ESTIMATION OF NON-LINEAR STIFFNESS PARAMETERS OF ROLLING ELEMENT BEARINGS FROM RANDOM RESPONSE OF ROTOR–BEARING SYSTEMS

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A technique for estimation of non-linear stiffness parameters of rolling element bearings in rotor systems, based on the analysis of the random response signals picked up from the bearing caps, is developed. The rotor–bearing system is modelled through the Fokker–Planck equation and the vibrations, resulting due to random imperfections of the bearing surfaces and assembly, are processed through a curve fitting algorithm to obtain the necessary bearing stiffness parameters. The technique has an advantage over existing ones in that it does not require an estimate of the excitation forces and works directly on the measured response signals of the system. The algorithm is illustrated for a laboratory rotor–bearing test rig and the results are compared with those obtained through an existing analytical model.

1. INTRODUCTION

Estimation of the elastic parameters of bearings is crucial to rotor design. Rolling element bearings are known to possess highly non-linear elastic characteristics and the complexities and approximate nature of the analytical determination of these properties are responsible for some of the unreliabilities in the prediction of response and stability of a rotor system. The early studies [1, 2] on bearings concern vibrations caused due to geometric imperfections of contact surfaces. Theoretical models [3, 4] are available for estimation of bearing stiffnesses under static loading conditions. Comprehensive investigations have been carried out on the high frequency response of bearings [5] and its relation to surface irregularities [6].

A method for determination of non-linear characteristics of bearings, using the procedure of Krylov–Bogoliubov–Mitropolsky, has been suggested by Kononenko and Plakhtienko [7], while Muszynska and Bently [8] developed a perturbation technique for estimation of these parameters. Both these techniques involve giving a controlled input excitation to the bearings. In the present study, a technique for on-line determination of non-linear bearing stiffness, based on the analysis of the random vibration signals picked up at the bearing caps, is described. The discussion is presently restricted to bearings supporting rigid rotors that can be treated as a single-degree-of-freedom systems. For a balanced rotor, the excitation, caused by a variety of undetermined sources, like bearing surface imperfections, inaccuracies in the rotor–bearing–housing assembly, etc., is treated as a white noise source. A curve fitting algorithm is developed to process the response of the Fokker–Planck equation to extract the bearing stiffness parameters. The algorithm is tested by Monte Carlo simulation.
2. RANDOM RESPONSE OF ROTOR–BEARING SYSTEMS

The governing equation for a balanced rigid rotor supported at ends in bearings (see Figure 1) with non-linear stiffnesses can be written as

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 [x + \lambda G(x)] = f(t), \] (1)

where \( G \) can be a polynomial in \( x \) and \( \lambda \) is the unknown non-linear stiffness contribution parameter. \( f(t) \) in equation (1) represents the random excitation to the system (a list of nomenclature is given in the Appendix). The bearing surface imperfections, caused by random deviations from their standard theoretical design and progressive surface and subsurface deterioration, are large enough to cause measurable levels of vibration and can be the primary source of these excitations. In addition excitation can be contributed by random inaccuracies in the rotor–bearing–housing assembly, etc. The approach to the solution of equation (1) is greatly simplified if the overall random excitation to the system, from the variety of sources, is treated as ideal white noise. While many engineering applications are based on this idealization, insofar as the excitation itself is concerned, it turns out that the responses obtained through such models are quite acceptable if the time scale of excitation is much smaller than the time scale of the response [10].

The time scale for the excitation is the correlation time, roughly defined as the length of time beyond which the excitation process is nearly uncorrelated. The time scale of the response is the measure of the memory duration of the system, which is generally about one quarter of the natural period of a mode which contributes significantly to the total response.

The joint probability, \( p(x, \dot{x}) \), of the displacement and velocity response to ideal white noise excitation is described by the Fokker–Planck equation [11]

\[ -\dot{x} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( 2\zeta \omega_n \dot{x} p \right) + \omega_n^2 [x + \lambda G(x)] \frac{\partial p}{\partial x} + \pi S_0 \frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t}, \] (2)
where \( S_0 \) is the uniform spectral density of excitation. For a stationary case equation (2) becomes

\[
-x \frac{\ddot{p}}{\ddot{X}} + \frac{\ddot{p}}{\ddot{X}} (2\zeta \omega_n \dot{x}p) + \omega_n^2 [x + \dot{x} G(x)] \frac{\dot{p}}{\ddot{X}} + \pi S_0 \frac{\ddot{p}}{\ddot{X}} = 0, \tag{3}
\]
the solution of which is

\[
p(x, \dot{x}) = c \exp[(2\zeta \omega_n / \pi S_0) \{\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{x} \omega_n^2 + \lambda \omega_n^2 g(x)\}], \tag{4}
\]
where \( g(x) = \int_0^x G(\xi) d\xi \). The probability density functions \( p(x) \) and \( p(\dot{x}) \) are obtained from equation (4) as [12]

\[
p(x) = \int_{-\infty}^{\infty} p(x, \dot{x}) d\dot{x} = c_1 \exp[-(2\zeta \omega_n / \pi S_0) \{\frac{1}{2} \dot{x}^2 + \dot{x} \dot{g}(x)\}],
\]
where

\[
c_1 = 1 \int_{-\infty}^{\infty} \exp[-2\zeta \omega_n / \pi S_0 \{\frac{1}{2} \dot{x}^2 + \dot{x} \dot{g}(x)\}] \int_{-\infty}^{x} \dot{x}^2 p(\dot{x}) d\dot{x} = \frac{\pi S_0}{2\zeta \omega_n}. \tag{5, 6}
\]

The variance of the velocity response is

\[
\sigma_\dot{x}^2 = \int_{-\infty}^{\infty} \dot{x}^2 p(\dot{x}) d\dot{x} = \frac{\pi S_0}{2\zeta \omega_n}. \tag{7}
\]

From equations (5), (6) and (7), the probability density function for the displacement response can be written as

\[
p(x) = c_1 \exp[-(\omega_n^2 / \sigma_\dot{x}^2) \{\frac{1}{2} \dot{x}^2 + \dot{x} \dot{g}(x)\}]
\]
with

\[
c_1 = 1 \int_{-\infty}^{\infty} \exp[-(\omega_n^2 / \sigma_\dot{x}^2) \{\frac{1}{2} \dot{x}^2 + \dot{x} \dot{g}(x)\}] d\dot{x}. \tag{8}
\]

3. EXTRACTION OF BEARING STIFFNESS PARAMETERS

The bearing stiffness parameters are obtained from the experimentally obtained random response in terms of the linear parameter \( \omega_n^2 \) and the non-linear stiffness parameter \( \lambda \). These parameters are obtained for both the vertical and horizontal directions, the problem formulation, in the horizontal direction, remaining identical to that in the vertical direction.

The probability density functions for any two displacements \( x_i \) and \( x_{i+1} (x_{i+1} > x_i) \), from equation (8), are

\[
p(x_i) = c_1 \exp[-(\omega_n^2 / \sigma_\dot{x}^2) \{\frac{1}{2} \dot{x}_i^2 + \dot{x}_i \dot{g}(x_i)\}]
\]
and
\[ p(x_{i+1}) = c_1 \exp \left[ -(\omega_0^2 / \sigma_x^2) \left\{ \frac{1}{2} x_{i+1}^2 + \tilde{g}(x_{i+1}) \right\} \right]. \tag{9, 10} \]

Defining
\[ \Delta x_i = x_{i+1} - x_i \quad \text{so that} \quad x_{i+1}^2 = (x_i + \Delta x_i)^2 \approx x_i^2 + 2x_i \Delta x_i \tag{11, 12} \]
for small \( \Delta x_i \), and
\[ g(x_{i+1}) = g(x_i + \Delta x_i) \approx g(x_i) + \frac{dg(x_i)}{dx} \bigg|_{x=x_i} \Delta x_i, \tag{13} \]
and substituting from equations (12) and (13) into equation (10), gives
\[ p(x_{i+1}) = c_1 \exp \left[ -\frac{\omega_0^2}{\sigma_x^2} \left\{ \frac{1}{2} x_i^2 + \tilde{g}(x_i) \right\} \exp \left[ -\frac{\omega_0^2}{\sigma_x^2} \left\{ x_i + \lambda \frac{dg(x_i)}{dx} \bigg|_{x=x_i} \right\} \Delta x_i \right] \right]. \tag{14} \]

Combining equations (14) and (9) gives
\[ p(x_{i+1}) = p(x_i) \exp \left[ -(\omega_0^2 / \sigma_x^2) \left\{ x_i + \lambda \frac{dg(x_i)}{dx} \bigg|_{x=x_i} \right\} \Delta x_i \right]. \tag{15} \]

For \( N \) displacement values, \( x_1, x_2, \ldots, x_N \), equation (15) can be expressed as a set of \( N-1 \) linear simultaneous algebraic equations,
\[ \frac{\sigma_x^2}{\Delta x_i} \ln \left[ \frac{p(x_i)}{p(x_{i+1})} \right] \left( \frac{1}{\omega_0^2} \right) \frac{dg(x_i)}{dx} \bigg|_{x=x_i} \lambda = x_i, \quad i = 1, 2, \ldots, N - 1. \tag{16} \]

Equations (16) are solved for \( 1/\omega_0^2 \) and \( \lambda \), by using the least squares fit technique. The variance, \( \sigma_x^2 \) and the probability function, \( p(x) \), are computed from the experimentally obtained displacement and velocity data \((x \text{ and } \dot{x})\), which are taken as zero mean Gaussian processes, and the nonlinear spring force provided by the rolling element bearings is taken to be cubic in nature; i.e., \( G(x) = x^3 \).

The laboratory rig for the experimental illustration of the technique is shown in Figures 2 and 3. The rig consists of a disc centrally mounted on a shaft supported into two identical ball bearings. The shaft is driven through a flexible coupling by motor and the vibration signals are picked up (after balancing rotor) in both the vertical and horizontal directions by accelerometers mounted on one side of the bearing housings. The signals from the accelerometers are digitized on a PC/AT after magnification.

Typical displacement and velocity signals, in the vertical direction, picked up by the accelerometer are given in Figures 4 and 5. The probability density function, \( p(x) \), of the displacement is shown in Figure 6. The bearing parameters estimated from equations (16) are as follows: vertical \( \omega_0^2 = 5.42 \times 10^7 \) (rad/s)^2, \( \lambda = -3.82 \times 10^6 \) (mm^-2); horizontal, \( \omega_0^2 = 3.21 \times 10^7 \) (rad/s)^2, \( \lambda = -3.86 \times 10^6 \) (mm^-2).

4. SIMULATION

The algorithm has been tested by Monte Carlo simulation. The experimentally obtained values of \( \omega_0^2 \) and \( \lambda \) were used in equation (1) to simulate the displacement and velocity responses, \( x \) and \( \dot{x} \), through a fourth order Runge–Kutta numerical technique, for a broadband excitation force, \( f(t) \), with zero mean and Gaussian probability distribution, as described in Figures 7 and 8. The simulated vertical displacement and velocity response is given in Figures 9 and 10. The probability distribution of the simulated vertical displacement
Figure 2. The laboratory rotor bearing rig.

Figure 3. The accelerometer mountings on the bearing cap.

Figure 4. The displacement signal in the vertical direction.
Figure 5. The velocity signal in the vertical direction.

Figure 6. The probability density distribution of the vertical displacement.

Figure 7. The simulated random force.
Figure 8. The probability density function of the simulated random force.

Figure 9. The simulated displacement signal.

Figure 10. The simulated velocity signal.
is shown in Figure 11. The simulated response was then fed into equation (16) to obtain the values of $\omega_i^2$ and $\lambda$. A similar exercise was carried out to obtain the parameters in the horizontal direction. These values are listed in Table 1.

The good agreement between the values of the bearing stiffness parameters, $\omega_i^2$ and $\lambda$, obtained by processing the experimental data and those from the Monte Carlo simulation, indicates the correctness of the experimental and algebraic exercises. It should be noted that the simulated values of the bearing stiffness parameters were obtained for an ideal white noise excitation, while the experimental ones were obtained by processing the actual response of the system, where the unknown excitation was idealized as white noise. It also needs to be pointed out that the value of the damping ratio, $\zeta$, is not required for the estimation procedure (equation (16)). Any convenient value of $\zeta$ can be employed in equation (1) for the purpose of simulation ($\zeta = 0.02$ has been assumed in the present simulation).

5. VALIDATION

The values of the bearing stiffness parameters $\omega_i^2$ and $\lambda$, obtained by the procedure outlined, can be compared with those obtained from the analytical formulations of Harris [4] and Ragulskis et al. [9], which are based on Hertzian contact theory.

The total elastic force at the points of contact of the $i$th ball with the inner and outer races is expressed as [9]

$$F_i = K_n(g + x \cos \eta_i + y \sin \eta_i)^{3/2}$$

and its projection along the line of action of the applied force is

$$F_i = K_n(g + x \cos \eta_i + y \sin \eta_i)^{3/2} \cos \eta_i,$$

where $g$ is the radial pre-load or pre-clearance between the ball and the races, and $x$ and $y$ are the displacements of the moving ring in the direction of the radial load and perpendicular to the direction of the radial load respectively. $\eta_i$ is the angle between the lines of action of the radial load (direction of displacement of the moving ring) and the radius passing through the center of the $i$th ball. $K_n$ is a coefficient of proportionality depending on the geometric and material properties of the bearing. The specifications of the test bearing are as follows: ball bearing type SKF6200; number of balls 6, ball diameter 6 mm, bore diameter 10 mm, outer diameter 30 mm, pitch diameter 20 mm, inner groove radius 3.09 mm, outer groove radius 3.09 mm, allowable pre-load 0–2 $\mu$m, rotor mass per bearing 0.41 kg. The value of $K_n$ for the test bearing with these specifications is estimated by the method suggested by Harris [4] as $2.82 \times 10^5$ N/mm$^{1.5}$.

The total elastic force in the direction of the applied force is

$$F = \sum_{i=1}^{n} F_i,$$

where $n$ is the total number of balls in the bearing.

By using the condition of zero elastic force in the direction perpendicular to the elastic load, the deformation, $y$, perpendicular to the radial force line, is expressed as

$$y = \sum_{i=1}^{n} \left[ g + x \cos (\eta_i) \right]^{1/2} \sin (\eta_i) / \left[ \sum_{i=1}^{n} \left[ g + x \cos (\eta_i) \right]^{1/2} \sin^2 (\eta_i) \right].$$
Figure 11. The probability density distribution of the simulated vertical displacement.

**Table 1**

*Experimental and simulated bearing stiffness parameters*

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Experimental</th>
<th>Simulated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega^2$ (rad/s)$^2$</td>
<td>$\lambda$ (mm$^{-1}$)</td>
</tr>
<tr>
<td>Vertical</td>
<td>$5.42 \times 10^7$</td>
<td>$-3.82 \times 10^6$</td>
</tr>
<tr>
<td>Horizontal</td>
<td>$3.21 \times 10^7$</td>
<td>$-3.86 \times 10^6$</td>
</tr>
</tbody>
</table>

Figure 12. A comparison of rolling element bearing stiffness results. 1, 2: Present study (in the vertical and horizontal directions respectively). 3–7: Harris [4] and Ragulskis *et al.* [9], with pre-load 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm, respectively (in the radial direction).
Table 2

<table>
<thead>
<tr>
<th>Pre-load (mm)</th>
<th>Theoretical stiffness (radial) (N/mm)</th>
<th>Experimental stiffness (N/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0·0002</td>
<td>1·20 × 10^4–4·01 × 10^8x^2</td>
<td>1·32 × 10^4–5·08 × 10^8x^2 (horizontal)</td>
</tr>
<tr>
<td>0·0003</td>
<td>1·47 × 10^4–2·18 × 10^8x^2</td>
<td></td>
</tr>
<tr>
<td>0·0004</td>
<td>1·69 × 10^4–1·42 × 10^8x^2</td>
<td></td>
</tr>
<tr>
<td>0·0005</td>
<td>1·89 × 10^4–1·02 × 10^8x^2</td>
<td></td>
</tr>
<tr>
<td>0·0006</td>
<td>2·08 × 10^4–6·09 × 10^8x^2</td>
<td>2·23 × 10^4–8·50 × 10^8x^2 (vertical)</td>
</tr>
</tbody>
</table>

Equations (18) and (20) are used in equation (19) and the bearing stiffness is determined as a function of the deformation \( x \) as

\[ k(x) = \frac{\partial F}{\partial x}. \]  

It can be seen that the bearing stiffness is critically dependent on the pre-loading, \( g \), of the balls. While the manufacturer, may, at times, provide the pre-load range, the exact value of the pre-loading of the bearing balls in the shaft–casing assembly, especially during operations which have involved wear and tear, would be difficult to determine. The stiffness of the test bearing is plotted in Figure 12 as a function of the radial deformation, \( x \), for various allowable pre-load values, \( g \). The bearing stiffness obtained experimentally, by using the procedure developed, also shown in Figure 12, shows a good resemblance to theoretically possible values. It is also to be noted that the theoretical stiffness calculations are based on formulations which analyze the bearing in isolation of the shaft. The comparison between the experimental and theoretically possible stiffnesses is also listed in Table 2. The expressions for the theoretical stiffnesses in Table 2 have been obtained by curve fitting the stiffness values obtained from equation (21), through a quadratic in \( x \).

6. CONCLUSIONS

The procedure developed for estimation of non-linear stiffness parameters of rolling element bearings by processing the random response of rotor–bearing systems shows good agreement with the analytical formulations developed for isolated bearings. It has a distinct advantage over other available techniques, for it does not involve measurement of the excitation forces and works directly on the random response signals which can be conveniently picked up at the bearing caps.

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REFERENCES


**APPENDIX: NOMENCLATURE**

\( c, c_1 \) normalization constants
\( f(t) \) excitation force
\( F \) elastic force
\( g(x), G(x) \) non-linear functions
\( k_1, k_2 \) stiffness parameters
\( K_n \) coefficient of proportionality
\( m \) rotor mass per bearing
\( N \) number of sample points
\( p(x), p(\dot{x}) \) probability density functions of displacement and velocity, respectively
\( x \sim y \sim z \) rectangular co-ordinate system
\( x, \dot{x}, \ddot{x} \) displacement, velocity and acceleration, respectively
\( S_0 \) spectral density
\( t \) time
\( \Delta t \) time increment
\( \Delta x \) displacement increment
\( \lambda \) non-linear stiffness parameter
\( \omega \) linear natural frequency
\( \sigma_x^2, \sigma_x^4 \) variance of displacement and velocity, respectively
\( \zeta \) damping ratio