumaran & Muralikrishnan (2000) conducted experiments to study the stability of viscous flow past a polymer gel in a related geometry, and the Reynolds number in their study was maintained below $Re \sim 10^{-1}$. The experiments were carried out using the parallel plate geometry of a rheometer, where a sheet of polyacrylamide gel of thickness about 4.5 mm was placed on the bottom plate, and a highly viscous fluid (a silicone oil) of thickness ranging from 300 μm to 1000 μm was placed on the surface of the gel, and the fluid was sheared at the top by a moving rigid plate. The rheometer was operated in the stress controlled mode, where the stress on the top rigid plate of the rheometer was increased at a constant rate. The strain rate and the apparent viscosity (assuming the flow in the gap is laminar) were recorded. These experiments showed that there was an anomalous increase in the apparent viscosity (determined by assuming the flow is laminar) above a certain strain rate. This implies that the laminar flow becomes unstable to a more complicated flow. The critical velocity required for initiating the instability in the experiments was found to be in good agreement with the theoretical predictions of Kumaran *et al.* (1994), with no adjustable parameters, for a wide range of gel thicknesses and elastic moduli. Thus, the linear instability of viscous flow past a flexible surface is now understood both from experimental and theoretical standpoints.

The objective of the present study is to determine the nature of the Couette flow past a flexible surface after it becomes linearly unstable in the limit of zero Reynolds number. The weakly nonlinear stability analysis is employed here to determine the nature of the bifurcation (supercritical/subcritical) that occurs after the linear instability. A majority of previous theoretical efforts based on the weakly nonlinear analysis have centred around the derivation of the 'Landau equation' for the amplitude of the most (linearly) unstable mode (Drazin & Reid 1981). The means to derive the Landau constants from the governing equations of fluid dynamics are based on the pioneering works of Stuart (1960) and Watson (1960) or variants of these theories. The usual source of nonlinearities in conventional fluid flow problems through rigid channels is the convective nonlinearities in the governing Navier–Stokes equations. In the present study, however, the interest is in the *zero* Reynolds number limit, and the convective nonlinearities are neglected this limit. Nonlinearities appear in the present problem due to the boundary conditions at the interface between the fluid and the wall medium, for the following reason. In the case of problems involving two adjacent continua (like fluid–fluid or fluid–flexible solid), the boundary conditions such as the velocity and stress continuities have to be applied at the interface between the fluid and the wall medium. However, the position of the interface itself is not known, and has to be determined as a part of the solution. In a weakly nonlinear treatment of the problem, the amplitude of the interfacial position is assumed to be small enough so that a perturbative expansion is valid, and the quantities at the unknown interface are expanded in a Taylor series about their values at the unperturbed interface (Joseph & Renardy 1993, p. 183) to the desired order. It is clear that such a procedure would yield nonlinearities in the boundary conditions. There have been some previous studies which have dealt with the weakly nonlinear stability of the two dimensional plane Poiseuille flow past compliant surfaces (Pierce 1992; Rotenberry & Saffman 1990; Rotenberry 1992; Thomas 1992). These studies have examined the limit of moderate to high Reynolds number, and in this limit nonlinearities are present in both the governing equations and the boundary conditions. In the present study, attention is restricted to the zero Reynolds number limit, and nonlinearities arise mainly due to the boundary conditions at the interface.

There are two major motivations for the present weakly nonlinear analysis. First, the possibility of a non-laminar stationary state in the limit of zero Reynolds number for a shear flow past a flexible surface has not been examined so far. Such non-laminar supercritically stable states, if they exist in the zero Reynolds number limit, are unique features of flow past flexible solid surfaces. Secondly, if the flow is subcritically unstable, the reduction in the critical velocity required for instability from its linear theory value can be calculated from the weakly nonlinear analysis by prescribing a particular level for the finite disturbances. This calculation can be used to determine whether the critical velocity predicted by the linear theory is an accurate estimate of the velocity at which the instability occurs in practice. In particular, such a study has been carried out for the case of plane Poiseuille flow in a rigid channel (Reynolds & Potter 1967), where it was found that even weak but finite disturbances result in a drastic reduction in the critical Reynolds number. It is therefore of interest to determine, when the flow is subcritically unstable in the present case, whether finite disturbances lead to significant reduction in the critical velocity required for instability. Apart from these two main motives, a detailed understanding of the asymptotic solutions in the zero Reynolds number limit could be used as a starting point to probe the stability of finite Reynolds number flows, where it is necessary to solve the governing equations numerically. The outline of the rest of this paper is as follows. In § 2, the governing equations and the boundary conditions are provided, and the linear stability characteristics of the system are briefly outlined, both for grafted and adsorbed gels. In § 3, the weakly nonlinear perturbation scheme is described in detail, and the method of determination of the Landau constant for the present problem is discussed. In § 4, the results for grafted and adsorbed gels are discussed, and § 5 summarizes the salient conclusions of the present study.

2. Problem formulation and governing equations

The system consists of a Newtonian fluid of density ρ, viscosity μ and thickness R (occupying the region $0 < y^* < R$), flowing past a viscoelastic material of finite thickness HR (occupying the region $-HR < y^* < 0$ in the unperturbed state) with shear modulus G and viscosity μ_g. A schematic of the configuration is shown in

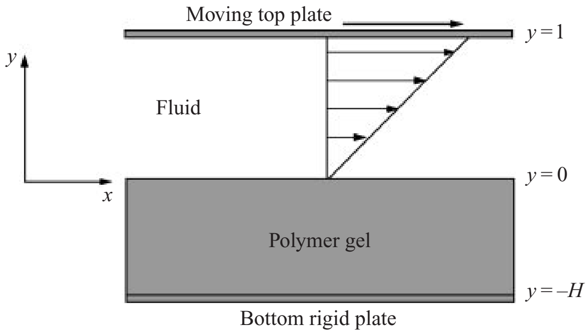

FIGURE 1. A schematic diagram of the configuration considered in the analysis.

figure 1. Here, quantities with a superscript * are dimensional, and quantities without the superscript are dimensionless unless stated otherwise. The wall material is assumed to be incompressible and impermeable to the fluid, and the conservation equations are similar to those used in earlier studies on polymer gels (Harden, Pleiner & Pincus 1991; Kumaran 1993) and in the previous linear stability analyses for flow through flexible channels and tubes. The fluid is sheared at the top boundary $y^* = R$ with a velocity V. This base laminar flow exerts a shear stress on the wall, and there is a mean strain in the wall in the base state. The wall medium (often referred to as 'gel' in the following discussion) is at rest in the unperturbed base state, since the velocity of the fluid at the unperturbed interface is zero. The velocities are scaled by GR/μ, time by G/μ, lengths by R, and the pressure in the fluid and the gel by G. The ratio between gel and fluid viscosities is denoted by $\mu_r = \mu_g/\mu$. The zero Reynolds number limit is considered, and inertial stresses are neglected in both the fluid and the gel. The scaled governing equations in the fluid and the wall medium are

$$\partial_i v_i = 0, \quad \partial_j \tau_{ij} = 0, \tag{1}$$

$$\partial_i u_i = 0, \quad \partial_j \sigma_{ij} = 0. \tag{2}$$

Here and in what follows, the indices i and j represent the Cartesian directions x and y, and repeated indices imply a summation over that index. In (1) τ_{ij} is the total stress tensor in the Newtonian fluid and it has the usual form:

$$\tau_{ij} = -p_f \delta_{ij} + (\partial_i v_j + \partial_j v_i), \tag{3}$$

where v_i denotes the velocity in the fluid. The gel is assumed to be an incompressible viscoelastic continuum, and the total stress tensor is given by the sum of an isotropic part, an elastic part proportional to the strain and a viscous part proportional to the strain rate:

$$\sigma_{ij} = -p_g \delta_{ij} + (\partial_i u_j + \partial_j u_i) + \mu_r(\partial_i v_j^g + \partial_j v_i^g). \tag{4}$$

Here $u_i(\boldsymbol{x})$ is the Eulerian displacement field and v_i^g is the Eulerian velocity.

The boundary conditions at the interface between the gel and the fluid are the continuity of normal and tangential velocities and stresses. At the bottom boundary $y = -H$ (see figure 1), two different types of boundary conditions are considered, following Kumaran (1993). For 'grafted gels', the polymer molecules in the gel are anchored to the rigid surface at $y = -H$ (see figure 1), and zero displacement conditions are appropriate:

$$u_y = 0, \quad u_x = 0. \tag{5}$$

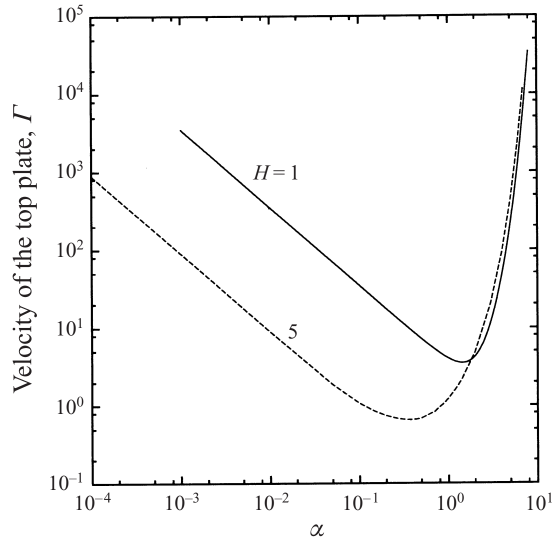

FIGURE 2. Neutral stability curve from the linear theory for grafted gels: $\mu_r = 0$ and $H = 1$ and $H = 5$.

This boundary condition was used in our previous studies (Kumaran *et al.* 1994; Kumaran 1995) on flow through flexible channels and tubes. For 'adsorbed gels' the polymer chains in the gel are permitted to move along the surface at $y = -H$ and hence the appropriate boundary conditions are

$$u_y = 0, \quad \partial_y u_x = 0. \tag{6}$$

This boundary conditions was used by Kumaran (1993) to study the thermodynamics of surface fluctuations in polymer gels. That study showed that there are significant differences between the behaviour of grafted and adsorbed gels in the equilibrium correlation functions at the free surface of the gel. Therefore, it is of interest here to study the stability of flow past flexible walls for both these boundary conditions at $y = -H$.

2.1. *Linear stability theory*

2.1.1. *Grafted gels*

For grafted gels, the base velocity profile is the Couette flow with velocity $\Gamma \equiv V\mu/GR$ at $y = 1$ and the base flow velocity is zero at the interface $y = 0$, and the base velocity profile is given by Γy. The linear stability of Couette flow past a flexible surface was analysed by Kumaran *et al.* (1994) for the case of grafted gels, using a temporal stability analysis. It was shown that for fixed values of μ_r and H, the flow becomes linearly unstable when Γ is greater than a critical value Γ_c. An important feature is the shape of the 'neutral stability curve', which shows the variation of the transition velocity Γ with wavenumber α. For grafted gels, the critical point, which is the minimum of the neutral stability curve, is at a non-zero value of α, as shown in figure 2. Perturbations of all wavelengths are stable below this value of Γ_c, while perturbations with wavenumber α_c become unstable at $\Gamma = \Gamma_c$. For grafted gels, α_c is $O(1)$ when H is $O(1)$, and α_c scales as H^{-1} for $H \gg 1$.

2.1.2. *Adsorbed gels*

For the case of fluid flow past adsorbed gels, with the top place at $y = 1$ moving with a non-dimensional velocity Γ, the base flow velocity profile is determined as follows. The base flow velocity in the fluid exerts a constant shear stress in the gel surface at $y = 0$. However, for adsorbed gels, since the bottom boundary at $y = -H$ has traction-free boundary conditions, it is necessary to postulate a constant (mean) pressure gradient in the gel medium which balances the fluid shear stress at the interface. This mean pressure gradient in the gel medium is transmitted to the fluid through the normal stress continuity condition, which gives rise to a Poiseuille flow component in addition to the base Couette flow. This combined Couette–Poiseuille flow is determined by solving the governing equations in the fluid and the gel coupled by the continuity conditions at the interface. Here, we look for unidirectional velocity and displacement fields in the x-direction, and hence the normal stress boundary condition requires the pressure in the fluid and the gel to be the same. This constant pressure gradient $\bar{P}$ is determined from the tangential stress continuity condition. It can be readily verified that the unidirectional solution to the base state velocity profile is given by

$$\bar{v}_x = \Gamma\,[y + (y^2 - 1)/(2H + 1) - (y - 1)/(2H + 1)]. \tag{7}$$

In the zero Reynolds number limit, the stability theory requires only the gradient of the base flow at the interface ($y = 0$). Using the above expression to calculate the velocity gradient of the base flow, the linear stability of flow past an 'adsorbed' gel is then easily determined using methods similar to that in Kumaran *et al.* (1994). The neutral stability curve, shown in figure 3, is significantly different from that for grafted gels, and has a minimum at $\alpha = 0$. This is because the strain energy required to deform the surface is much lower in the adsorbed gel when compared to a grafted gel (Kumaran 1993). The behaviour of the neutral curve as $\alpha \to 0$ can be obtained through an asymptotic analysis in small α, and this shows that Γ for neutral modes behaves as $\Gamma = \Gamma^{(0)} + \alpha^2\Gamma^{(1)} + \cdots$ for small α. The real part of the growth rate scales as α^2 and the imaginary part scales as α in the limit $\alpha \ll 1$. This behaviour was observed for a wide range of parameters μ_r and H.

3. Weakly nonlinear analysis

In this section, the weakly nonlinear analysis for the present problem is outlined, and the differences between the present analysis and the previous studies are highlighted. As mentioned in § 1, the nonlinearities arise in the present problem due to the boundary conditions at the interface, and the formulation of the boundary conditions is first discussed. The velocity perturbation is zero at the top plate $y = 1$, and the 'grafted' or 'adsorbed' gel boundary conditions discussed in the previous section are applied at $y = -H$. The boundary conditions at the interface between the gel and the fluid are the continuity of normal and tangential velocities and stresses. The position of the interface in the perturbed state is different from that in the base state $y = 0$, and this has to be obtained as a part of the solution. Consider a material point on this unperturbed interface $(x, 0)$. The position changes to $x + \xi(x), \eta(x)$ due to the perturbations, where $\xi(x)$ and $\eta(x)$ are the x- and y-components of the *Lagrangian* displacement of the material point at the interface. The components of the *Eulerian* displacement field $\boldsymbol{u}$ at the perturbed surface are given by

$$u_x(x + \xi, \eta) = \xi, \quad u_y(x + \xi, \eta) = \eta. \tag{8}$$

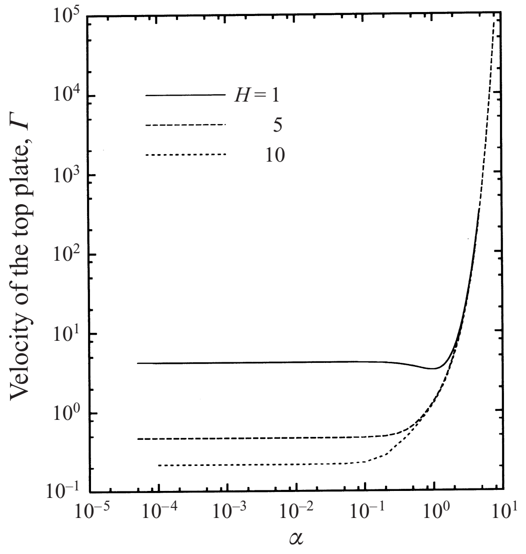

FIGURE 3. Neutral stability curve from the linear theory for adsorbed gels: $\mu_r = 0$ and $H = 1, 5$ and 10.

An infinite series representation for ξ and η can be obtained using the above definition, and these expansions can be used to the desired order in the weakly nonlinear theory. The matching conditions (stress and velocity continuity) at the interface are applied at the perturbed interface, $(x + \xi, \eta)$, and these are given by

$$v_x^f|_{x+\xi,\eta} = v_x^g|_{x+\xi,\eta}, \quad v_y^f|_{x+\xi,\eta} = v_y^g|_{x+\xi,\eta}, \tag{9}$$

$$\tau_{xy}|_{x+\xi,\eta} = \sigma_{xy}|_{x+\xi,\eta}, \quad \tau_{yy}|_{x+\xi,\eta} = \sigma_{yy}|_{x+\xi,\eta}, \tag{10}$$

where τ_{ij} is the stress tensor in the fluid and σ_{ij} is the stress tensor in the gel. The boundary conditions at the interface have the generic form $F|_{x+\xi,\eta} = G|_{x+\xi,\eta}$ where F and G respectively denote fluid and gel quantities, and ξ, η are obtained as a part of the solution. For small perturbations, an asymptotic expansion can be used where the conditions at the perturbed interface are expressed using a Taylor expansion about their values at the unperturbed interface at $(x, 0)$. In this case, the interface conditions are correct only to within a certain order of the small parameter. This procedure is called the 'method of domain perturbations', and this has widely been used in problems involving moving interfaces (Joseph & Renardy 1993). The x- and y-components of the Eulerian gel velocity v_i^g are defined as the substantial derivative of the displacement field:

$$v_x^g = \partial_t u_x + v_x^g \partial_x u_x + v_y^g \partial_y u_x, \quad v_y^g = \partial_t u_y + v_x^g \partial_x u_y + v_y^g \partial_y u_y. \tag{11}$$

These two equations are linear in v_x^g and v_y^g, and hence can be solved to obtain expressions for v_x^g and v_y^g in terms of the displacement fields u_x and u_y and their derivatives. The resulting expression is nonlinear in the displacement quantities, and in the present weakly nonlinear analysis the Eulerian velocities are expanded in a series for small u_i. The resulting expansions for the Eulerian velocities in the gel are nonlinear in terms of the Eulerian displacements, and hence the gel momentum equations are nonlinear in general. However, the gel equations become linear in the absence of viscosity in the gel ($\mu_r = 0$).

The weakly nonlinear analysis for grafted gels is discussed first, and the modifications required for the case of adsorbed gels are outlined at the end of this section. For grafted gels, the neutral curve has a minimum at non-zero α. The values of Γ and α corresponding to this minimum are denoted by Γ_c and α_c. When Γ is slightly above Γ_c, the dynamics will be dominated by the most unstable mode with wavenumber α_c and its higher harmonics generated due to self-interactions. It is convenient to define $E(x,t) = \exp \mathrm{i}(\alpha_c x + \omega t)]$ where ω is the frequency of the perturbations. In the weakly nonlinear expansion, the dynamical quantities are separated into the laminar component (ϕ_l) and the fluctuating component, and the fluctuating component is expressed as an amplitude and harmonic expansion (Stuart 1960; Watson 1960):

$$\phi(x,y,t) = \phi_l(y) + \sum_{k=0}^{\infty} \sum_{n=k,n\neq 0}^{\infty} (A_1(\tau))^n [E^k \tilde{\phi}^{(k,n)}(y) + E^{-k} \tilde{\phi}^{\dagger(k,n)}(y)], \tag{12}$$

where superscript † denotes a complex conjugate, $A_1(\tau) = \epsilon A(\tau)$ is the amplitude of the wave which varies in the slow time scale τ (to be defined below), ϵ is a small parameter that characterizes the initial amplitude of the perturbations, and $A(\tau)$ is an $O(1)$ quantity. In (12) and in the following analysis, the superscript to any dynamical quantity $\tilde{\phi}^{(k,n)}$ implies that it has a harmonic index k and asymptotic order n. $A_1(\tau)$ is a real quantity, since the temporal oscillations are included in $E(x,t)$, and $E(x,t) = \exp[\mathrm{i}(\alpha_c x + \omega t)]$, where t is the fast time scale at which the wave oscillations take place. The following Landau expansion is valid in the vicinity of the critical point (Γ_c, α_c) of the neutral curve:

$$A_1(\tau)^{-1} d_t A_1(\tau) = s_r^0 + A_1(\tau)^2 s_r^1 + \cdots. \tag{13}$$

Here, $A_1(\tau) = \epsilon A(\tau)$ where $A(\tau)$ is $O(1)$, and ϵ is the small parameter that characterizes the amplitude of the disturbances. Here s_r^0 is the real part of the linear growth rate s^0, and s_r^1 is the real part of the first Landau constant s^1. Since $s_r^0 \sim (\Gamma - \Gamma_c)$ near the neutral curve, s_r^0 is expressed as $s_r^0 = (\mathrm{d}s_r^0/\mathrm{d}\Gamma)(\Gamma - \Gamma_c)$. If s_r^1 is $O(1)$, then the second term on the right-hand side of (13) is $O(\epsilon^2)$, and consistency requires that $(\Gamma - \Gamma_c)\mathrm{d}s_r^0/\mathrm{d}\Gamma \sim \epsilon^2$. For definiteness, let $(\Gamma - \Gamma_c) = \Gamma_2 \epsilon^2$, where Γ_2 determines whether the flow is stable or unstable. For a balance between the right- and left-hand sides of (13), it is necessary to introduce the slow time scale τ in the time derivative as $\mathrm{d}_t = \mathrm{d}_t + \epsilon^2 \mathrm{d}_\tau$. Since $A_1(\tau)$ is independent of the fast time scale t, (13) becomes

$$A^{-1}\mathrm{d}_\tau A = \Gamma_2 \mathrm{d}s_r^0/\mathrm{d}\Gamma + s_r^1 A^2, \tag{14}$$

where the Landau equation is now independent of ϵ near the critical point. Similarly, the frequency of oscillations ω is also expanded in a series

$$\omega = s_i^0 + A(\tau)^2 s_i^1 + \cdots, \tag{15}$$

where s_i^0 is the frequency of perturbations according to the linear theory and s_i^1 is the modification to the frequency of the perturbations due to self-interactions generated due to the nonlinearities.

The objective of the rest of the analysis is to determine s^1 from the governing equations of the fluid and the gel, and thus to determine whether the instability is supercritical or subcritical. All the dynamical quantities are expanded in the amplitude and harmonic expansion as in (12). In the limit of zero inertia and zero gel viscosity ($\mu_r = 0$), the governing equations at different orders are linear and do not contain any inhomogeneous terms. The boundary conditions for the problem at order (k,n) contain inhomogeneous terms of order (k,m) for reasons explained above, where

$m < n$. Thus, the original nonlinear problem with an unknown interface is reduced to a hierarchy of linear but inhomogeneous problems, and these are solved sequentially beginning from $k = 1, n = 1$. For $\mu_r \neq 0$, the governing equations in the gel also have nonlinearities due to the Eulerian representation of the gel velocities in terms of the Eulerian displacements u_i. The differential operators at different k and n are simple enough to allow analytical solutions at each order. In the interests of brevity, we do not provide the expressions for governing equations and boundary conditions at various orders, and the interested reader is referred to Shankar (2000) for further details. Here we merely provide an outline of the (by now standard) solution procedure.

(i) The problem at order $k = 1, n = 1$ corresponds to the linear stability analysis (Kumaran *et al.* 1994). The eigenvalue of the linear problem Γ and the constants in the eigenfunctions are determined from the boundary conditions at the interface $y = 0$. To determine all the constants, an additional 'normalization' condition is required, which is specified as $|\tilde{v}_y^{(1,1)}|_{y=0} = \sqrt{2}$.

(ii) The velocity at the order $k = 0, n = 2$ represents the correction to the mean flow due to nonlinearities. The fluid continuity equation takes the form $\partial_y \tilde{v}_y^{(0,2)} = 0$, and after using the boundary condition $\tilde{v}_y^{(0,2)} = 0$ at the top boundary $y = 1$, it can easily be seen that $\tilde{v}_y^{(0,2)} = 0$ throughout the domain. Similarly, it can easily be concluded that $\tilde{u}_y^{(0,2)} = 0$ in the gel. Further, $\tilde{p}_f^{(0,2)} = 0$ and $\tilde{p}_g^{(0,2)} = 0$ since the mean pressure gradient in the Couette flow is zero. The solution for $\tilde{v}_x^{(0,2)}$ is readily obtained using the condition $\tilde{v}_x^{(0,2)} = 0$ at $y = 1$ and the inhomogeneous tangential velocity condition at $y = 0$, and the solution for $\tilde{v}_x^{(0,2)}$ is a linear function of y. The solution for $\tilde{u}_x^{(0,2)}$ is obtained using the boundary condition at $y = -H$ and the inhomogeneous tangential stress condition at $y = 0$.

(iii) Similarly, the eigenfunctions for $\tilde{v}_y^{(2,2)}$ and $\tilde{u}_y^{(2,2)}$ at the order $k = 2, n = 2$ are obtained analytically, and the inhomogeneous boundary conditions are used to determine the constants appearing in $\tilde{v}_y^{(2,2)}$ and $\tilde{u}_y^{(2,2)}$.

(iv) The variation of the amplitude $A(\tau)$ with the slow time scale appears in the inhomogeneous terms in the boundary conditions at order $k = 1, n = 3$. The governing equations of the fluid are identical to those for $k = 1, n = 1$, but the equations for the gel contain inhomogeneous terms due to the nonlinear nature of the Eulerian gel velocity, which are absent when $\mu_r = 0$. The homogeneous part of the differential operator and boundary conditions at the order $k = 1, n = 3$ are identical to the $k = 1, n = 1$ problem. Consequently, the Fredholm solvability conditions (Drazin & Reid 1981) should be satisfied for the $k = 1, n = 3$ problem to have non-trivial solutions. It is necessary to determine the adjoint eigenfunctions of the $k = 1, n = 1$ problem to this end. On substituting the adjoint eigenfunctions in the solvability condition, the Landau equation is obtained after taking the real part of the solvability condition, from which the Landau constant s_1 can be read off. For further details, the interested reader is referred to Shankar (2000). If the real part of the first Landau constant s_r^1 is positive, then the instability is subcritical, while if s_r^1 is negative the instability is supercritical. The imaginary part s_i^1 is the correction to the frequency of the basic wave due to nonlinear self-interactions.

3.1. *Modifications for the adsorbed gel problem*

The analysis outlined above carries through for the case where the gel is permitted to move at the bottom plate ($y = -H$), except for a few modifications which are briefly explained below. The governing equations at the order $k = 0, n = 2$ for the fluid are

$$\partial_y \tilde{v}_y^{(0,2)} = 0, \quad -P + \partial_y^2 \tilde{v}_x^{(0,2)} = 0, \quad \partial_y^2 \tilde{v}_y^{(0,2)} = 0. \tag{16}$$

	μ_r	α_c	Γ_c	Re $[s_1]$	Im $[s_1]$
$H=1$	0	1.45	3.52680	32.713388	4.508961
	0.5	1.3	5.23957	30.526829	−5.414918
	0.75	1.25	7.57164	28.247901	−6.268525
	0.80	1.225	8.50067	27.211807	−5.989048
	0.90	1.21	12.11915	25.066276	−4.585820
	0.99	1.201	38.59668	22.417563	−1.4728612
$H=2$	0	0.875	1.66764	28.388916	−0.5882257
	1.0	0.7	2.164447	22.987794	−9.1962394
	2.0	0.5	3.31957	20.136629	−14.288666
	3.0	0.25	6.91270	23.772963	−32.969592
	3.2	0.22	8.69238	22.763930	−37.3070072
	3.5	0.16	14.0363	24.0741715	−54.651585
$H=5$	0	0.375	0.65835	45.5548444	−4.0702738
	1	0.35	0.67642	42.4199290	−4.78980903
	5	0.25	0.79707	42.8017640	−16.3943086
	10	0.18	1.06362	38.1968291	−33.8443030
	20	0.075	3.20798	32.6955271	−90.877898
$H=10$	0	0.17	0.326877	105.13686	−29.7340444
	1	0.17	0.327903	100.238669	−22.860737
	5	0.16	0.334138	91.008872	−14.365128
	10	0.15	0.345974	80.458307	−17.736651
	20	0.125	0.381342	74.88226	−32.289085
$H=20$	0	0.083	0.161800	179.1429090	−109.8956
	1	0.0825	0.161859	184.304374	−109.091013
	10	0.081	0.162705	198.574548	−74.968425
	20	0.08	0.164258	182.367762	−47.90102080
$H=50$	0	0.0325	6.4210701×10^{-2}	364.61357	−313.768022
	5	0.0325	6.4218755×10^{-2}	372.83500	−311.126917
	10	0.0325	6.4228535×10^{-2}	379.35433	−309.304507
	20	0.0325	6.4253274×10^{-2}	396.34186	−299.231062

TABLE 1. Landau coefficients for the flow past a grafted gel.

The governing equations for the gel are

$$\partial_y \tilde{v}_y^{(0,2)} = 0, \quad -P + \partial_y^2 \tilde{v}_x^{(0,2)} = 0, \quad \partial_y^2 \tilde{u}_y^{(0,2)} = 0. \tag{17}$$

Here P is the constant pressure gradient that is required to obtain non-trivial solutions at this order ($k=0, n=2$), and the reason for this pressure gradient is similar to that given in § 2.1.2 for the base flow past an adsorbed gel. Once the $k=0, n=2$ problem is solved using the above governing equations, the rest of the analysis, including the form of the solvability condition, remains the same for both grafted and adsorbed gels, and the Landau constant is found in an identical manner.

4. Results

4.1. *Grafted gel*

As mentioned before, the neutral curve for the grafted gel problem exhibits a minimum at a finite α_c, and the Landau coefficient was calculated for a wide range of μ_r and H for values for Γ slightly above Γ_c. The first term in the Landau equation is the linear growth rate, and this should be the same as that obtained from the linear stability theory. This serves as a useful consistency check in the analysis, and this consistency was verified for all the results reported here. The results for Landau constants are provided in table 1, where the real and imaginary parts of the Landau coefficients are presented along with the Γ_c and α_c values (the critical Γ and wavenumber α, respectively) of the linear neutral curve. It is important to note here that the numerical values of the Landau constant depend on the normalization condition used for determining the eigenfunctions of the linear theory. In the present study, the normalization that $\tilde{v}_y^{(1,1)} = (1 + \mathrm{i})$ at $y = 0$ has been used. The results of table 1 show that the real part of the Landau constants ($\mathrm{Re}[s_1]$) is always positive for a wide range of parameter values (μ_r and H), and the bifurcation is subcritical. This implies that there do not exist any neighbouring supercritically stable stationary states in the immediate vicinity of the linear neutral curve (to the order to which the present calculations are performed). The question arises whether this result is observed experimentally. In the experiments of Kumaran & Muralikrishnan (2000), when attempts were made to maintain the stress just above the value at which instability occurred, it was observed that there were large oscillations in the gel of the same amplitude as the gap width, and these oscillations damaged the surface of the gel. However, the transition point itself was obtained repeatably if the experiment was stopped before the damage was caused to the polymer gel. This experimental observation is consistent with the results from the present weakly nonlinear analysis for the following reason. If there existed an experimental supercritical stable state very close to the linear instability, the flow would have settled to a new non-laminar state, and this would not have led to the oscillations in the polymer gel. Nonetheless, we are still operating in the Stokes flow limit, and increasing the fluid velocity cannot cause a transition to turbulence in the conventional sense of the term, and the nature of the eventual state that would be reached by the flow after the instability remains unclear at present.

A quantity of interest in subcritically unstable flows is the reduction in the critical velocity required to destabilize the flow (Γ) from the value predicted by the linear theory, Γ_c, due to the finite-amplitude nature of the disturbances. In these calculations, the normal displacement of the gel at the interface was taken to be a representative quantity for the amplitude of the disturbances. The fluctuation in the normal displacement at the unperturbed interface ($y = 0$) is, correct to $O(\epsilon)$,

$$u_y|_{y=0} = u_y^*/R = A_{1c}|\tilde{u}_y^{(1,1)}|_{y=0}. \tag{18}$$

Here, u_y is the non-dimensional y-displacement which is defined as the ratio of the dimensional y displacement and the dimensional fluid thickness R. As defined in the previous section, $A_1 = \epsilon A$, and A is the $O(1)$ amplitude while A_1 is $O(\epsilon)$. The physical quantity u_y is given by the product of the modulus of the $k = 1, n = 1$ eigenfunction $\tilde{u}_y^{(1,1)}$ and the critical amplitude A_{1c}, which is obtained by setting $\mathrm{d}A/\mathrm{d}t$ to zero in the Landau equation. A_{1c} is therefore given by

$$(A_{1c})^2 = \frac{\mathrm{d}s_r^0}{\mathrm{d}\Gamma}(\Gamma_c - \Gamma)/s_r^{(1)}, \tag{19}$$

where $\Gamma_c > \Gamma$, and the flow is stable according to the linear theory. From equations (18) and (19) the actual Γ required for instability to finite-amplitude disturbances can be calculated by prescribing a particular level of disturbances (u_y). Table 2 provides the results of this calculation, where the percentage reduction in Γ is calculated as a function of percentage increase in $|u_y|_{y=0}$. This table shows that, for all the parameter

	$H = 20$		$H = 10$	
% increase in $\|u_y\|_{y=0}$	μ_r	% reduction in Γ	μ_r	% reduction in Γ
2.5	0	0.69	0	0.8337
5.0	0	2.76	0	3.335
10.0	0	11.04	0	13.34
2.5	10	0.774	5	0.7637
5.0	10	3.0	5	3.055
10.0	10	12.38	5	12.22
2.5	20	0.732	10	0.744
5.0	20	2.931	10	2.978
10.0	20	11.726	10	11.9138

TABLE 2. Reduction in the critical velocity Γ required for instability when the flow is subcritical: results for grafted gels. Here, $|u_y|_{y=0}$, which is the amplitude of the displacement in the gel at the interface, is taken to be the value representative of the actual perturbations. The percentage reduction in Γ is calculated as $(\Gamma_c - \Gamma)/\Gamma_c \times 100$, where Γ_c is the critical velocity predicted by the linear theory, and Γ is the actual critical velocity due to finite amplitude disturbances.

values shown, the reduction in critical velocity is less than 1% for a disturbance level of 2.5%, and the reduction is around 3% for a disturbance level of 5%. These results imply that even though the bifurcation is subcritical in the present case, relatively weak but finite disturbances do not result in a significant decrease in the critical velocity required to destabilize the flow, when compared to the predictions of the linear stability theory. This could be the reason for the consistent agreement between the experimental results of Kumaran & Muralikrishnan (2000) with the results of the linear stability theory (Kumaran *et al.* 1994). This result is in marked contrast with the behaviour of finite disturbances in plane-Poiseuille flow in a rigid channel, which is also subcritically unstable. Reynolds & Potter (1967) calculated the Landau constants for the plane-Poiseuille flow in a rigid channel, and found that even very weak finite disturbances resulted in a drastic reduction of the critical Reynolds number. For example, when the disturbance level was assumed at 2.5%, the reduction in the critical *Re* from the linear theory value due to finite disturbances was found to be around 65%. It is also experimentally known that the plane-Poiseuille flow in a rigid channel does become unstable at values of *Re* well below the Re_c of the linear stability theory. However, in the present case, the extent of reduction of the critical velocity is very small for weak but finite disturbances, and hence the flow can be expected to become unstable only in the vicinity of the critical velocity predicted by the linear theory.

4.2. *Adsorbed gel*

For adsorbed gels, the most (linearly) unstable modes have $\alpha \to 0$ (see figure 3). However, in an experiment this just implies that the wavelength ($\sim 1/\alpha$) of the most unstable mode is cut off by the system size. This is in contrast to the case of grafted gels, where the most unstable modes have $\alpha_c \sim O(1)$, and consequently the weakly nonlinear analysis should be carried out about this (unique) critical wavenumber. Therefore, it is important to carry out the weakly nonlinear calculations for a range of values of α_c for adsorbed gels. In the present study, the Landau constants have been calculated for small values of α (typically $\alpha \leqslant 0.1$) in the Γ, α neutral curve. A qualitative summary of the results obtained from the analysis is presented in table 3.

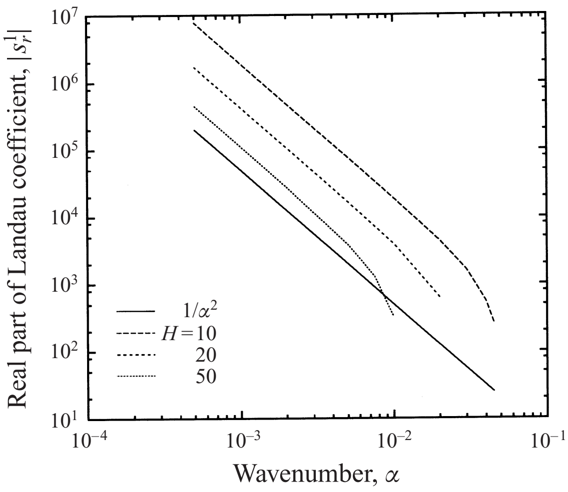

FIGURE 4. The variation of the modulus of real part of the Landau coefficient with α (Note that Re $[s_1]$ is negative and only the magnitude of the quantity is plotted above): $\mu_r = 0$ and for different values of H.

H	Re $[s_1]$ $\mu_r = 0$	Re $[s_1]$ $\mu_r = 5$	Re $[s_1]$ $\mu_r = 10$
1	positive	negative	negative
5	positive	positive	positive
10	negative	negative	positive
20	negative	negative	negative
50	negative	negative	negative

TABLE 3. Qualitative summary of Landau coefficients for the flow past an adsorbed gel for different values of μ_r and H.

The term 'positive' in table 3 implies that Re$[s_1]$ is always positive for all values of α for a fixed H and μ_r, and the term 'negative' refers to the fact that Re$[s_1]$ is negative for sufficiently low values of α. This is illustrated in detail for the specific case $\mu_r = 0$ in table 4, and it can be seen that Re$[s_1]$ changes its sign from positive to negative as α is decreased to lower values and remains negative for $\alpha \rightarrow 0$. The normalization condition used for determining the eigenfunctions of the linear stability theory is given by $\tilde{v}_y^{(1,1)}(y = 0) = (1 + \mathrm{i})$. In table 4, only the data for those values of H for which the Landau constants are negative are presented; the data for the cases where Re$[s_1]$ is always positive are not presented here. The variation of Re$[s_1]$ with α is shown in figure 4 only for the cases where the real part of Landau constant is negative. This figure clearly shows that Re$[s_1] \sim 1/\alpha^2$ in the limit $\alpha \ll 1$. This behaviour was observed for other values of μ_r as well. As seen from table 3, there is no systematic trend for the dependence of the sign of Re$[s_1]$ on μ_r and H. However, from the results obtained, it can be concluded that for larger values of $H (H = 20$ and 50 in table 3) Re$[s_1]$ always becomes negative for sufficiently low values of α for all μ_r. Thus, the weakly nonlinear analysis reveals that there could be supercritically stable states in the immediate vicinity of the neutral curve in the limit of low wavenumbers.

The results of the weakly nonlinear analysis also clearly show that when the Landau

	α	Re $[s_1]$
$H = 10$	0.1	153.0135175
	0.07	202.702799
	0.06	142.509125
	0.05	−67.398321
	0.0475	−157.300434
	0.045	−266.8097432
	0.04	−557.460738
	0.03	−1598.92959
	0.02	−4357.610656
	0.01	−18732.27321
	0.005	−75963.77863
	0.002	−476461.086018
	0.001	-1.90985×10^6
	0.0005	-7.7726185×10^6
$H = 20$	0.1	108.282520
	0.05	229.229763
	0.02	−625.233424
	0.01	−3909.087884
	0.005	−16667.46921
	0.002	−105782.41004
	0.001	−424124.31417
	0.0005	-1.70194363×10^6
$H = 50$	0.1	105.466041
	0.05	205.486326
	0.01	−319.89544
	0.0075	−1281.914646
	0.005	−3852.405875
	0.002	−27409.70771
	0.001	−111331.88248
	0.00075	−199273.3158
	0.0005	−447149.5143

TABLE 4. Landau coefficients for the flow past an adsorbed gel: $\mu_r = 0$.

constant is negative, there exists an asymptotic regime where $s_r^1 \sim 1/\alpha^2$ in the limit $\alpha \to 0$, where $s_r^1 \equiv \mathrm{Re}[s_1]$. However, this behaviour of $\mathrm{Re}[s_1]$ with α is dependent on the normalization condition used to determine the eigenfunctions of the linear theory. For example, a different normalization condition $\tilde{v}_x^{(1,1)}(y=0) = (1+\mathrm{i})$ yields constant $\mathrm{Re}[s_1]$ in the limit $\alpha \to 0$. The physical velocity components are related to the product of the amplitudes and the eigenfunctions, and this product is independent of the normalization used to determine the eigenfunctions of the linear theory. In the supercritical equilibrium state, the perturbation velocities are given (to $O(\epsilon)$) by

$$v_y = \epsilon A_{eq} \tilde{v}_y^{(1,1)} E(x,t) + \cdots, \quad v_x = \epsilon A_{eq} \tilde{v}_x^{(1,1)} E(x,t) + \cdots. \tag{20}$$

The expression for the equilibrium amplitude (A_{eq}) that occurs in (20) can be obtained from the Landau equation by setting the time derivative of A to zero:

$$(\epsilon A_{eq})^2 = \frac{\mathrm{d}s_r^0}{\mathrm{d}\Gamma}(\Gamma - \Gamma_c)/|s_r^1|, \tag{21}$$

where $\Gamma > \Gamma_c$ and the flow is linearly unstable. As noted above, $\mathrm{Re}[s_1] \sim \alpha^2$, and $\mathrm{d}s_r^0/\mathrm{d}\Gamma \sim \alpha^2$ from the low-α asymptotic analysis of the linear stability theory. This

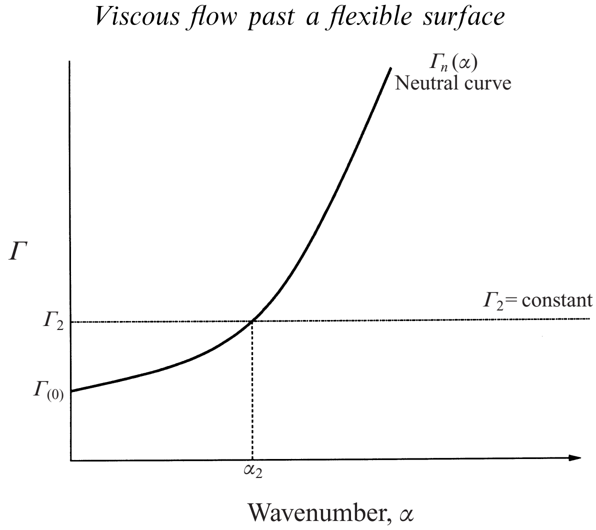

FIGURE 5. Schematic illustration of the neutral curve for adsorbed gels. The scales are exaggerated for clarity; see text for explanation.

implies that $A_{eq} \sim \alpha^2$. From equation (20), the physical perturbation velocity v_y should then scale as α^2 in the low-α limit, since the normalization used is $\tilde{v}_x \sim \alpha^{-1}\tilde{v}_y$, and hence the physical perturbation velocity $\tilde{v}_x \sim \alpha$.

In the expression for A_{eq} (equation (21)), the numerator goes to zero as α^2 in the limit $\alpha \to 0$, and the denominator diverges as α^{-2} in the same limit. This implies that the most (linearly) unstable mode ($\alpha \to 0$) will not have the largest amplitude in the supercritical equilibrium state; rather a mode with finite wavenumber will have the maximum amplitude. This is explained in some detail in the following discussion. Consider the schematic neutral curve for adsorbed gels shown in figure 5. Here, $\Gamma_n(\alpha)$ is the functional relation for the linear neutral curve where $s_r^0 = 0$. As stated before, a low-α asymptotic analysis of the linear stability problem reveals that, for $\alpha \to 0$,

$$\Gamma_n(\alpha) = \Gamma^{(0)} + \alpha^2\Gamma^{(1)} + O(\alpha^4). \tag{22}$$

The quantities $\Gamma^{(0)}$ and $\Gamma^{(1)}$ can be readily obtained from a low-α asymptotic analysis of the linear problem. In the low-α limit, the neutral curve can be truncated to $O(\alpha^2)$, and $O(\alpha^4)$ terms are neglected in this analysis. The weakly nonlinear analysis assumes that we are operating slightly above or below the minimum Γ of the linear neutral curve. The threshold Γ of the linear neutral is denoted by $\Gamma^{(0)} \equiv \Gamma_n(\alpha \to 0)$. If the velocity of the top plate is fixed at Γ_2 given by

$$\Gamma_2 = \Gamma^{(0)} + \epsilon^2\Gamma_{10}, \tag{23}$$

where Γ_{10} is an 'experimentally' fixed $O(1)$ quantity, and ϵ is the small parameter discussed in the previous section, and if $\Gamma_{10} > 0$, the linear growth rate can be expanded as

$$s_r^0(\Gamma_2, \alpha) = \frac{\mathrm{d}s_r^0}{\mathrm{d}\Gamma}(\Gamma_2 - \Gamma_n(\alpha)). \tag{24}$$

The quantity $\mathrm{d}s_r^0/\mathrm{d}\Gamma$ is readily obtained from the linear stability analysis, and it scales as α^2 for $\alpha \to 0$. The line Γ_2 = constant intersects the neutral curve $\Gamma_n(\alpha)$ at $\alpha = \alpha_2$ (see figure 5), and the linear growth rate is identically zero at this point. In addition, the linear growth rate also goes to zero as α^2 in the limit $\alpha \to 0$, and consequently the equilibrium amplitude is zero for both $\alpha \to 0$ and $\alpha = \alpha_2$. This implies that the maximum amplitude will occur for modes with wavenumbers somewhere in between $\alpha = 0$ and $\alpha = \alpha_2$. The functional form for the dependence of A_{eq} on α can obtained as follows. Since $\Gamma_2 = \Gamma^{(0)} + \alpha_2^2\Gamma^{(1)}$ and $\Gamma_n(\alpha) = \Gamma^{(0)} + \alpha^2\Gamma^{(1)}$, we have

$$\Gamma_2 - \Gamma_n(\alpha) = \alpha_2^2\Gamma^{(1)}(1 - \alpha^2/\alpha_2^2). \tag{25}$$

The line $\Gamma_2 =$ constant intersects the neutral curve at $\alpha = \alpha_2$, i.e. $\Gamma_2 = \Gamma_n(\alpha = \alpha_2)$, and so $\Gamma_2 - \Gamma^{(0)} = \alpha_2^2\Gamma^{(1)}$. Using the fact that $\Gamma_2 - \Gamma^{(0)}$ is also equal to $\epsilon^2\Gamma_{10}$ (see equation (23)), gives $\alpha_2^2\Gamma^{(1)} = \epsilon^2\Gamma_{10}$. We are free to choose Γ_{10} which is an 'experimentally' fixed $O(1)$ quantity, and if we choose $\Gamma_{(10)} = \Gamma^{(1)}$, we then have $\alpha_2 = \epsilon$. Therefore, equation (25) becomes

$$\Gamma_2 - \Gamma_n(\alpha) = \epsilon^2\Gamma_{10}(1 - \alpha^2/\alpha_2^2). \tag{26}$$

The growth rate of the linear theory in the vicinity of the neutral curve then becomes

$$s_r^0 = \epsilon^2 \frac{\mathrm{d}s_r^0}{\mathrm{d}\Gamma}\Gamma_{10}(1 - \alpha^2/\alpha_2^2). \tag{27}$$

The low-α asymptotic analysis of the linear theory indicates that s_r^0 and (hence) $\mathrm{d}s_r^0/\mathrm{d}\Gamma$ scale as α^2. Therefore, the equilibrium amplitude in equation (21) scales as (after noting that $s_r^1 \sim \alpha^{-2}$)

$$A_{eq} \sim \alpha^2(1 - \alpha^2/\alpha_2^2)^{1/2}. \tag{28}$$

The equilibrium amplitude (for a given fixed Γ_{10}) goes to zero both as $\alpha \to 0$ and for $\alpha = \alpha_2$, and the maximum amplitude occurs at $\alpha^* \approx 0.82\alpha_2$. Thus, the physical perturbation velocity v_y (see equation (20)) will vary as A_{eq}, since $\tilde{v}_y^{(1,1)}$ was fixed to be $O(1)$ in the normalization, and the maximum of v_y will also occur at $\alpha^* \approx 0.82\alpha_2$. However, v_x will vary as $\alpha^{-1}A_{eq} \equiv \alpha(1 - \alpha^2/\alpha_2^2)^{1/2}$, and this will exhibit a maximum at $\alpha^* = \alpha_2/\sqrt{2}$.

5. Concluding remarks

The weakly nonlinear stability of viscous flow past a flexible surface was analysed in the limit of zero Reynolds number for two different types of polymer gels: (i) grafted gels where the polymer chains are anchored to the bottom wall and (ii) adsorbed gels where the polymer chains are free to slide across the bottom wall. The results from the weakly nonlinear analysis show that the bifurcation at the linear instability is always subcritical for the grafted gel for a wide range of parameters (μ_r and H), while it could be supercritical for the adsorbed gel in the limit of low wavenumbers. For grafted gels, it is shown that the reduction in the critical velocity from the linear theory value due to nonlinear effects of the finite-amplitude perturbations is small. For example, the reduction in the critical velocity is about 1% for perturbations of amplitude 2.5% of the channel width. This is in contrast to the weakly nonlinear studies of the plane-Poiseuille flow in rigid channels, where there is a reduction of about 65% in the critical Reynolds number due to nonlinear effects when the velocity perturbation is 2.5% of the maximum velocity. Therefore, the linear stability studies provide an accurate estimate of the critical velocity in the present case. The subcritical nature of the bifurcation also implies that there are no new non-laminar steady states in the vicinity of the critical velocity of the linear theory. While there is some experimental support for this scenario (Kumaran & Muralikrishnan 2000), the present analysis was carried out in the low Reynolds number limit and hence a transition to turbulence at such low Reynolds number seems implausible. Clearly further experimental studies are needed to clarify this picture, and there still remain unanswered questions regarding the nature of the transition. One possible approach to clarify the nature of transition after the linear instability is to numerically stimulate the flow past a flexible surface in the limit of low Reynolds numbers. Such a study can be used to verify the asymptotic results of the present study, and to examine the actual structure of the non-laminar flow in the limit of zero Reynolds number, which is a unique feature in flow past flexible solid surfaces. For adsorbed gels, the weakly nonlinear analysis reveals that there could be supercritically stable states in the immediate vicinity of the linear neutral curve in the limit of low wavenumber, for a wide range of parameters. When the bifurcation is supercritical, it is further shown that the physical quantities will exhibit a maximum equilibrium amplitude at non-zero wavenumber, even though the most unstable mode in the linear theory has zero wavenumber. For adsorbed gels, the present study also offers the experimentally verifiable prediction that the nature of the bifurcation changes from subcritical (at higher values of α) to supercritical for lower values of α. It is possible to test this prediction in an experiment by changing the length of the experimental set-up and thereby varying the range of accessible wavenumbers.

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