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EQUATIONS OF MOTION OF A BLADE ROTATING WITH VARIABLE ANGULAR VELOCITY

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The equations of motion of a blade mounted on a disk rotating with variable angular velocity are derived. The acceleration of the disk is taken as constant and the calculus of variations is employed to obtain the equation governing the ensuing free vibrations. Coriolis forces are included in the derivation and the higher order terms due to shear deflection and rotary inertia are also considered.

1. INTRODUCTION

Equations of motion of cantilever blades mounted on a disk rotating at a constant speed have been considered by various authors, particularly by Carnegie [1] and Rao and Rao [2]. The effect of Coriolis forces, which give rise to several non-linear terms, was considered by Carnegie [3]. Rao and Carnegie [4] have shown that the influence of Coriolis forces is of importance only for long and slender blades and high speeds of rotation.

The importance of non-linear terms arising due to Coriolis forces and other higher order effects on the stability of helicopter rotor blades has been considered by several researchers; e.g., Ormiston and Hodges [5] and Subrahmanyam and Kaza [6]. However, all these studies pertain to blades rotating at constant angular velocity.

Irretier [7] and Rao and Vyas [8] have obtained the forced vibration response of turbine blades during operations such as step-up and -down, involving variation in angular velocity with time. They have shown that the acceleration values have a significant influence on the blade response. Rao and Vyas [9] have shown that with proper acceleration rates of a rotor the life of a blade can be significantly improved.

In the turbine blade studies mentioned above [7–9], higher order effects due to Coriolis forces, rotary inertia and shear deformation have been neglected. Thus, the resulting equations are linear in nature. These higher order effects can be significant for the response of long rotor blades. Furthermore, the influence of acceleration of the rotor blade may also be significant, as in the case of turbomachine blades. In this paper, the governing equations of motion of cantilever blades mounted on a disk rotating with constant angular acceleration are derived by considering the Coriolis forces and higher order terms of shear deflection and rotary inertia.

2. KINETIC ENERGY

Consider a blade of length l and uniform cross-section with area A, mounted on a rotating disk of radius R. The x axis is located at the root of the blade, parallel to the axis

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of rotation, and the y axis in the plane of the disk. The z axis is along the length of the blade passing through the centroids of all the cross-sections. For simplicity the blade is considered to be straight, without any pretwist and asymmetry: that is, the blade is executing pure bending oscillations in the y–z plane. $\bar{M}$ and $\bar{N}$ are axes fixed in space. At the instant shown, the blade is at an angle Γ, as shown in Figure 1, given by $\Gamma=\omega_0 t+\frac{1}{2}\alpha t^2$, where ω_0 is the initial angular velocity of the disk and α is the constant angular acceleration (a list of nomenclature is given in Appendix C).

With reference to Figure 1, the displacements of an element at a distance z from the blade root, in the $\bar{M}$ and $\bar{N}$ directions with respect to the disk center, are

$$m=(R+z-\Delta)\sin\Gamma+y\cos\Gamma, \qquad n=(R+z-\Delta)\cos\Gamma-y\sin\Gamma. \tag{1}$$

The corresponding velocities are

$$\begin{aligned}\dot{m}&=\{\dot{y}+\dot{\Gamma}(R+z-\Delta)\}\cos\Gamma-(\dot{\Delta}+\dot{\Gamma}y)\sin\Gamma,\\ \dot{n}&=\{-\dot{y}+\dot{\Gamma}(R+z-\Delta)\}\sin\Gamma-(\dot{\Delta}+\dot{\Gamma}y)\cos\Gamma.\end{aligned} \tag{2}$$

The kinetic energy due to translation of the blade element of length dz is $\mathrm{d}T_t=\frac{1}{2}\rho A\,\mathrm{d}z(\dot{m}^2+\dot{n}^2)$ and hence

$$T_t=\int_0^l \tfrac{1}{2}\rho A(\dot{m}^2+\dot{n}^2)\,\mathrm{d}z. \tag{3}$$

The kinetic energy due to rotation of the blade element is $\mathrm{d}T_r=\frac{1}{2}\rho I_{xx}\,\mathrm{d}z(\dot{\Gamma}+\dot{\phi}_b)^2$, and hence

$$T_r=\int_0^l \tfrac{1}{2}\rho I_{xx}(\dot{\Gamma}+\dot{\phi}_b)^2\,\mathrm{d}z. \tag{4}$$

Thus, the total kinetic energy is

$$T=T_t+T_r=\int_0^l \{\tfrac{1}{2}\rho A(\dot{m}^2+\dot{n}^2)+\tfrac{1}{2}\rho I_{xx}(\dot{\Gamma}+\dot{\phi}_b)^2\}\,\mathrm{d}z. \tag{5}$$

With the help of equation (2) and $\Gamma=\omega_0 t+\frac{1}{2}\alpha t^2$ and $\dot{\Gamma}=\omega_0+\alpha t$, the kinetic energy expression (5) can be expressed as

$$\begin{aligned}T=&\int_0^l \tfrac{1}{2}\rho A(\dot{y}^2+\dot{\Delta}^2)\,\mathrm{d}z+\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)^2\{(R+z-\Delta)^2+y^2\}\,\mathrm{d}z\\ &+\int_0^l \rho A(\omega_0+\alpha t)\{\dot{y}(R+z-\Delta)+y\dot{\Delta}\}\,\mathrm{d}z+\int_0^l \tfrac{1}{2}\rho I_{xx}\{(\omega_0+\alpha t)+\dot{\phi}_b\}^2\,\mathrm{d}z.\end{aligned} \tag{6}$$

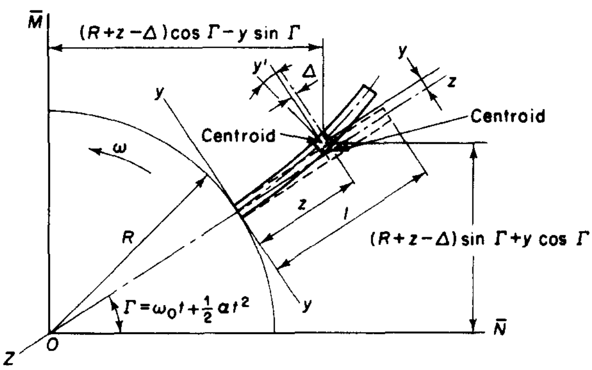

Figure 1. Displacement of a blade element mounted on a disk rotating at constant angular acceleration.

In equation (6), the first term on the right side accounts for blade inertia, the second term accounts for the effect of centrifugal force, the third term accounts for Coriolis force, while the fourth refers to rotary inertia.

The inward displacement of an inextensional blade is given by $\Delta = 1 - \int_0^z \cos\phi_b(\xi)\,d\xi$. Noting that $y' = \sin\phi_b$, one has $\Delta = 1 - \int_0^z \sqrt{1-[y'(\xi)]^2}\,d\xi$. Expanding the square root term and truncating the series yields

$$\Delta = \tfrac{1}{2}\int_0^z y'^2\,dz \qquad \text{and} \qquad \Delta = \tfrac{1}{2}(\partial/\partial t)\int_0^z y'^2\,dz. \tag{7}$$

Hence, equation (6) can be written as

$$T = \int_0^l \tfrac{1}{2}\rho A\left[\dot{y}^2 + \left\{\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}^2 + (\omega_0 + \alpha t)^2\left\{\left(R + z - \tfrac{1}{2}\int_0^z y'^2\,dz\right)^2 + y^2\right\}\right.$$
$$\left. + 2(\omega_0 + \alpha t)\left\{\dot{y}\left(R + z - \tfrac{1}{2}\int_0^z y'^2\,dz\right) + \tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}\right]dz$$
$$+ \int_0^l \tfrac{1}{2}\rho I_{xx}\{(\omega_0 + \alpha t) + \dot{\phi}_b\}^2\,dz. \tag{8}$$

3. POTENTIAL ENERGY

The potential energy expression, with account taken of the shear deformation, can be shown to be

$$V = \int_0^l \{(AG/k)(y' - \phi_b)^2 + \tfrac{1}{2}EI_{xx}\phi_b'^2\}\,dz. \tag{9}$$

4. APPLICATION OF HAMILTON'S PRINCIPLE

Hamilton's principle states that if $L = T - V$, then $\int_{t_1}^{t_2} L\,dt$, taken between any arbitrary intervals of time (t_1, t_2), is stationary for a dynamic trajectory. Therefore,

$$\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A\left[\dot{y}^2 + \left\{\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}^2\right]dz\,dt$$
$$+ \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0 + \alpha t)^2\left\{\left(R + z - \tfrac{1}{2}\int_0^z y'^2\,dz\right)^2 + y^2\right\}dz\,dt$$
$$+ \int_{t_1}^{t_2}\int_0^l \rho A(\omega_0 + \alpha t)\left\{\dot{y}\left(R + z - \tfrac{1}{2}\int_0^z y'^2\,dz\right) + \tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}dz\,dt$$
$$+ \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho I_{xx}\{(\omega_0 + \alpha t) + \dot{\phi}_b\}^2\,dz\,dt$$
$$- \int_{t_1}^{t_2}\int_0^l \left\{\frac{AG}{k}(y' - \phi_b)^2 + \tfrac{1}{2}EI_{xx}\phi_b'^2\right\}dz\,dt. \tag{10}$$

The subject of the calculus of variations (see, e.g., the book by Fox [10]) has been fully exploited by many researchers in the field of vibration problems, as it offers a very powerful tool in deriving the equations of motion and the corresponding boundary conditions. Dym and Shames [11] have given a general treatment of this subject as applied to the problems of solid mechanics. It can be observed here that it is most expedient to use the calculus of variations in deriving the equation of motion for the blade under consideration, by applying Hamilton's principle to equation (10). With the help of the transformation (see reference [3])

$$\frac{\partial}{\partial t}\int_0^z y'^2\,dz = \frac{\partial}{\partial t}y'^2\int_z^l dz,$$

the first term on the right side of equation (10) can be written as

$$\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A\left[\dot{y}^2+\left\{\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}^2\right]dz\,dt = \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A\left[\dot{y}^2+\left\{\frac{1}{2}\frac{\partial}{\partial t}y'^2\int_z^l dz\right\}^2\right]dz\,dt. \tag{11}$$

This can be re-expressed as

$$\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A\left[\dot{y}^2+\left\{\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}^2\right]dz\,dt = \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A[\dot{y}^2+R_1^2(y'\dot{y}')^2]\,dz\,dt, \tag{12}$$

where $R_1 = l - z$. With higher powers of Δ neglected, the second term on the right side of equation (10) can be written as

$$+\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)^2\left\{\left(R+z-\tfrac{1}{2}\int_0^z y'^2\,dz\right)^2+y^2\right\}dz\,dt$$

$$=\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)^2\left\{y^2+(R+z)^2-(R+z)\int_0^z y'^2\,dz\right\}dz\,dt. \tag{13}$$

The third term on the right side of equation (10) is

$$\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{\dot{y}\left(R+z-\tfrac{1}{2}\int_0^z y'^2\,dz\right)+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}dz\,dt$$

$$=\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\dot{y}(R+z)\,dz\,dt$$

$$+\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{-\tfrac{1}{2}\dot{y}\int_0^z y'^2\,dz+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}dz\,dt. \tag{14}$$

From equations (10), (12), (13) and (14), one has

$$\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A[\dot{y}^2+R_1^2(y'\dot{y}')^2]\,dz\,dt$$

$$+\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)^2\left\{y^2+(R+z)^2-(R+z)\int_0^z y'^2\,dz\right\}dz\,dt$$

$$+\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\dot{y}(R+z)\,dz\,dt$$

$$+\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{-\tfrac{1}{2}\dot{y}\int_0^z y'^2\,dz+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}dz\,dt$$

$$+\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho I_{xx}\{(\omega_0+\alpha t)+\dot{\phi}_b\}^2\,dz\,dt$$

$$-\int_{t_1}^{t_2}\int_0^l\left\{\frac{AG}{k}(y'-\phi_b)^2+\tfrac{1}{2}EI_{xx}\phi_b'^2\right\}dz\,dt. \quad (15)$$

Carrying out the extremization on the first term on the right side of equation (15) yields,

$$\delta\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A[\dot{y}^2+R_1^2(y'\dot{y}')^2]\,dz\,dt=\int_{t_1}^{t_2}\int_0^l \rho A[\dot{y}\delta\dot{y}+R_1^2 y'\dot{y}'\{y'\delta\dot{y}'+\dot{y}'\delta y'\}]\,dz\,dt.$$

Integrating by parts, neglecting higher order terms and noting that variations are zero at $t=t_1$ and $t=t_2$, one obtains

$$\delta\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A[\dot{y}^2+R_1^2(y'\dot{y}')^2]\,dz\,dt=-\int_{t_1}^{t_2}\int_0^l \rho A\ddot{y}\delta y\,dz\,dt. \quad (16)$$

Similarly, the second term on the right side of equation (15) yields

$$\delta\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)^2\left\{y^2+(R+z)^2-(R+z)\int_0^z y'^2\,dz\right\}dz\,dt$$

$$=\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)^2\{y+R_2y''-(R+z)y'\}\delta y\,dz\,dt, \quad (17)$$

where

$$R_2=\int_z^l (R+z)\,dz=R(l-z)+\tfrac{1}{2}(l^2-z^2).$$

The third term on the right side of equation (15) yields

$$\delta\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\dot{y}(R+z)\,dz\,dt=-\int_{t_1}^{t_2}\int_0^l \rho A\alpha(R+z)\delta y\,dz\,dt. \quad (18)$$

For the variation of the fourth term in equation (15), one has (see Appendix A)

$$\delta\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{-\tfrac{1}{2}\dot{y}\int_0^z y'^2\,dz+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}dz\,dt$$

$$=-\int_{t_1}^{t_2}\int_0^l \rho A\Bigg[\{\alpha y+(\omega_0+\alpha t)\dot{y}\}y'-y''\int_z^l\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\,dz$$

$$+(\omega_0+\alpha t)\left\{-\int_0^z y'\dot{y}'\,dz+\dot{y}y'-y''\int_z^l \dot{y}\,dz-\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}$$

$$-\frac{\alpha}{2}\int_0^z y'^2\,dz\Bigg]\delta y\,dz\,dt. \quad (19)$$

The variation operation on the fifth and sixth terms in equation (15) gives

$$\delta\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho I_{xx}\{(\omega_0+\alpha t)+\dot{\phi}_b\}^2\,dz\,dt=-\int_{t_1}^{t_2}\int_0^l \rho I_{xx}(\alpha+\ddot{\phi}_b)\delta\phi_b\,dz\,dt, \tag{20}$$

$$\delta\int_{t_1}^{t_2}\int_0^l \left\{\frac{AG}{k}(y'-\phi_b)^2+\tfrac{1}{2}EI_{xx}\phi_b'^2\right\}dz\,dt$$

$$=\int_{t_1}^{t_2}\left\{\left[-\frac{AG}{k}(y'-\phi_b)\delta y\right]_0^l-[-EI_{xx}\phi_b'\delta\phi_b]_0^l+\int_0^l\frac{AG}{k}(y'-\phi_b)\delta\phi_b\,dz\right.$$

$$\left.+\int_0^l\frac{AG}{k}(y''-\phi_b')\delta y\,dz+\int_0^l EI_{xx}\phi_b''\delta\phi_b\,dz\right\}dt. \tag{21}$$

5. EQUATIONS OF MOTION

Using Hamilton's principle, one has, from equations (16)–(21), the following equations of motion and corresponding boundary conditions:

$$\rho A\ddot{y}-\rho A(\omega_0+\alpha t)^2\{y+R_2y''-(R+z)y'\}$$

$$+2\rho A(\omega_0+\alpha t)\left\{\dot{y}y'-y''\int_z^l\dot{y}\,dz-\int_0^z y'\dot{y}'\,dz\right\}$$

$$+\rho A\alpha\left\{(R+z)+yy'-\tfrac{1}{2}\int_0^z y'^2\,dz-y''\int_z^l y\,dz\right\}-\frac{AG}{k}(y''-\phi_b')=0, \tag{22}$$

$$EI_{xx}\phi_b''+(AG/k)(y'-\phi_b)-\rho I_{xx}(\alpha+\ddot{\phi}_b)=0, \tag{23}$$

$$y=\phi_b=0 \quad \text{at } z=0, \qquad (AG/k)(y'-\phi_b)=0 \quad \text{and} \quad EI_{xx}\phi_b'=0 \quad \text{at } z=l. \tag{24}$$

With the help of the equations given in Appendix B, the equations of motion (22) and (23) can be combined to give

$$EI_{xx}y''''+\rho A\left[\ddot{y}-(\omega_0+\alpha t)^2\{y+R_2y''-(R+z)y'\}\right.$$

$$+2(\omega_0+\alpha t)\left\{\dot{y}y'-y''\int_z^l\dot{y}\,dz-\int_0^z y'\dot{y}'\,dz\right\}$$

$$\left.+\alpha\left\{(R+z)+yy'-\tfrac{1}{2}\int_0^z y'^2\,dz-y''\int_z^l y\,dz\right\}\right]-\left(\rho I_{xx}+\frac{k\rho EI_{xx}}{G}\right)\ddot{y}''+I_{xx}k\rho^2\ddddot{y}=0. \tag{25}$$

Equation (25) can be rewritten as

$$EI_{xx}y''''+\rho A\left[\ddot{y}-(\omega_0+\alpha t)^2\{y+R_2y''-(R+z)y'\}\right.$$

$$\left.+2(\omega_0+\alpha t)\left\{\dot{y}y'-y''\int_z^l\dot{y}\,dz-\int_0^z y'\dot{y}'\,dz\right\}+\alpha\left\{yy'-\tfrac{1}{2}\int_0^z y'^2\,dz-y''\int_z^l y\,dz\right\}\right]$$

$$-\left(\rho I_{xx}+\frac{k\rho EI_{xx}}{G}\right)\ddot{y}''+I_{xx}k\rho^2\ddddot{y}=-\rho A\alpha(R+z). \tag{26}$$

This form of the equation of motion reveals the presence of a pseudo-static force term $[-\rho A\alpha(R+z)]$ arising due to the acceleration conditions. The origin of the pseudo-static force term can be readily traced back to the Coriolis effects.

For a disk rotating with constant angular speed ($\alpha=0$), and with shear deformation ignored ($k=0$), equation (26) reduces to

$$EI_{xx}y'''' + \rho A[\ddot{y} - \omega_0^2\{y + R_2 y'' - (R+z)y'\} + 2\omega_0\left\{\dot{y}y' - y''\int_z^l \dot{y}\,\mathrm{d}z - \int_0^z y'\dot{y}'\,\mathrm{d}z\right\} - \rho I_{xx}\ddot{y}'' = 0.$$

This equation is the same as that derived by Carnegie [3] for the case of constant speed of rotation.

6. COMMENTS

The governing equations of motion of a cantilever blade mounted on a disk rotating with variable angular velocity have been, derived with account taken of Coriolis forces and higher order terms due to shear deformation and rotary inertia. The equation is a non-linear integro-partial differential equation. A solution of this equation can be obtained by using a Ritz averaging principle to ascertain the importance of Coriolis forces and the resultant pseudo-static force term under transient conditions of operation. Such an analysis may be carried out on long flexible helicopter rotor blades, to investigate the dynamic behavior under Coriolis forces and transient conditions of operation.

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APPENDIX A

Let

$$I=\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{-\tfrac{1}{2}\dot{y}\int_0^z y'^2\,\mathrm{d}z+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^z y'^2\,\mathrm{d}z\right\}\mathrm{d}z\,\mathrm{d}t. \tag{15a}$$

Upon replacing one of the z variables by θ, this becomes

$$I=\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{-\tfrac{1}{2}\dot{y}\int_0^\theta y'^2\,\mathrm{d}\theta+\tfrac{1}{2}y\frac{\partial}{\partial t}\int_0^\theta y'^2\,\mathrm{d}\theta\right\}\mathrm{d}z\,\mathrm{d}t. \tag{15b}$$

For the variation of $\delta(y)$ of y it follows that

$$\begin{aligned}I+\delta I=\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)&\left\{-(\dot{y}+\delta\dot{y})\int_0^\theta (y'+\delta y')^2\,\mathrm{d}\theta\right.\\ &\left.+(y+\delta y)\frac{\partial}{\partial t}\int_0^\theta (y'+\delta y')^2\,\mathrm{d}\theta\right\}\mathrm{d}z\,\mathrm{d}t.\end{aligned} \tag{A1}$$

From these two equations (15b) and (A1), one can write δI neglecting the second order terms, as

$$\begin{aligned}\delta I=&\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\int_0^\theta (\dot{y}'\delta y'+y'\delta\dot{y}')\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\delta y\int_0^\theta y'\dot{y}'\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t\\ &-\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\dot{y}\int_0^\theta y'\delta y'\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t\\ &-\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)\delta\dot{y}\int_0^\theta y'^2\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t.\end{aligned} \tag{A2}$$

Now the first term on the right side of equation (A2) is

$$\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\int_0^\theta (\dot{y}'\delta y'+y'\delta\dot{y}')\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t=\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\frac{\partial}{\partial t}\int_0^\theta y'\delta y'\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t.$$

Integrating by parts and noting that the variations are zero at $t=t_1$ and $t=t_2$, one obtains

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\int_0^\theta (\dot{y}'\delta y'+y'\delta\dot{y}')\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t\\ &\qquad=-\int_{t_1}^{t_2}\int_0^l \rho A\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\left\{[y'\delta y]_0^\theta-\int_0^\theta y''\delta y\,\mathrm{d}\theta\right\}\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

Replacing θ by z and since $\delta y=0$ at $z=0$, one has

$$\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\int_0^\theta (\dot{y}'\delta y'+y'\delta\dot{y}')\,d\theta\,dz\,dt$$

$$=-\int_{t_1}^{t_2}\int_0^l \rho A\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\left\{y'\delta y-\int_0^z y''\delta y\,dz\right\}dz\,dt.$$

Effecting the transformation (see reference [3])

$$\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\int_0^z y''\delta y\,dz=y''\delta y\int_z^l\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\,dz,$$

one can write

$$\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)y\int_0^\theta (\dot{y}'\delta y'+y'\delta\dot{y}')\,d\theta\,dz\,dt$$

$$=-\int_{t_1}^{t_2}\int_0^l \rho A\left[\{\alpha y+(\omega_0+\alpha t)\dot{y}\}y'\delta y-y''\delta y\int_z^l\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\,dz\right]dz\,dt. \qquad \text{(A3)}$$

Similarly, the third term on the right side of equation (A2) is

$$-\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\dot{y}\int_0^\theta y'\delta y'\,d\theta\,dz\,dt=-\int_{t_1}^{t_2}\int_0^l \rho A(\omega_0+\alpha t)\left\{\dot{y}y'-y''\int_z^l \dot{y}\,dz\right\}\delta y\,dz\,dt \qquad \text{(A4)}$$

and the fourth term is

$$-\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A(\omega_0+\alpha t)\delta\dot{y}\int_0^\theta y'^2\,d\theta\,dz\,dt$$

$$=\int_{t_1}^{t_2}\int_0^l \tfrac{1}{2}\rho A\left\{\alpha\int_0^z y'^2\,dz+(\omega_0+\alpha t)\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}\delta y\,dz\,dt. \qquad \text{(A5)}$$

Using equations (A3)–(A5) in equation (A2) then yields

$$\delta I=-\int_{t_1}^{t_2}\int_0^l \rho A\left[\{\alpha y+(\omega_0+\alpha t)\dot{y}\}y'-y''\int_z^l\{\alpha y+(\omega_0+\alpha t)\dot{y}\}\,dz+(\omega_0+\alpha t)\right.$$

$$\left.\times\left\{\dot{y}y'-\int_0^z y'\dot{y}'\,dz-y''\int_z^l\dot{y}\,dz-\frac{1}{2}\frac{\partial}{\partial t}\int_0^z y'^2\,dz\right\}-\frac{\alpha}{2}\int_0^z y'^2\,dz\right]\delta y\,dz\,dt. \qquad \text{(A6)}$$

APPENDIX B

The total slope of the beam is given by

$$y'=\phi_s+\phi_b, \qquad \text{(B1)}$$

The shearing force on the beam is given as

$$F=AG\phi_s/k, \qquad \text{(B2)}$$

where k is a factor allowing for non-uniform shear stress distribution over the beam cross-section.

The bending moment M is given by

$$M = EI_{xx}\phi_b'. \tag{B3}$$

Equating the forces and moments on the beam elements results in the following equations:

$$F' = \rho A\ddot{y}, \qquad M' = I_{xx}\rho\ddot{\phi}_b - F. \tag{B4}$$

APPENDIX 3: NOMENCLATURE

A	area of cross-section
E	Young's modulus
F	shear force
G	shear modulus
I_{xx}	second moment of area about xx axis
k	shear distribution factor
l	length
L	Lagrangian function
M	bending moment
m, n	linear displacements in $\bar{M}$ and $\bar{N}$ directions
R	disk radius
T	kinetic energy
t	time
V	potential energy
xx, yy	co-ordinate axes
y	displacement in y direction
zz	longitudinal axis
z	co-ordinate measured along the blade
α	constant angular acceleration of the disk
ω	instantaneous angular velocity of the disk
ω_0	initial angular velocity of the disk
Δ	inward displacement of the blade element
ρ	mass density
ϕ_b	slope due to bending
ϕ_s	slope due to shear
Γ	angular displacement of the disk

A dot denotes differentiation with time: thus, $\dot{y}$.
A prime denotes differentiation with z: thus, y'.