Rumor Dynamics and Inoculation of Nodes in Complex Networks

A *Thesis Submitted*

in Partial Fulfilment of the Requirements

for the Degree of

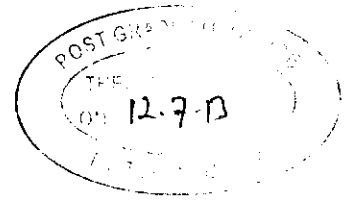**DOCTOR OF PHILOSOPHY**

by

ANURAG SINGH

to the

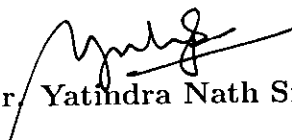**DEPARTMENT OF ELECTRICAL ENGINEERING**

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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CERTIFICATE

It is certified that the work contained in the thesis entitled " **Rumor Dynamics and Inoculation of Nodes in Complex Networks**" being submitted by **Mr. Anurag Singh** has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted elsewhere for the award of any degree or diploma.

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Synopsis

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Online social network has become one of the most important medium to propagate the information among communities. The topologies of these have been found as complex network arrangements. Thus, the study of information diffusion in the complex networks is of great importance. If any information circulates without officially publicized confirmation, it is called rumor. Most of the researchers have focused on the rumor dynamics, primarily deriving from earlier epidemic studies. The susceptible-infected-removed (SIR) model and its variants for rumor spreading were introduced many years ago by Daley-Kendal and Maki-Thomsan [1, 2] without considering the topology of the underlying networks. In these models, population of nodes are divided into three groups: ignorants, spreaders and stifler. The models assume that the rumor propagates through pairwise contacts of the spreaders with ignorants. By considering the topology of network, rumor model on small world network [3, 4, 5] and scale free networks [6] have been defined. In today's information oriented society, a mechanism to

suppress harmful rumors in social web has become very important. In order to control the spread of rumors, inoculating the nodes is an option. Although, random inoculation strategy works very well in homogeneous random networks, but this strategy is not effective in preventing a rumor in scale free networks [7]. Hence, the new inoculation strategies need to be developed which are able to recover from the rumor spreading. One of the efficient approach is to inoculate the high degrees nodes, or more specifically, to inoculate those nodes (hereafter termed as hubs or hub nodes) which have degrees higher than a preset cut-off value k_c . Such a strategy is known as targeted inoculation [8, 9, 10, 11, 12, 13]. Targeted inoculation is successful in arresting the rumor spread in scale free networks [7]. Random inoculation usually requires inoculation of large number of nodes for being effective. If nodes with higher connectivity are targeted for inoculation, the same effectiveness can be achieved with smaller number of inoculated nodes. But it requires the knowledge of nodes which have higher connectivity [14].

The classical rumor spreading model based on SIR model was proposed by Nekovee *et al.* [3] for small world (homogeneous) and scale free (heterogeneous) networks. When a spreader meets with an ignorant node, ignorant node becomes a spreader with rate λ. When a spreader meets with a spreader or stifler, the spreader becomes a stifler who accepts the rumor with rate σ. A spreader can become stifler spontaneously with rate δ. In further improvement of this model, the real life scenarios can be added. When a person passes rumor to his friends then, some of them accept the rumor and spread it, some of them accept it but do not spread it, while some of them reject the rumor. Therefore in the thesis, two new compartments for the population of nodes have been added in the classical model: nodes who accept the rumor and become stifler, nodes who reject the rumor with rate ρ and become stifler. Here, $1/\rho$ has been considered as acceptability factor. Therefore, acceptability factor is considered as a new parameter

in the model and its impact has been investigated on the rumor spreading. It has been found that it helps to control the rumor spreading when simultaneously applies with the inoculation of nodes. The new models have been considered for small world network which are homogeneous [15], as well as scale free network which are heterogeneous [16]. In both the scenarios, networks have been considered uncorrelated (or no degree-degree correlation).

The model has been further modified, to incorporate the links with varying tie-strengths. The tie strength models the weights for the edges between the pair of nodes for rumor spreading with the varying spreading and stifling rates [17]. We have further considered the rumor spreading that will vary nonlinearly with the nodal degree of the neighbors of an informed node. In the classical model, a node contacts all of its neighbors in a single time step for rumor spreading. This scenario can not be always true for social networks. A person can spread the rumor only to some of his friends in a single time step. Therefore, number of contacted neighbors in a single time step by a node for rumor spreading may be nonlinear e.g. k^{α}, with exponent α where, $0 \leq \alpha \leq 1$. Further, in the classical model, spreading rate λ and stifling rate σ remain constant throughout the population. But λ and σ may vary according to the tie strength between the nodes. It can be understood by an example where a person has higher probability to spread the rumor to his more close ignorant friends in comparison with the less close ignorant friends. The tie strength between two nodes is $(k_i k_j)^{\beta}$, where k_i and k_j are degrees of node i and j, and β is tie strength exponent. Considering α and β exponents, a new rumor spreading model can be defined. Here, the new models has been considered for uncorrelated scale free networks. The existing rumor spreading model has also been used for inoculation for correlated scale free network to study the rumor dynamics [18].

In order to control the rumor spread, the inoculation of nodes is an effective tech-

nique. We have looked into the random, targeted, and neighbor inoculation mechanisms for their effectiveness in all proposed rumor spreading models. We have further looked into the selection of nodes for inoculation on the basis of structural centrality conditions [19]. It has been observed that the rumor spread can be effectively controlled by inoculation of nodes if chosen judiciously.

The thesis has been organized in the following eight chapters.

Chapter 1, defines the basic theory behind the complex networks. It describes the metrics used in complex networks, e.g. degree distribution, excess degree etc.. This chapter also differentiates the correlated and uncorrelated complex networks. Here, homogeneous and heterogeneous networks have been defined.The construction methods for both of them are also presented. The given theory of complex network will help in the design of underlying topology to study the rumor dynamics.

In chapter 2, we have discussed about the classic rumor spreading model and also discussed the different inoculation strategies.

In chapter 3, we have proposed a new rumor spreading model using the acceptability factor. The dynamics of the rumor spreading has been studied using this model for small world network (homogeneous networks). Inoculation of the nodes has been considered to stop rumor spreading. Random and targeted inoculation techniques have been applied on the proposed rumor spreading model for small world network. Rumor threshold has been found for different fraction of inoculation in small world network. Finally, the role of acceptability factor has been studied to control the rumor spreading in small world network. Further, acceptability factor in rumor spreading has been investigated for scale free (which are heterogeneous) networks. As random inoculation does not work satisfactorily, we have investigated neighbor inoculation for arresting the

rumor spread for the networks with unknown topologies.

Chapter 4, describes the rumor dynamics and inoculation of nodes for correlated networks. Different scale free networks have been generated with given correlation coefficients. Effects of correlation coefficients in controlling of rumor spread in the scale free networks have been studied. The threshold variation with the change of correlation coefficients in the scale free networks has also been discussed.

In chapter 5, we have proposed rumor spreading model for more realistic cases of real world complex networks. Let us assume that in a social network, some of the friends are closer than others. Therefore, strong tie strengths are there between them. A node is expected to spread a rumor with high probability to a neighbor connected strongly than to a neighbor which is loosely connected. The strength of connection is modeled by the tie strength. Hence, rumor spreading rate may vary with the tie strength of the edges in the social networks. In this chapter, variable spreading rates and stifling rates have been included. In more realistic scenario, it depends on a person that he may spread the information only to some of his friends and not to all at a time. Thus, the rumors will spread nonlinearly to the neighboring nodes with increase in the nodal degree. The dynamics of rumor spreading has been studied as a function of time for different spreading rates. The effects of the tie strength exponent β and nonlinear rumor spreading exponent α on rumor threshold have been studied. Some nonzero rumor thresholds have been found with some specific conditions on α and β values. With certain conditions, constant rumor threshold has been found, which is independent of the network size. To control the rumor spreading, random and targeted inoculations have been applied on proposed rumor spreading model and effect of α and β parameters has been studied.

In chapter 6, we have discussed a new technique to find influential node based on structural centrality.

In the last, chapter 7 of the thesis presents the conclusions of the work done. Some of the possible future research directions are also suggested.

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I feel immense pleasure in expressing my profound sense of gratitude to my thesis supervisor Dr. Y. N. Singh under whose supervision and inspiring guidance, I was privileged to carry out my research work. I am indebted to him for his constant and ungrudging encouragement, valuable suggestions and ingenious ideas. Words won't be sufficient to quantify his immense knowledge and understanding of the subject which has helped me in better understanding of theoretical aspects. He showed me different ways to approach a research problem and the need to be persistent to accomplish any goal.

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never ending support.

(Anurag Singh)

List of Symbols

k	Degree of the node.
$P(k)$	Degree distribution of the network.
$\langle k \rangle$	Average degree of the network.
σ	Variance of the degree distribution.
N	Total number of the nodes in the network.
q_k	Probability that a node at the end of the random edge having excess degree k.
μ_q	Mean of q_k.
σ_q	Variance of q_k.
$P(k,l)$	the joint probability that two nodes of degree k and degree l are connected.
r	Degree-degree correlation function.
L	Total number of edges.
x	Number of local connections in the WS model.
P_r	Rewiring probability in the WS model.
γ	Power law degree exponent.
$\langle k^2 \rangle$	Second moment of degree.
$\mathbb{I}(k,t)$	Expected value of ignorant nodes with degree k at time t.
$\mathbb{S}(k,t)$	Expected value of spreader nodes with degree k at time t.
$\mathbb{R}(k,t)$	Expected value of stifler nodes with degree k at time t.
$I(k,t)$	Fraction of ignorant nodes with degree k at time t.
$S(k,t)$	Fraction of ignorant nodes with degree k at time t.
$R(k,t)$	Fraction of ignorant nodes with degree k at time t.
λ	rumor spreading rate.
σ	Stifling rate.
δ	Rate at which a spreader turns into stifler spontaneously.
λ_c	Critical rumor threshold.
$P(l\|k)$	Probability of a randomly chosen edge emanating from k degree node to l degree node.
g	Fraction of inoculated nodes.
g_k	Fraction of inoculated nodes with degree k.
k_t	Cut off degree threshold.
ρ	Rate at which a ignorant node becomes stifler directly after rejecting the rumor.
η	Rate at which a ignorant node becomes stifler directly after accepting the rumor.

R_{rej}	Fraction of stifler nodes who rejects the rumor.
R_{acc}	Fraction of stifler nodes who accepts the rumor.
α	Nonlinear rumor spreading exponent.
β	Tie strength exponent between two nodes.
$\Phi(l)$	Rumor spreadness, given by k^{α}.
w_{ij}	Weight of the edge connecting vertices i and j.
k_{min}	Minimum degree of the node in a network.
k_{max}	Maximum degree of the node in a network.
D	Diagonal matrix of degrees of all nodes $[d_i]$.
A	Adjacency matrix of a network.
L	Laplace matrix of a network.
Z	Matrix of eigenvectors.
Λ	Diagonal matrix of eigenvalues.
L^{+}	Pseudo inverse matrix.
$n(i,j)$	Distance measure.
V_G	Volume of a graph.
$\vec{y_i}$	Transformation node vector.

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Chapter 1

Introduction

In this thesis, the dynamic process of the rumor spread in the complex networks has been studied. We have investigated the dynamic behavior of rumor spread using different strategies for inoculation of nodes in homogeneous and heterogeneous complex networks. The rumor spreading model was introduced many years ago by Daley and Kendal (DK) [1] and its variant was given by Maki-Thomsan (MK) [2]. DK and MK models have an important shortcoming that they do not take into account the topology of the underlying social interconnection networks along which rumors spread. It is important to consider the topology of underlying network. Many researchers have discussed the properties of the networks which affect the dynamical processes taking place in the networks. Recently, the complex network structures and their dynamics have been studied extensively [3, 7, 20, 21, 22, 23, 24, 25]. By analyzing different real world networks e.g. Internet, WWW, social network and so on, researchers have identified different topological characteristics of complex networks such as the small world phenomenon and scale free property. Therefore, we need to study the basics of complex network before starting to talk about the rumor dynamics and inoculation of nodes.

1.1   Network

In the context of networks, graphs are used to model pairwise connections between components nodes. A complex network is defined as graph, $G = (V, E)$, where, V denotes the set of nodes and E the set of connections (links or edges) between them. These links can be either directed or undirected (Figure 1.1). Each link may have different weights.

(a) Undirected network   **(b)** Directed network
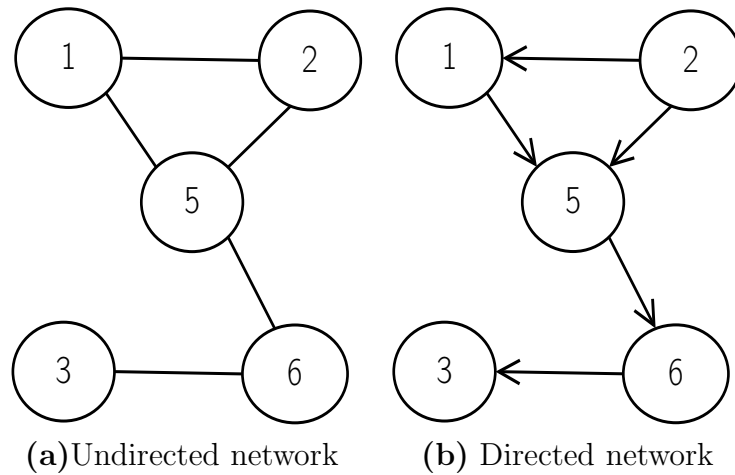
Figure 1.1: Example of a network (graph) with 5 nodes.

1.2   Network Metrics

A network is considered as **connected** if it is possible to travel between any pair of individual nodes by moving along edges of the network. An epidemiological interpretation of connectedness is that a single individual node can transmit infection to any other individual node in the population, typically via a number of intermediate nodes. Clearly, connectedness can only be determined from global knowledge of the network.

The **degree** or **connectivity** of a node, often written as k, is equal to the number of

neighbors that an individual node has on the graph (that is, the number of other nodes to whom it is directly connected). Since different individual nodes may have different numbers of neighbors, we talk about the **degree distribution** of the network, often written as $P(k)$. If X_k is the number of nodes having the degree k, then $P(k) = X_k/N$ is called the degree distribution, where N is the total number of nodes. From this distribution, the average degree, written as $\bar{k}$ or $\langle k \rangle$, can be calculated as $\sum_k kP_k$. The variance of the degree distribution is given by $\sigma^2 = \sum_k (k - \bar{k})^2 P_k$. This variance nearly equals zero if every individual has the same number of neighbors and we say that the network is homogeneous. Otherwise, the network will to be classified as heterogeneous. All of these quantities are local measures i.e, they can be calculated once we know the connectivity of an individual node with other nodes.

Several metrics have been used to describe the size of the network. The **distance** between two nodes is the length of the shortest path that connects them. The **diameter** of a graph is the largest of these values when all pairs of nodes are examined. The **average path length** can be calculated and provides some idea of the typical number of steps between individuals on the network . Clearly, one needs to have global knowledge of the network in order to calculate these quantities

Assortative mixing describes situations in which individuals are more likely to interact with other individuals who are similar to themselves in some respect [26]. **Disassortative mixing** describes the opposite situation, in which individuals tend to interact with dissimilar individuals. **Proportionate mixing** (also known as random mixing) occurs when interactions have no particular preference.

1.3   Complex Networks

A complex network is a network with non-trivial topological features, with patterns of connection between their elements that are neither purely regular nor purely random. The structure of complex networks is irregular, complex and dynamically evolving with time. Thus complex networks resembles the real networks e.g. transportation networks, phone call networks, social network, the Internet, the World Wide Web , the actors collaboration network, scientific co-authorship and citation networks, neural network, metabolic and protein networks [27].

1.4   Properties of Complex Networks

1.4.1   Degree Distributions

The probability of randomly chosen node of degree k is $P(k)$, where $1 \leq k \leq N$ (N number of nodes in a network). The distribution of the probability $P(k)$ is called degree distribution in the network. The directed networks have in-degree distributions and out-degree distributions.

1.4.2   Excess Degree

Except a randomly chosen link to reach on to the node, all other links connected to the node are considered to be excess degree in the network. q_k is the probability of a node at the end of a random link having excess degree k [24] (Figure 1.2). The distribution of this probability is the excess degree distribution q_k [26, 28] in the network, and is given by

$$q_k = \frac{(k+1)P(k+1)}{\sum_{k=1}^{\infty} kP(k)}. \tag{1.4.1}$$
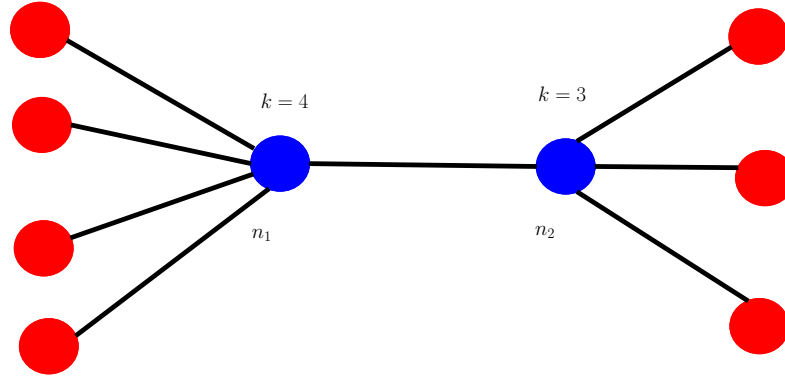
Figure 1.2: Excess degree of nodes, edge between n_1 and n_2 nodes has excess degree 4 at one end and 3 at another end.

Therefore, average number of outgoing edges of a neighbor vertex is

$$\begin{aligned}
\sum_{k=0}^{\infty} kq_k &= \frac{\sum_{k=0}^{\infty} k(k+1)P_{k+1}}{\sum_j jP_j} \\
&= \frac{\sum_{k=0}^{\infty} (k-1)kP_k}{\sum_j jP_j} \\
&= \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \tag{1.4.2}
\end{aligned}$$

Number of Next-Nearest Neighbors : The number of neighbors m hops away is denoted by z_m. Thus

$$z_1 = \langle k \rangle, \tag{1.4.3}$$

z_1 is also commonly written as z. Equation (1.4.2) gives the average number of nodes two hops away from the starting node via a specific neighbor vertex. Multiplying the

Equation (1.4.2) by average degree of starting node, $z_1 \equiv z$, the mean number of second neighbors, z_2 is

$$z_2 = \langle k^2 \rangle - \langle k \rangle. \tag{1.4.4}$$

The average number of edges emerging from a second neighbor (but not leading back), is also given by Equation (1.4.2). Hence, the average number of neighbors at the m hops away is

$$z_m = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} z_{m-1} = \frac{z_2}{z_1} z_{m-1}. \tag{1.4.5}$$

using recursion, the relation can be given by

$$z_m = \left[\frac{z_2}{z_1} \right]^{m-1} z_1. \tag{1.4.6}$$

Giant Connected Component

The set of nodes reachable from a given node is called a component. The convergence or divergence of Equation (1.4.6) will depend on whether z_2 is lesser than z_1 or not i.e.,

$$\lim_{m \to \infty} z_m = \begin{cases} \infty, & \text{if } z_2 > z_1 \\ 0, & \text{if } z_2 < z_1 \end{cases} \tag{1.4.7}$$

The $z_1 = z_2$ is the percolation point. For $z_2 < z_1$, the total number of neighbors

$$\sum_m z_m = z_1 \sum_{m=1}^{\infty} \left[\frac{z_2}{z_1} \right]^{m-1} \tag{1.4.8}$$

$$= \frac{z_1}{1 - z_2/z_1} \tag{1.4.9}$$

$$= \frac{z_1^2}{z_1 - z_2}, \tag{1.4.10}$$

is finite in the macroscopic limit, whereas for $z_2 > z_1$, it is infinite. The complex network will decay for $N \to \infty$, into non-connected components when the total number of neighbors is finite.

When the largest component of a graph encompasses a finite fraction of all vertices, in the macroscopic limit, it is said to form a giant connected component. Only when the number of neighbors is infinite then there must be a giant connected component.

The Percolation Threshold : If a system has two or more possibly macroscopically states, then they are said to have a phase transition. The phase transition happens at $z_2 = z_1$. Therefore, from Equation (1.4.4), the condition can be found

$$\langle k^2 \rangle - 2\langle k \rangle \;\; = \;\; 0, \tag{1.4.11}$$

$$\sum_{k=0}^{\infty} k(k-2)P_k \;\; = \;\; 0. \tag{1.4.12}$$

Joint Degree Distribution

In undirected networks, for a link having excess degree k on one end and excess degree l at another end [26, 29], we can define joint probability distribution $P(k,l)$ of the excess degrees of two nodes at either end of a randomly chosen link. For undirected networks, $P(k,l)$ and $P(l,k)$ will be symmetric [28] i.e.,

$$P(k,l) = P(l,k). \tag{1.4.13}$$

For the above equation, the sum rule will be true i.e.,

$$\sum_{k} P(k,l) = q_l. \tag{1.4.14}$$

1.5   Assortativity

Assortativity [26, 28, 30, 31, 32, 33, 34, 35] is a property of the complex networks where nodes mostly make connections with nodes of the similar degree [36, 37, 38].

In the real world, most of the complex networks show assortativity where high degree nodes are likely to be connected with other high degree nodes. Similarly, some complex networks show the disassortativity where high degree nodes are likely to be connected with low degree nodes. Therefore, for the assortative and disassortative complex networks the probability for existence of an edge between the two nodes depend on the degrees of both nodes, respectively.

The assortative mixing can be measured by correlation function in terms of the degrees of the nodes in the network [26, 29]. The value of this correlation function will be zero for non-assortative mixing, positive for assortative mixing and negative for dis-assortative mixing.

In the undirected networks, for non-assortative mixing then

$$P(k, l) = q_k q_l. \tag{1.5.15}$$

If there is assortative mixing then, $P(k, l)$ will be different from Equation (1.5.15). The correlation function with assortative mixing can be defined as

$$
\begin{aligned}
P(k, l) &= \langle kl \rangle - \langle k \rangle \langle l \rangle \\
&= \sum_{kl} kl(P(k, l) - q_k q_l).
\end{aligned}
\tag{1.5.16}
$$

The correlation function [26] can be defined as

$$r = \frac{1}{\sigma_q^2} \sum_{kl} kl(P(k,l) - q_k q_l), \tag{1.5.17}$$

where, $P(k,l)$ is the joint probability distribution of the excess degrees of the nodes on the both sides of randomly chosen edge. The σ_q is the variance of excess degree distribution, q_k, and is given by $\sigma_q^2 = \sum_k k^2 q_k - [\sum_k kq_k]^2$. Similarly, mean of the q_k can be defined as

$$\mu_q = \sum_l lq_l. \tag{1.5.18}$$

Now, correlation function can be defined as

$$r = \frac{1}{\sigma_q^2} \left[\left(\sum_{kl} klP(k,l)\right) - \mu_q^2 \right]. \tag{1.5.19}$$

In the above Equation (1.5.19), $-1 \leq r \leq 1$, where, $r = 1$ shows full assortative networks and $r = -1$ full disassortative networks. For $r = 0$, network will be non-assortative. If correlations between adjacent nodes exist then, the probability that a randomly chosen edge connecting two node of degrees k and l respectively, is $(2 - \delta_{kl})P(k,l)$.

The probability that a node with excess degree k is reached by any randomly chosen edge emanating from a node with degree l is

$$P(k|l) = \frac{P(k,l)}{q_l}. \tag{1.5.20}$$

1.6   Models of Complex Networks

1.6.1   Homogeneous Networks

There are some types of complex networks exist, which have similar degrees. Alternatively, we can say that degree distributions of these type of networks have very small variations. These properties can be seen in random networks (Erdos Renyi graph) and in small world networks. The complex networks with the given properties of their degree distributions are called homogeneous networks.

1.6.1.1   Erdos-Renyi Model of Random Networks

In the Erdos-Rnyi (ER) model [39], we start from N vertices without edges. Subsequently, edges connecting two randomly chosen vertices are added until the total number of edges becomes L. It generates random networks with no particular structural bias. The only restriction in the model is that no multiple edges are allowed between two vertices. In this study,it has been chosen that the average degree $\langle k \rangle \equiv 2L/N$ as a control parameter in the ER model. The ER model graphs have a logarithmically increasing l , a Poisson-type degree distribution, and a clustering coefficient close to zero.

1.6.1.2   Watts-Strogatz Model of Small-World Networks

In the Watts-Strogatz (WS) model [25] one starts by constructing a regular network with only local connections of range x. For example, $x = 2$ means that each vertex is connected to its two nearest neighbors and two next nearest neighbors (see Figure 1.3). Then each edge is visited once, and with the rewiring probability P_r is detached at the opposite vertex and reconnected to a randomly chosen vertex forming a shortcut

(Figure 1.3). For $P_r = 0$ the network is a regular local network, with high clustering, but without the small-world behavior. The average geodesic length in this case grows linearly with the network size. In the opposite limit of $P_r = 1$, where every edge has been rewired, the generated random graph has vanishing clustering, but shows a logarithmic behavior of the average geodesic length $l \propto logN$. In an intermediate range of P_r (typically $P \sim O(1/N)$), the network generated by the WS model displays both high clustering and small-world behavior the commonly found characteristics of real social networks.
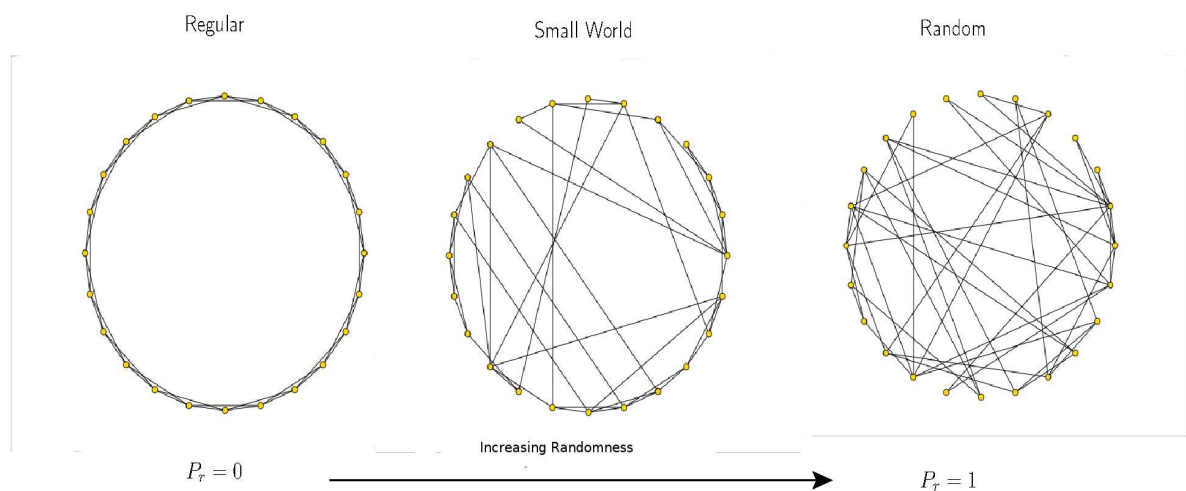
Figure 1.3: The Watts-Strogatz (WS) model. The starting point is a regular network with the range $x = 2$ of the connections. First the local regular network when $P_r = 0$, with the high clustering but with the large average geodesic length, and last, the fully random network when $P_r = 1$, with low clustering but with very short geodesic length. In the intermediate region of P_r, the WS model has both high clustering and the small-world behavior (more specifically, the average geodesic length $l \propto logN$ for the network with the size N).

1.6.2 Heterogeneous Networks

In adverse of homogeneous networks, some networks have larger degree variations in their degree distributions. These categories of complex networks are called heterogeneous networks. Internet, WWW, email, power grid and other large networks define the class of heterogeneous networks.

1.6.2.1 Barabsi-Albert Model of Scale-Free Networks

It has been found that degree distribution of the most of the large scale networks like Internet, Web network etc. follows the power law, hence most of the theories developed to understand the topology of the emerging network are directed towards explaining the emergence of scale free networks. In this context, several complex network theoretical models are proposed. In Barabasi and Albert(BA) model, the appearance of scale free networks is explained with the help of preferential attachment rules [20].

Apart from the average geodesic length and clustering, the degree distribution is a structural bias that has received much attention. Many (but not all) real networks are known to have a power-law distribution of degrees [3, 28], manifesting a scale-free nature of the network. The BA model of scale-free network [3, 20] is defined by the following ingredients:

- Initial condition: To start with the network, consists of m_0 vertices and no edges.

- Growth: One vertex v with m edges is added every time step.

- Preferential attachment: An edge is added to an old vertex with the probability proportional to its degree. More precisely, the probability P_u for a new vertex v

to be attached to u [21].

1.6.2.2   Other Models of Scale Free Networks

Kleinberg *et al.* proposed vertex coping model of network growth [40] where, the network grows stochastically by constant addition of nodes and replicating the edges from another existing nodes. Fabrikant *et al.* [41] proposed a plausible explanation of the power law distributions observed in the graphs arising in the Internet topology. In this growth paradigm (FKP growth model),the incoming node i stochastically connects to an existing node j such that the node j is physically close to node i (small Euclidean distance d_{ij} between the node i and j) and at the same time the node j is centrally located in the network (hop distance of j, (h_j) to other nodes is minimum). Callaway *et al.* [42] proposed the model of evolving networks that is initially scattered into disconnected components and eventually merged with each other to form large components. In this model, nodes may join to network without necessarily connecting with some existing nodes. Then, with probability d, two nodes are chosen uniformly at random and joined by an undirected edge. This may result in growing network containing isolated nodes along with component of various sizes. However, among these various growth models, the BA model is the simplest one, widely studied.

1.6.2.3   Configuration Model for Scale Free Networks

The scale free networks can also be generated by configuration model [24]. In this, we define a random number (nodes' degree k_i) to each node i with $i \leq i \leq N$ by using a probability distribution (degree distribution). The given random number (nodes degree k_i) for each node i is called 'subs' or 'half edges'. After this, we find the stubs of random

pairs of the nodes i and j and combine to make an edge (i, j) by assuming that there is no duplicate edge between the nodes i and j and node i should not be equal to node j. The generation of edges will go on until there are no more stubs to combine. For making the scale free networks we give the random numbers to nodes using the power law degree distribution,

$$P(k) \ \propto \ \begin{cases} k^{-\gamma}, & \text{if } k_{min} \le k \le k_{max} \\ 0, & \text{otherwise}, \end{cases} \qquad (1.6.21)$$

where, $k_{min} = 1$ is the minimum degree and $k_{max} = N - 1$ is the maximum degree of the scale free network and γ is the power law exponent with, $2 < \gamma \le 3$. For generation of degree-degree correlated networks each node each edge is generated with the given degree-degree correlation matrix. In the thesis, the scale free network has been generated with the configuration model.

1.7 Problem Definition and Organization of Thesis

In chapter 2, we have discussed about the classic rumor spreading model and also discussed the different inoculation strategies.

In chapter 3, we have proposed a new rumor spreading model using the acceptability factor. The dynamics of the rumor spreading has been studied using this model for small world network (homogeneous networks). Inoculation of the nodes has been considered to stop rumor spreading. Random and targeted inoculation techniques have been applied on the proposed rumor spreading model for small world network. Rumor threshold has been found for different fraction of inoculation in small world network. Finally, the role of acceptability factor has been studied to control the rumor spreading in small world network. Further, acceptability factor in rumor spreading has been investigated for

scale free (which are heterogeneous) networks. As random inoculation does not work satisfactorily, we have investigated neighbor inoculation for arresting the rumor spread for the networks with unknown topologies.

Chapter 4, describes the rumor dynamics and inoculation of nodes for correlated networks. Different scale free networks have been generated with given correlation coefficients. Effects of correlation coefficients in controlling of rumor spread in the scale free networks have been studied. The threshold variation with the change of correlation coefficients in the scale free networks has also been discussed.

In chapter 5, we have proposed rumor spreading model for more realistic cases of real world complex networks. Let us assume that in a social network, some of the friends are closer than others. Therefore, strong tie strengths are there between them. A node is expected to spread a rumor with high probability to a neighbor connected strongly than to a neighbor which is loosely connected. The strength of connection is modeled by the tie strength. Hence, rumor spreading rate may vary with the tie strength of the edges in the social networks. In this chapter, variable spreading rates and stifling rates have been included. In more realistic scenario, it depends on a person that he may spread the information only to some of his friends and not to all at a time. Thus, the rumors will spread nonlinearly to the neighboring nodes with increase in the nodal degree. The dynamics of rumor spreading has been studied as a function of time for different spreading rates. The effects of the tie strength exponent β and nonlinear rumor spreading exponent α on rumor threshold have been studied. Some nonzero rumor thresholds have been found with some specific conditions on α and β values. With certain conditions, constant rumor threshold has been found, which is independent of the network size. To control the rumor spreading, random and targeted inoculations have been applied on proposed rumor spreading model and effect of α and

β parameters has been studied.

In chapter 6, we have discussed a new technique to find influential node based on structural centrality.

In the last, chapter 7 of the thesis presents the conclusions of the work done. Some of the possible future research directions are also suggested.

Chapter 2

Rumor Dynamics: A Study

2.1 Background

In today's world, the Internet has become the most important medium to circulate information. We use online social network sites almost every day to express our locations, emotions and to communicate with friends. For most of the events, the information first spreads over the Internet than on any other medium. Twitter and Facebook have become most important mechanisms for information spread. Twitter has more than 500 million registered users and Facebook has more than 955 million registered users currently. Huge number of users share information on Twitter and Facebook. Lot of research has been carried out to provide valuable insight into the information diffusion over social networks. If any information circulates without officially publicized confirmation, it is called rumor [43]. In other words, rumors are unreliable information. Rumor may change its meaning when it moves from one person to another. The rumor may evolve (change) as they propagate in an real scenario. This makes the study of information diffusion quite complex. In this thesis, we have restricted ourselves to the study of rumor which does not change with time. To stop the rumors in the network

inoculation of nodes can be used.

Dynamics is the process for the propagation in the networks as seen in temporal domain. The networks carrying rumor normally belong to the real world networks e.g. Internet, World Wide Web, Social networks etc. These real world networks follow the complex network topologies and can be defined in the form of contact networks, e.g. social networks. It can be formed after the contacts/interactions have taken place between the friends in a social network. The dynamics of the flow of some entities (rumor, epidemics, viruses etc.) in the network is influenced by the contact pattern among the nodes. Epidemic spreading also has similar features, hence epidemic models can be used to study the rumor dynamics. Most of the work reported so far in propagation dynamics has been based on the studies done in epidemics. The simplest of all, susceptible-infected (SI) model for dynamic process has only two states for any node. A node can be infected or not infected. An uninfected node can be infected by an infected neighbor permanently with some spreading rate. Finally, all the nodes become infected in the end. There are two more models for epidemic spreading, susceptible-infected-susceptible (SIS) [8, 44] and susceptible-infected-recovered (SIR) [23, 45]. SIS model allows nodes to recover and become susceptible again. Therefore, it is difficult for the disease to infect all the population. The SIR model introduces a new refractory state in which nodes cannot be infected again. The SIR model for rumor spreading was introduced many years ago by Daley and Kendal [1] and its variant was given by Maki-Thomsan [2]. In the given model, all the population has been considered to be homogeneous. In the homogeneous population all nodes will have same degree. The epidemic spreads with certain rate throughout the network. Simultaneously, the infected nodes are also cured with certain rate. By using the Daley Kendal model [1], Kephart and White [46] have studied the propagation of the computer viruses in the

network.

Studies of rumor spreading in complex networks are of interest and the results have largely changed the views on the issue of rumor spreading. The theory and the method of transmission dynamics being applied to the analysis of structure and characteristics of rumor spreading play a vital role in the design of rumors prevention and control system.

In order to improve the resistance of the community against undesirable rumors, it is essential to develop deep understanding of the mechanism and underlying laws involved in rumor spreading and to establish an appropriate prevention and control strategy to generate social stability. First time, Sudbury studied the spread of rumors based on SIR model [47]. In Daley-Kendal (DK) model [48] homogeneous population is subdivided into three groups: ignorant (who don't know about the rumor), spreaders (who know about the rumor) and stifler (who know about the rumor but do not want to spread it). The rumor is propagated throughout the population by pairwise contacts between spreaders and other individuals in the population. Any spreader involved in a pairwise meeting attempts to infect other individual with the rumor. In case other individual is an ignorant, it becomes a spreader. If the other individual is a spreader or stifler, it finds that rumor is already known to it, and decides not to spread rumor anymore, thereby turning into stifler. In Maki Thomsan (MK) model, when spreader contacts another spreader, only the initiating spreader becomes a stifler. DK and MK models have an important shortcoming that they do not take into account the topology of the underlying social interconnection networks along which rumors spread. These models are restricted while explaining real world scenario for rumor spreading. By considering the topology of network, rumor model on small world network [3, 4, 5] and scale free networks [6] have been defined. Therefore, as long as, one knows the structure of spreading networks, he

can figure out variables and observables to conduct quantitative analysis, forecast and control the rumor spreading. The most important conclusion of classical propagation theory is existence of critical point of rumor transmission intensity. When an actual intensity is greater than critical value, the rumors can spread in networks and exist persistently. When the actual intensity is less than the critical value, rumors decay at an exponential rate. This critical value is called rumor threshold. Each informed node can be assumed to make contacts with all of its neighbors in a single time step. In other words we can say that each informed node can spread information to nodes equal to its degree in single time step. Studies on small world networks have found that compared to regular network, small world network has smaller transmission threshold and faster dissemination. Even at smaller spreading rates, rumors can exist for long. Studies on infinite-size scale free networks have also revealed that no matter how small transmission intensity may be, rumors can be persistent as positive critical threshold does not exist [7, 9].

In the previous studies on the rumor spreading in the scale free networks, it has been assumed that larger the node degree, greater the rumor spreading from the informed node, i.e. the rumor spread is proportional to the nodal degree. With these assumptions for SIR model, in scale free networks of sufficiently large size, the rumor threshold λ_c can be zero. In these studies, the dynamical differential equations are used to represent the models for information spread. These equations are used to find the threshold and study the rumor propagation behavior. The results are usually verified by the simulations.

In the scale free networks, rumors first affect individuals who have more social contacts, then the general individuals and finally those with less social contacts. It has been found that in the scale free networks, rumors spread at a relatively low speed for a very short period of time starting from the outbreak and then rise rapidly to a high

peak, followed by a rapid decline exponentially.

To consider the topology of network, the rumor spreading models on small world network [3, 4] and scale free networks [6] have been defined. Using mean field theory, Nekovee *et al.* [3] discovered that threshold (below which a rumor can not be spread) was small in homogeneous networks e.g. small world networks, and networks following Erdos Renyi (ER) model. On the other hand, as found by Liu and his associates [6], the heterogeneous networks e.g. scale free networks, are more robust against spreading of rumors as compared to homogeneous model. In the previous studies on rumor spreading in scale free networks, it has been found that larger the nodal degree, the greater the rumor spread from the informed node. Therefore, in scale free networks with sufficiently large size, the rumor threshold λ_c can be zero [9]. Few studies have been reported to stop the rumor spreading. These studies are more important since false and fatal rumors have negative impacts on the society during disasters. There is a threshold value on spreading rate below which the disease(or rumor) cannot propagate in the system.

2.2   Rumor Threshold

If the value of spreading rate λ is above threshold λ_c, the rumor spreads. When $\lambda \leq \lambda_c$, the rumor quickly dies out exponentially. For scale free networks with connectivity exponent γ $(2 < \gamma \leq 3)$ threshold, λ_c is $\frac{\langle k \rangle}{\langle k^2 \rangle}$ [7]. If $\langle k^2 \rangle \to \infty$ then $\lambda_c = 0$. It shows that for any value of λ, the infection can pervade a large network with finite prevalence. Statistically speaking, a rumor can easily survive and cause an outbreak in an infinitely large scale-free network no matter how weak its spreading capability is. Further studies on the finite-size scale-free networks show that the threshold remains low and decreases with an increasing network size [7] . Such analytical results help to explain our real-life

experiences, e.g., persistent rumor spreading in the scale free networks. For finite size network, $\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle} \sim \frac{1}{ln(N)}$, where N is the network size.

2.3   Classical Rumor Spreading Model

Classical rumor spreading model using SIR model is one of the most investigated rumor spreading models for the complex networks. In this model, nodes are in one of the three categories —ignorants (the nodes who are ignorant of the rumor), spreaders (those who hear the rumor and also actively spread it) and stifler (the nodes who hear the rumor but do not spread it further). The rumor is propagated through the nodes by pairwise contacts between the spreaders and other nodes in the network. Following the law of mass action, the spreading process evolves with direct contact of the spreaders with others in the population. These contacts can only take place along the edges of undirected graph of complex network. If the other node is the spreader or stifler then the initiating spreader becomes the stifler. The classical SIR model has been studied by M. Nekovee *et al.* [3, 22] for heterogeneous population (nodes having different degrees). In this chapter, $\mathbb{I}(k,t), \mathbb{S}(k,t), \mathbb{R}(k,t)$ are the expected values of ignorants, spreaders and stifler nodes in network with degree k at time t. Let $I(k,t) = \mathbb{I}(k,t)/N(k), S(k,t) = \mathbb{S}(k,t)/N(k), R(k,t) = \mathbb{R}(k,t)/N(k)$ be the fraction of ignorant, spreaders and stifler nodes, respectively with degree k at time t. These fractions of the nodes satisfy the normalization condition, $I(k,t) + S(k,t) + R(k,t) = 1$. Here, $N(k)$ represents the total number of nodes with degree k, in the network. Above rumor spreading process can be summarized by the following set of pairwise interactions (see Figure 2.1).

$$S_1 + I_2 \xrightarrow{\lambda} S_1 + S_2,$$

(when spreader meets with the ignorant, it makes them spreader at rate λ)

$$S_1 + R_2 \xrightarrow{\sigma} R_1 + R_2,$$

(when a spreader contacts a stifler, the spreader becomes a stifler at

the rate σ)

$$S_1 + S_2 \xrightarrow{\sigma} R_1 + S_2,$$

(when a spreader contacts with another spreader, initiating spreader

becomes a stifler at the rate σ) and

$$S \xrightarrow{\delta} R.$$

(δ is the rate at which spreaders change their state to stifler spontaneously

and stop spreading the rumor).

Nekovee *et al.* [3] proposed the formulation of this model for analyzing complex networks as interacting Markov chains. They used the framework to derive from the first-principles, the mean-field equations for the dynamics of rumor spreading in the complex networks with arbitrary correlations. These are given below.

$$\frac{dI(k,t)}{dt} = -k\lambda I(k,t) \sum_l P(l|k)S(l,t). \tag{2.3.1}$$

$$\frac{dS(k,t)}{dt} = k\lambda I(k,t) \sum_l P(l|k)S(l,t) - k\sigma S(k,t) \sum_l \left(S(l,t) + R(l,t)\right)P(l|k)$$
$$-\delta S(k,t). \tag{2.3.2}$$

$$\frac{dR(k,t)}{dt} = k\sigma S(k,t) \sum_l (S(l,t) + R(l,t))P(l|k) + \delta S(k,t). \tag{2.3.3}$$
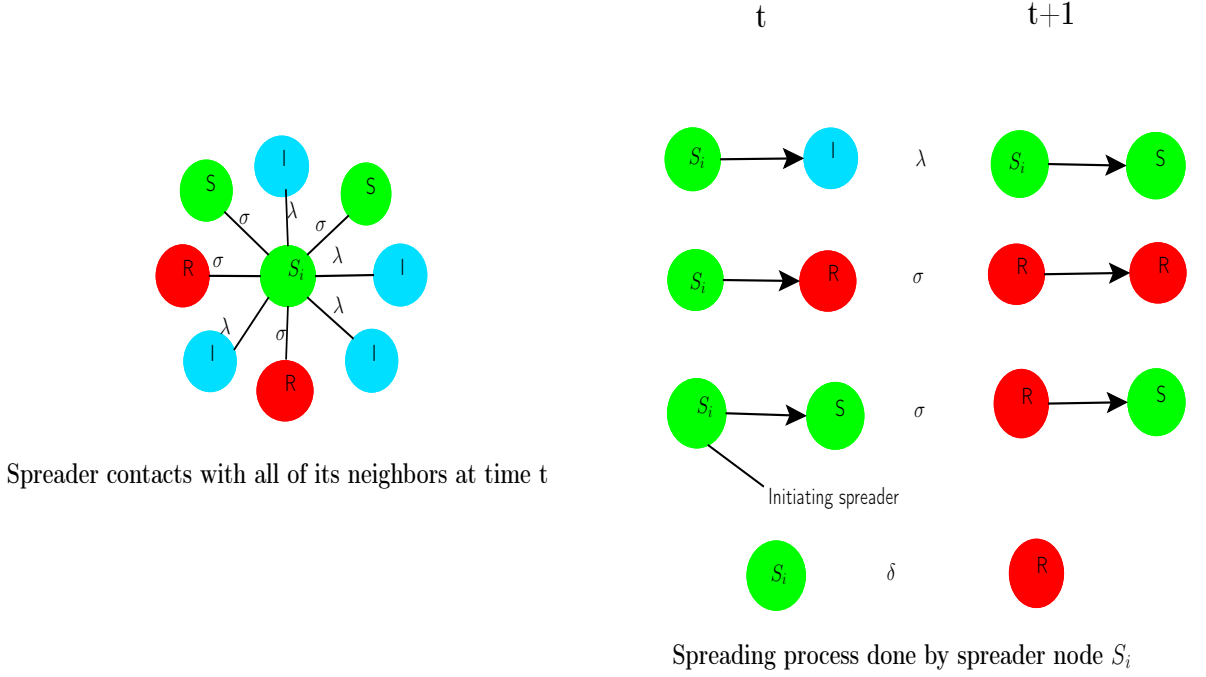
Figure 2.1: Pairwise interactions in classical model.

Where, the conditional probability $P(l|k)$ is the degree-degree correlation function that a randomly chosen edge emanating from a node of degree k, leads to a node of degree l. Here, it has been assumed that the degree of nodes in the whole network are uncorrelated. Thus $P(l|k)$ will be the probability that a randomly chosen link is terminating on a node with degree k. Therefore, degree-degree correlation is $P(l|k) = \frac{lP(l)}{\langle k \rangle}$ where, $P(l)$ is the degree distribution and $\langle k \rangle$ is the average degree of the network (the edge will be biased to fall on vertices of high degree, therefore conditional probability $P(l|k)$ is proportional to $kP(k)$ and after normalizing this gives us the result). Nekovee *et al.* [3] have shown that the critical threshold for rumor spreading is independent of the stifling mechanism. The critical threshold found by them was $\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$. It is same as found for SIR model [45, 49]. Hence, it implies that epidemic threshold is absent in large size scale free networks ($\langle k^2 \rangle \to \infty, \lambda_c \to 0$) irrespective of stifling mechanism. This result is not good for epidemic control, as the epidemics will exist in the real networks for any

non zero value of spreading rate λ.

2.4   Inoculation Strategies

Inoculating the nodes is an option to control the spread of rumors. Inoculated nodes cannot be made to believe on rumor thus cannot be made spreaders. Therefore, they do not help in spread of rumors to their neighboring nodes. The inoculation process is similar to percolation. Each inoculated node can be considered as a site which is disconnected from the network. The goal of inoculation strategy is to remain below the percolation threshold. So that, we can get minimum number of nodes who are infected with rumor. Therefore, inoculation strategy is successful if the network is operating below the percolation threshold [50]. Although, random immunization strategy works very well in homogeneous random networks, but this strategy is not effective in preventing a rumor in the scale free networks [7]. Hence, the new immunization strategies need to be explored which are able to recover from the rumor spreading in the scale free networks. One of the efficient approaches is to inoculate the high degrees nodes or, more specifically, to inoculate those nodes (hereafter termed as hubs or hub nodes) which have degree higher than a preset cut-off value k_c . Such a strategy is known as targeted inoculation [8, 9, 10, 11, 12, 13, 15]. Targeted inoculation is successful in arresting the rumor spread in scale free networks [7]. Random inoculation usually requires inoculation of much large number of nodes for being effective. If nodes with higher connectivity are targeted for inoculation, the same effectiveness can be achieved with smaller number of inoculated nodes. But it needs the knowledge of nodes which have higher connectivity [14]. D.Chen *et al.* have suggested that identifying influential nodes using betweenness centrality and closeness centrality for spreading an information

can lead to faster and wider spreading in complex networks [51]. The best spreaders using various measures of centrality can be identified in a network to ensure the more efficient spread of information. The inoculation of these efficient spreaders can also efficiently stop the rumor spreading [52, 53]. But this approach cannot be applied in large-scale networks due to the computational complexity involved in identifying such nodes. On the other hand, higher degree nodes can be identified with much less efforts.

2.4.1   Random Inoculation

This approach inoculates a fraction of the nodes randomly, without any information of the network (Figure 2.2 a). Here, variable g $(0 \leq g \leq 1)$ defines the fraction of inoculated nodes. In the presence of random inoculation rumor spreading rate λ is reduced by a factor $(1 - g)$. Therefore,

$$\lambda \rightarrow \lambda(1 - g).$$

Random inoculation is successful in the homogeneous networks, as there is no large degree variation in them. The degree of all the nodes is closer to the average degree of the network. But random inoculation is not successful in the case of heterogeneous network due to the large degree variation. Therefore, we need to inoculate almost 80-90 % nodes to make the random inoculation successful [8, 11, 54].

2.4.2   Targeted Inoculation

Scale free networks permit efficient strategies and depend on the hierarchy of nodes. It has been shown that scale free networks show robustness against random inoculation.
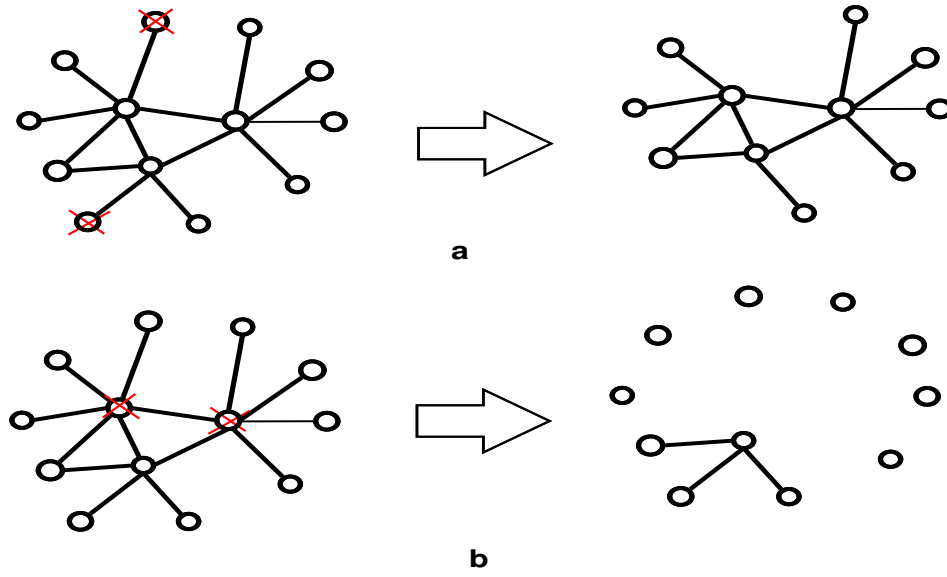
Figure 2.2: Modified network after inoculation (a) Random inoculation (red crossed nodes inoculated) (b) Targeted inoculation (red crossed nodes inoculated).

It shows that the high fraction of inoculation of nodes can be resisted without loosing its global connectivity. But on the other hand SF networks are strongly affected by targeted inoculation of nodes (Figure 2.2 b). The SF network suffers an interesting reduction of its robustness to carry information due to targeted inoculation. In targeted inoculation, the high degree nodes are inoculated progressively as these are more likely to spread the information. In SF networks, the robustness of the network decreases even with a tiny fraction of inoculated individuals [7, 12, 50, 55].

Let us assume the fraction of g_k of nodes with degree k are successfully inoculated. An upper threshold of degree k_t , can be defined, such that all nodes with degree $k > k_t$ get inoculated. This fraction g_k is given by,

$$g_k = \begin{cases} 1, & k > k_t, \\ f, & k = k_t, \\ 0, & k < k_t. \end{cases} \qquad (2.4.4)$$

Ahere, $0 < f \leq 1$, and $\sum_k g_k P(k) = \bar{g}$, where $\bar{g}$ is the average inoculation fraction.

2.4.3 Neighbor Inoculation

For random inoculation, it is necessary to inoculate almost all the nodes in the network in order to stop the rumor. The targeted inoculation is very effective but it needs the global information of the network. At least, the knowledge of most of the high degree nodes is required. Because of large, complex and time varying nature of most of the networks as well as the Internet, it is very difficult to determine the target nodes. Therefore, Cohen *et al.* proposed an inoculation strategy known as acquaintance immunization [56]. In the acquaintance inoculation strategy, some nodes are being randomly selected with probability p in the network from the N nodes. Then, neighbors are being selected randomly by the selected nodes for inoculation. The probability that a specific neighbor node with degree k is selected for inoculation is $kP(k)/(N\langle k \rangle)$. In this inoculation strategy, we only need the information of randomly selected nodes and neighbor nodes attached with them. In scale free networks, if a node has been selected randomly then probability of choosing one of its neighbor nodes with higher degree is higher than a node with lesser degree [57]. This inoculation strategy is referred to as neighbor inoculation.

Chapter 3

Rumor Dynamics with Acceptability Factor

In this chapter, the dynamics of rumor spreading has been studied with the addition of a new compartment of ignorant nodes who rejects rumors for homogeneous network (small world network) [1] and heterogeneous network (scale free networks) [2]. The rumor acceptability factor is being introduced and the effects on propagation of rumor in small world and scale free networks have been observed. Here, small world and scale free network topology have been considered because real world social networks follow these properties. The inoculation of nodes has been introduced to control the rumor with the variation in rumor acceptability factor $(1/\rho)$.

Nekovee, *et al.* [3] gave a general stochastic model for the rumor spreading. In this model, the total population is divided into three compartments: ignorant individuals, spreaders and stifler. Ignorant population is susceptible to being informed, spreaders

[1]**Anurag Singh** and Y. N. Singh, "Rumor spreading and inoculation of nodes in complex networks," in *Proceedings of the 21st international conference companion on World Wide Web*, ser. WWW '12 Companion. New York, NY, USA: ACM, 2012, pp. 675–678.

[2]**Anurag Singh**, R. Kumar, and Y. N. Singh, "Rumor dynamics with acceptability factor and inoculation of nodes in scale free networks," in *Signal Image Technology and Internet Based Systems (SITIS), 2012 Eighth International Conference on*, nov. 2012, pp. 798–804.

spread the rumor and stifler know the rumor but they are not interested in spreading it. In this chapter, stifler have been further divided into two compartments: one population of stifler which accepts the rumor but is not interested in spreading it, the other one rejects the rumor (i.e., not interested in accepting it). The other part of population can be considered inoculated as in epidemic spreading model. Let there be N nodes and each node can be in one of the compartments of ignorant, spreaders and stifler. When a spreader meets with an ignorant node, ignorant node becomes a spreader with rate λ, or a stifler who accepts the rumor with rate η, or a stifler who rejects the rumor with rate ρ. Here, λ, ρ and η rates satisfy the condition $\lambda + \rho + \eta \leq 1$. When a spreader meets with a spreader or stifler, the spreader becomes a stifler who accepts the rumor with rate σ. The specialty of this model is that it allows ignorant nodes to become stifler when it is contacted by a spreader. This is similar to real life social network examples, when a person wants to spread rumor to his friends then some of his friends may not be interested in spreading it further after hearing the rumor. The friends may reject the rumor and become stifler, it depends on the acceptability of rumor $(1/\rho)$ [58, 59, 60]. Here $1/\rho$ can be considered as acceptability of the rumor. If the value of ρ increases, ignorant nodes are more likely to become stifler who reject the rumor. On the other hand, the friends may accept the rumor and decide not to spread further because of his limited energy. When a spreader contacts another spreader or a stifler, forgetting and stifling mechanism mutually result in the cessation of rumor spreading. The individuals no longer spread a rumor when they know that the rumor is out dated or wrong. If for a rumor, ρ is small, the rumor has more acceptable information [15]. In this chapter $I(t), S(t), R_{acc}(t), R_{rej}(t)$ represent the fraction of ignorant nodes, spreaders, stifler who accept the rumor and stifler who reject the rumor respectively, as the function of time t for homogeneous networks. $R(t)$ is the total number of stifler (includes both, the one who accept the rumor and the other who reject the rumor) at a time t. On the

other hand for heterogeneous networks, $I(k,t), S(k,t), R_{acc}(k,t), R_{rej}(k,t)$ are defined as the density of ignorants, spreaders, stifler who accepts the rumors and the stifler who rejects the rumors, respectively belonging to connectivity class k at time t. Above rumor spreading process can be summarized by the following set of pairwise interactions with the condition that,

$$S + I \xrightarrow{\lambda} S + S,$$

$$S + I \xrightarrow{\rho} S + R_{rej},$$

$$S + I \xrightarrow{\eta} S + R_{acc},$$

$$S + R_{acc} \xrightarrow{\sigma} R_{acc} + R_{acc},$$

$$S + R_{rej} \xrightarrow{\sigma} R_{acc} + R_{rej}, \text{and}$$

$$S + S \xrightarrow{\sigma} R_{acc} + S.$$

In heterogeneous networks, when a rumor is being propagated on the network from spreaders, stifling is not the only way to stop it and we have to consider the factor of forgetting mechanism with rate δ. Therefore, one more pairwise interaction has been added for heterogeneous networks,

$$S \xrightarrow{\delta} R_{acc}.$$

Normalizing condition for all three types of fractions,

$$I(t) + S(t) + R_{acc}(t) + R_{rej}(t) = 1, \tag{3.0.1}$$

$$I(k,t) + S(k,t) + R_{acc}(k,t) + R_{rej}(k,t) = 1. \tag{3.0.2}$$

In this chapter, the rumor spreading model defined for homogeneous networks with acceptability factor in section 3.1. Random and targeted inoculation strategies have been applied to observe the changes in rumor dynamics. The rumor spreading model for heterogeneous networks has been defined in section 3.2. Random, targeted and neighbor inoculation strategies have been applied to control the rumor. After inoculating some fraction of nodes, new degree distributions have been generated [61, 62].

3.1 Rumor Dynamics with Acceptability Factor in Homogeneous Networks

3.1.1 Proposed Rumor Spreading Model

The mean field rate equations are defined for our model as

$$\frac{dI(t)}{dt} = -(\lambda + \rho + \eta)\langle k \rangle I(t)S(t), \tag{3.1.3}$$

$$\frac{dS(t)}{dt} = \lambda\langle k \rangle I(t)S(t) - \sigma\langle k \rangle S(t)(S(t) + R_{acc}(t) + R_{rej}(t)), \tag{3.1.4}$$

$$\frac{dR_{acc}(t)}{dt} = \sigma\langle k \rangle S(t)(S(t) + R_{acc}(t) + R_{rej}(t)) + \eta\langle k \rangle I(t)S(t), \tag{3.1.5}$$

$$\frac{dR_{rej}(t)}{dt} = \rho\langle k \rangle I(t)S(t). \tag{3.1.6}$$

$$R(t) = R_{acc}(t) + R_{rej}(t). \tag{3.1.7}$$

Here, $\langle k \rangle$ is the average degree of the Watts-Strogatz (WS) [25] network and initial conditions of above equations are $I(0) \approx 1, S(0) \approx 0, R_{acc}(0) = 0, R_{rej}(0) = 0$. In Equations (3.1.3)-(3.1.6), $\lambda + \eta + \rho \leq 1$. Here I, S, R_{acc}, R_{rej} are not the function of k as we are discussing homogeneous network. Instead average of node degree $\langle k \rangle$ has been multiplied on the right hand side of Equations (3.1.3)-(3.1.7)

By using Equations (3.0.1) -(3.1.7), one get the following transcendental equation (Appendix A.1).

$$R(\infty) = 1 - e^{-\frac{\lambda+\sigma}{\sigma}R(\infty)}, \tag{3.1.8}$$

where, $R(\infty) = \lim_{t\to\infty} R(t)$ and $R(\infty) = R_{acc}(\infty) + R_{rej}(\infty)$. One can solve the Equations (3.1.3) -(3.1.6) with simulink and get a relation between fraction of different population with the time as shown in Figure 3.1. We know that for a nonzero solution
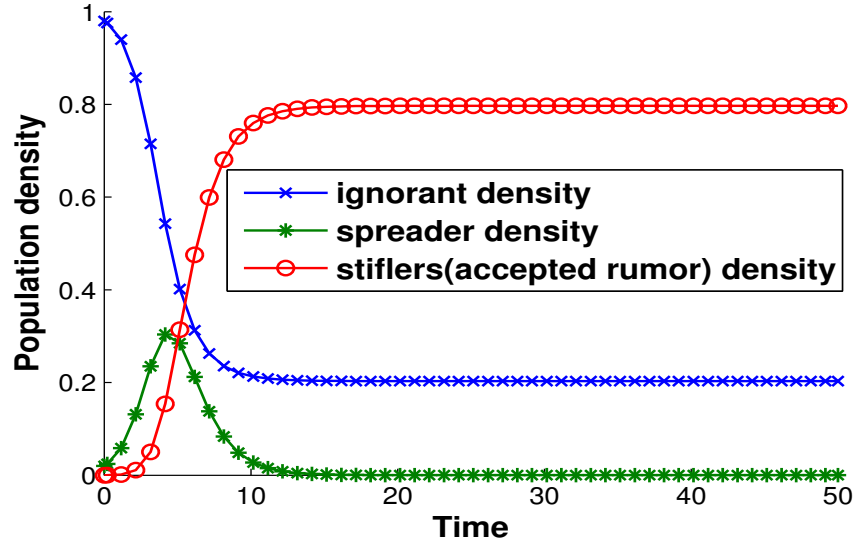
Figure 3.1: Ignorant, Spreaders and R_{acc} densities with time.

of Equation (3.1.8), $(\lambda + \sigma)/\sigma \geq 1$. This inequality is always valid, except for $\sigma=0$. There will not be any threshold for λ. It is different from SIR [14, 63] model. Now we can also solve expressions for $R_{acc}(\infty)$ and $R_{rej}(\infty)$ (Appendix A.3) :

$$R_{acc}(\infty) = \frac{\eta + \lambda}{\lambda + \rho + \eta} R(\infty)$$

$$R_{rej}(\infty) = \frac{\rho}{\rho + \eta + \lambda} R(\infty) \tag{3.1.9}$$

Using Equation (3.1.9):

$$\frac{R_{acc}(\infty)}{R_{rej}(\infty)} = \frac{\eta + \lambda}{\rho} \tag{3.1.10}$$

From Equation (3.1.10) it is evident that if we increase ρ (decrease the acceptability) and fix other parameters, $R_{acc}(\infty)$ will decrease. Thus, if we decrease the acceptability of the rumor, density of populations who accept the rumor will also decrease.

3.1.2   Random Inoculation on Small World Network

A Random inoculation strategy inoculates a fraction of the nodes randomly, using no knowledge of the network. The g defines the fraction of inoculated nodes. In mean field level, in case of uniform inoculation, the initial conditions of Equations (3.1.3) -(3.1.6) will be modified as: $I(0) \approx 1 - g, S(0) \approx 0, R_{acc}(0) = 0, R_{rej}(0) = g$. Solving the Equations (3.0.1) -(3.1.7) under these initial conditions, following transcendental equation (Appendix A.2) is obtained.

$$R(\infty) = 1 - (1 - g)e^{\frac{\lambda+\sigma}{\sigma}g}e^{-\frac{\lambda+\sigma}{\sigma}R(\infty)} \tag{3.1.11}$$

For any desirable value of $R(\infty)$, one can always find a nonzero g using Equation (3.1.11), (Appendix A.4). Defining an auxiliary function using Equation (3.1.11).

$$f(R(\infty)) = 1 - (1 - g)e^{\frac{\lambda+\sigma}{\sigma}(g-R(\infty))} - R(\infty) \tag{3.1.12}$$

$$f'(R(\infty)) = \frac{\lambda + \sigma}{\sigma}(1 - g)e^{\frac{\lambda+\sigma}{\sigma}(g-R(\infty))} - 1$$

There are three possible cases,

Case I $(1 + \frac{\lambda}{\sigma})(1 - g) > 1$

If $0 < R < g$ then,

$$\frac{\lambda + \sigma}{\sigma}(g - R(\infty)) > 0. \tag{3.1.13}$$

Thus,

$$e^{\frac{\lambda + \sigma}{\sigma}(g - R(\infty))} > e^0 = 1. \tag{3.1.14}$$

Therefore,

$$f'(R(\infty)) > 0, \text{ as } \frac{\lambda + \sigma}{\sigma}(1 - g)e^{\frac{\lambda + \sigma}{\sigma}(g - R(\infty))} > 1. \tag{3.1.15}$$

Thus $f(R(\infty))$ is an increasing function for $0 < R(\infty) < g$.

This case can be understood by solving Equation (3.1.12) by graphical method (Figure 3.2). Equation (3.1.12) can be broken as,

$$
\begin{aligned}
y1 &= R(\infty), \\
y2 &= 1 - (1 - g)e^{\frac{\lambda + \sigma}{\sigma}(g - R(\infty))}, \\
y &= y2 - y1.
\end{aligned}
\tag{3.1.16}
\tag{3.1.17}
$$

Plotting the above, we get Figure 3.2.

In Figure 3.2, we get $R(\infty) = g$ as one of the solutions. At this point the curves for $y1$ and $y2$ are intersecting and slope of y is positive. We will also get another solution when $R(\infty) > g$ and curve y starts decreasing and again cuts the $x - axis$ (since the value of curve y will go down after getting some maximum value). We can show that y is less than 1 for $R(\infty) = 1$ and hence the curve y has to cross x axis at some points s.t. $R(\infty) > g$.
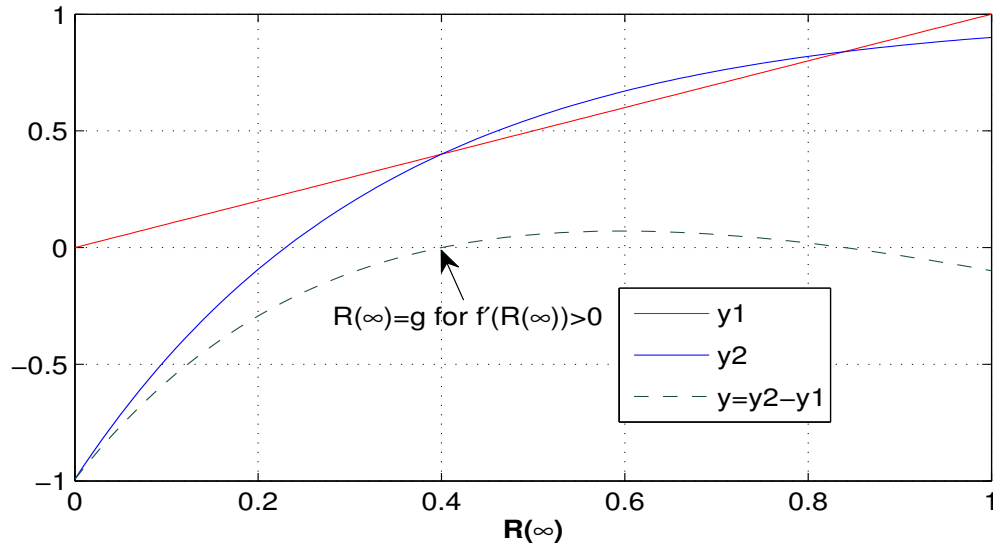
Figure 3.2: Graphical solution for case I, for $g = 0.3$.

Case II

Similarly, if $(1 + \frac{\lambda}{\sigma})(1 - g) < 1$ then $f(R(\infty))$ will be a decreasing function for $g < R(\infty) < 1$.

This case can be understood by solving Equation (3.1.12) after considering Equations (3.1.17) (Figure 3.3).

In Figure 3.3, we got one of the solution, $R(\infty) = g$ when curve y is intersecting x axis with negative slope. If value of y at $R(\infty) = 0$ is less than zero, then another solution which is less than g exists. In case value of y at $R(\infty) = 0$, is more than zero, then $R(\infty) = g$ will be the only solution, as the other solution will be invalid as $0 \leq R(\infty) \leq 1$.
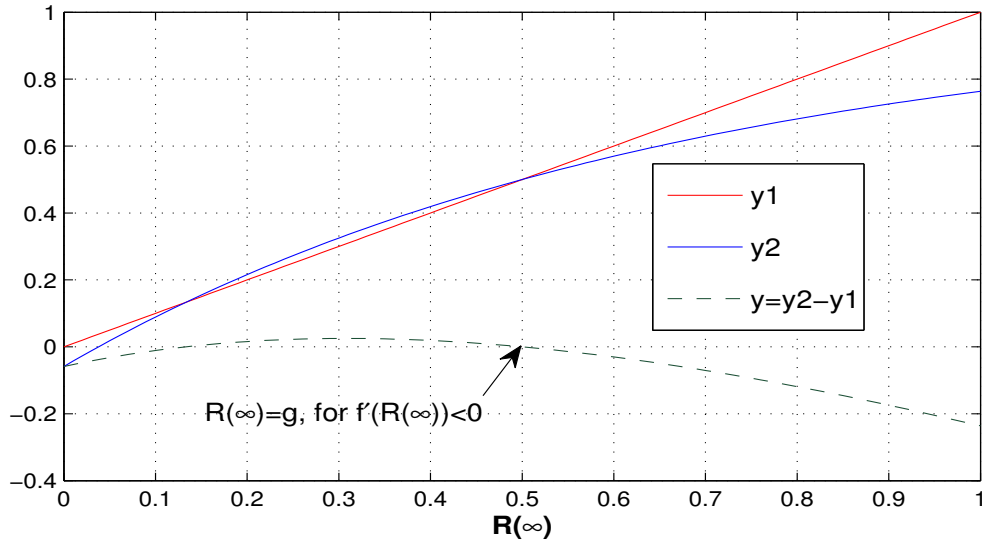
Figure 3.3: Graphical solution for case II, for $g = 0.7$.

Case III

If $(1 + \frac{\lambda}{\sigma})(1 - g) = 1$, then we will get only one solution identified as critical value of g. We can represent this as g_c. If $g < g_c$, then $(1 + \frac{\lambda}{\sigma})(1 - g) > 1$. This scenario is case I and one of the solutions is $R(\infty) = g$. The other solution will be higher than g and will be largest.

As g is the only nonzero solution of Equation (3.1.11) with $(1 + \frac{\lambda}{\sigma})(1 - g) = 1$,

$$f'(R(\infty))|_{R(\infty)=g} = \frac{\lambda + \sigma}{\sigma}(1 - g) - 1 \leq 0$$

If $g > g_c$, then $(1 + \frac{\lambda}{\sigma})(1 - g) < 1$. This is same as case II and one of the solutions will be $R(\infty) = g$. This solution will be the largest (Figure 3.4). The final worst case solution will be largest solutions of all possible ones.
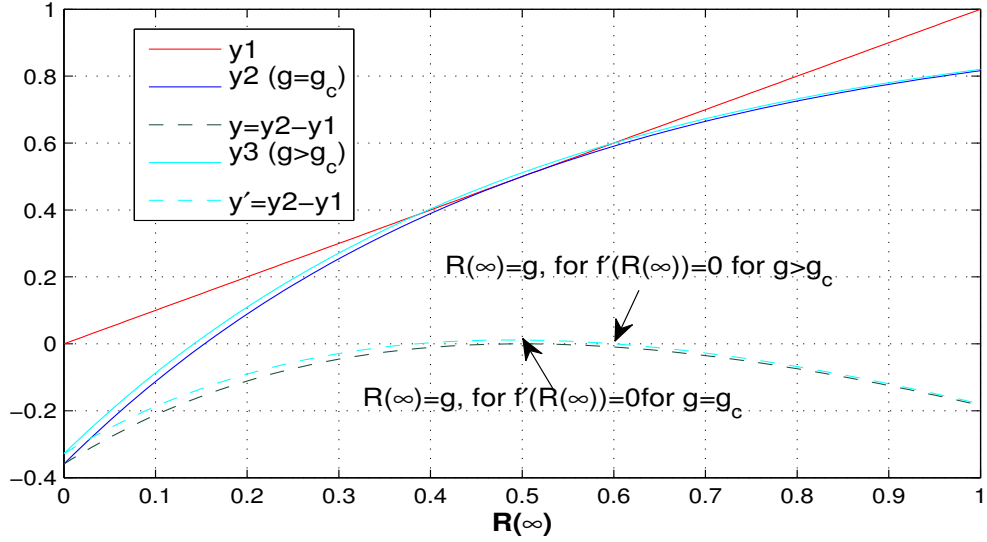
Figure 3.4: Solution for case III, for $g = g_c = 0.5$ and $g = 0.6$.

Therefore, $g_c = \frac{\lambda}{\lambda+\sigma}$ is the critical fraction of inoculation. When, $g > g_c$, $R(\infty) = g$, is only the nonzero solution of Equation (3.1.11). Therefore, $R_{acc}(\infty) = R(\infty) - R_{rej}(\infty) = R(\infty) - g \equiv 0$. This model shows that, by using random inoculation the density of stifler who accept the rumor can be brought down to zero. For the other values of g ($g < g_c$), we will get another solution for $R(\infty)$, for which $R(\infty) > g$. Therefore, $R_{acc}(\infty) = R(\infty) - R_{rej}(\infty)$ will have some value greater than zero (Figure 3.5).

The above analysis can be validated by numerical simulations on WS model. Let N=10,000, $\langle k \rangle = 4$ and rewiring probability (P_r)=0.8. Set the other parameters as, $\lambda = 0.25, \eta = 0, \rho = 0$ and $\sigma = 0.25$. These simulations are performed for 100 different initial configurations of proposed rumor models on at least 10 different realizations of WS model. Results are shown in Figure 3.5, it can be analyzed that if fraction of inoculation g increases, the fraction of stifler who accept the rumor decreases monotonically. For the values of g such that $g < g_c$, we will get $R(\infty) > g$ as a worst case solution as per
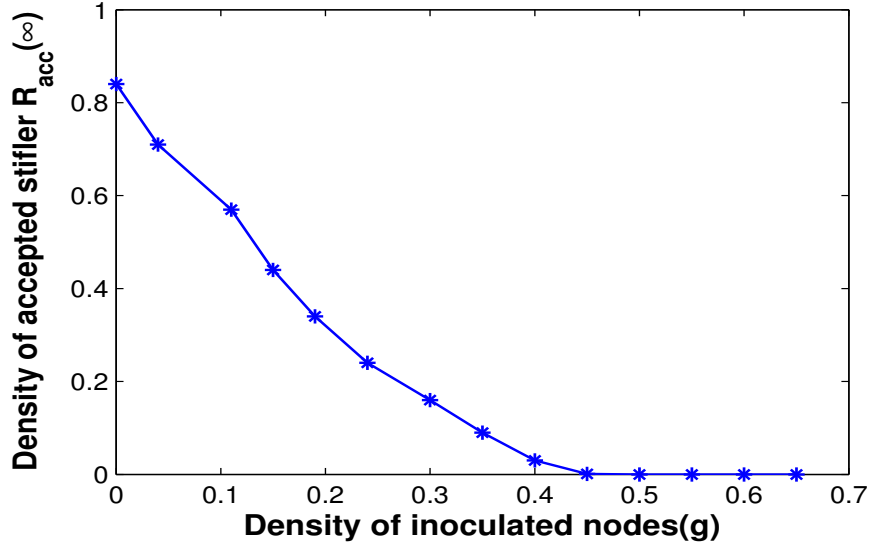
Figure 3.5: Comparison between R_{acc} and fraction of inoculation nodes g in random inoculation with a small average degree of 4.

case I. After the critical fraction, the both solution of $R(\infty)$ will be merged into a single solution $R(\infty) = g$. In Figure 3.5, critical inoculation g_c is approximately 0.5 , which is in agreement with the calculated value of $g_c = \frac{\lambda}{\lambda+\sigma} = 0.5$ considered.

By this analysis, it can be seen that critical inoculation g_c does not depends on $\langle k \rangle$ in homogeneous networks. In [48], it has been shown that the condition $0 \leq (\lambda + \eta + \rho)\langle k \rangle S(t) \leq 1$ and $0 \leq \sigma\langle k \rangle(S(t) + R_{acc}(t) + R_{rej}(t)) \leq 1$ should be satisfied for the mean field equations representing the proposed model. Therefore, when $\langle k \rangle$ is too large the rumor equations will not be valid and g_c cannot be calculated using $\frac{\lambda}{\lambda+\sigma}$. Too large fraction of nodes will need to be inoculated to stop the rumor. It can be decreased by decreasing the acceptability factor $(1/\rho)$. The simulation has been again performed on WS model with same parameters as used earlier for different values of ρ, $0 \leq \rho \leq 0.75$. The results are plotted in Figure 3.6. It has been observed from Figure 3.6 that when $\lambda = 0.25, \eta = 0, \rho = 0, \sigma = 0.25$, analysis will fail to calculate
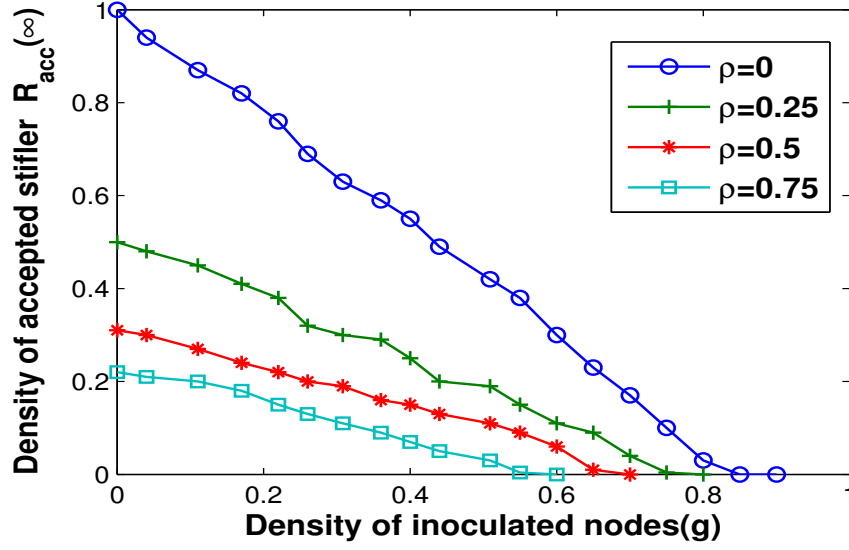
Figure 3.6: Comparison between R_{acc} and density of inoculation nodes g in random inoculation with large degree 12.

correct g_c. The obtained g_c is approx 0.85 which is much higher than the previously obtained value of 0.5 for $\rho = 0$. If mean field rate equations are valid, then constraint $0 \leq \sigma \langle k \rangle (S(t) + R_{acc}(t) + R_{rej}(t)) \leq 1$ should be satisfied, while for $\sigma = 0.25, \langle k \rangle = 12$ and $S(0) + R_{acc}(0) + R_{rej}(0) = g_c = 0.5$, we have $\sigma \langle k \rangle (S(0) + R_{acc}(0) + R_{rej}(0)) = 1.5 > 1$. Therefore mean field rate equation will not be accurate when $\langle k \rangle$ is large and random inoculation is no longer efficient. The problem can be solved by increasing the parameter ρ (decrease the acceptability of rumor) and apply random inoculation method at same time. In Figure 3.6, when $\rho = 0.75$ and $g=0.55$, $R_{acc}(\infty)$ near to zero, which means that the given model is valid for larger values of ρ ($\rho > 0.75$).

3.1.3 Targeted Inoculation on Small World Network

To arrest the spread of rumors in heterogeneous networks (e.g., scale free networks), targeted inoculation were introduced. If we have information about degrees of all nodes,

we may rank nodes by degree, and use targeted inoculation to inoculate nodes in order of descending degree. When a high-degree node is inoculated, the effective degree of its neighbors drop. This inoculation strategy is more effective on heterogeneous networks e.g. scale free networks. The results for numerical simulations for targeted inoculation for a network built using WS model with same parameters as used earlier, are given in Figure 3.7. It can be analyzed from Figure 3.7 that the target inoculation
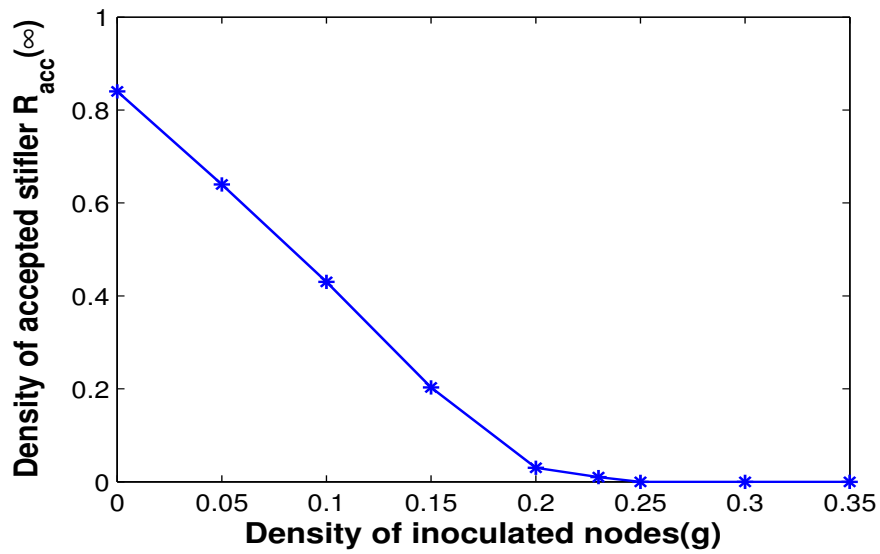
Figure 3.7: Comparison between R_{acc} and density of inoculation nodes g in targeted inoculation with small degree 4.

is better than the random inoculation. Here, g_c is approx 0.25 and rumor spreading is almost zero for inoculation $g \geq 0.25$. The degree distribution in WS model is Poisson degree distribution, which is not strictly homogeneous and possesses some heterogeneity property. If a small world network is strictly homogeneous then random inoculation will be equivalent to targeted inoculation.

When $\langle k \rangle$ is high, Figure 3.8 shows the simulation results for WS model with $\langle k \rangle = 12$ and all parameters as take earlier. ρ has been varied to determine its impact. When

$\langle k \rangle$ is large, targeted inoculation is not effective. Like before, the rumor spreading can be suppressed by increasing the parameter ρ with targeted inoculation also.
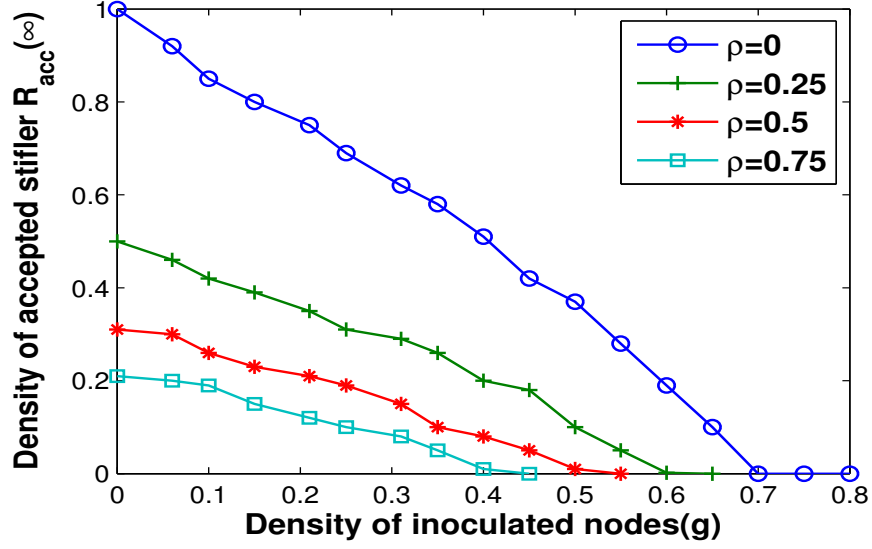
Figure 3.8: Comparison between R_{acc} and density of inoculation nodes g in targeted inoculation with large degree 12.

3.2 Rumor Dynamics with Acceptability Factor in Heterogeneous Networks

3.2.1 Proposed Rumor Spreading Model

An ignorant node with degree k is influenced by informed neighbors, and the average density of informed neighbors over connectivity class l is $P(l|k) = lP(l)/\langle k \rangle$ [3]. Here, we are considering uncorrelated networks only where conditional probability satisfies

$P(l|k) = lP(l)/\langle k \rangle$. The rate equations for the rumor diffusion model are,

$$\frac{dI(k,t)}{dt} = -k(\lambda + \rho + \eta)I(k,t)\sum_l P(l|k)S(l,t), \tag{3.2.18}$$

$$\frac{dS(k,t)}{dt} = \lambda k I(k,t)\sum_l P(l|k)S(l,t) - k\sigma S(k,t)\sum_l (S(l,t) + R_{acc}(l,t) +$$
$$R_{rej}(l,t))P(l|k) - \delta S(k,t), \tag{3.2.19}$$

$$\frac{dR_{acc}(k,t)}{dt} = \sigma k S(k,t)\sum_l (S(l,t) + R_{acc}(l,t) + R_{rej}(l,t))P(l|k) +$$
$$\eta k I(k,t)\sum_l P(l|k)S(l,t) + \delta S(k,t), \tag{3.2.20}$$

$$\frac{dR_{rej}(k,t)}{dt} = \rho k I(k,t)\sum_l P(l|k)S(l,t). \tag{3.2.21}$$

The Equation (3.2.31) can be integrated to get,

$$I(k,t) = I(k,0)exp\left(\frac{-k(\lambda + \rho + \eta)}{\langle k \rangle}\Theta(t)\right). \tag{3.2.22}$$

Here, $I(k,0)$ is the initial fraction of ignorant nodes with degree k and
$\Theta(t) = \int_0^t \sum_l S(l,t')P(l)ldt'$. The initial conditions are taken as, $I(k,0) \approx 1$, $S(k,0) \approx 0$, $R_{acc}(k,0) \approx 0$ and $R_{rej}(k,0) \approx 0$. At $t \to \infty$ i.e., at the end of rumor spread, $S(k,\infty) = 0$ as system achieves steady state, and consequently, $\lim_{t \to \infty} d\Theta(t)/dt = 0$. After solving Equations (3.2.31)-(3.2.21) to leading order in σ,

$$\Theta = \frac{\left((\lambda + \rho + \eta)\frac{\langle k^2 \rangle}{\langle k \rangle} - \delta\right)}{\lambda^2 \frac{\langle k^3 \rangle}{\langle k \rangle}(1/2 + \sigma\delta\frac{\langle k^2 \rangle}{\langle k \rangle}I)}. \tag{3.2.23}$$

Here, $\lim_{t\to\infty} \Theta(t) = \Theta$ and I is a finite positive integral in the form $I = \int_0^t e^{\delta(t-t')} f(t') dt'$.
Hence rumor threshold can be calculated for positive value of Θ from Equation (3.2.23),

$$\frac{(\lambda + \rho + \eta)}{\delta} \geq \frac{\langle k \rangle}{\langle k^2 \rangle} \tag{3.2.24}$$

$$\tag{3.2.25}$$

It has been assumed that after each time step spreaders are going into stifler state
spontaneously, i.e. $\delta = 1$. Therefore critical rumor threshold is

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle} - \rho - \eta. \tag{3.2.26}$$

When ignorants are not converting into stifler state directly after being contacted by
spreaders then $\rho = 0$ and $\eta = 0$, in this case, the critical rumor threshold is

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}. \tag{3.2.27}$$

In other words, if the value of λ is above the threshold, $\lambda \geq \lambda_c$, the rumor can spread in
the network. For $\lambda < \lambda_c$, the rumor dies out exponentially. This finding also suggests
that in infinite scale free networks with, $2 < \gamma \leq 3$, for which, $\langle k^2 \rangle \to \infty$, we have
$\lambda_c = 0$. The final size of rumor spread is given by the fraction of nodes which hear the
rumor by $t \to \infty$ i.e. ,

$$
\begin{aligned}
R(\infty) &= \sum_l P(l) R(l, \infty), &\tag{3.2.28} \\
R(\infty) &= R_{acc}(\infty) + R_{rej}(\infty), &\tag{3.2.29} \\
&= \sum_l P(l)\left(1 - exp\left(\frac{-(\lambda + \rho + \eta)l\Theta}{\langle k \rangle}\right)\right). &\tag{3.2.30}
\end{aligned}
$$

From Equations (3.2.21) and (3.2.28)-(3.2.30), we obtain (Appendix A.5) the following.

$$R_{acc}(\infty) \;\; = \;\; = \frac{\lambda + \eta}{\lambda + \eta + \rho}\left[1 - \sum_l P(l)exp\left(\frac{-(\lambda + \eta + \rho)l\Theta}{\langle k \rangle}\right)\right], \qquad (3.2.31)$$

$$R_{rej}(\infty) \;\; = \;\; \frac{\rho}{\lambda + \eta}R_{acc}(\infty), \qquad (3.2.32)$$

$$R_{rej}(\infty) \;\; = \;\; \frac{\rho}{(\lambda + \eta + \rho)}\left[1 - \sum_l P(l)exp\left(\frac{-(\lambda + \eta + \rho)l\Theta}{\langle k \rangle}\right)\right]. \qquad (3.2.33)$$

After observing the Equations (3.2.31)-(3.2.33), we can easily understand that the size of population of nodes with accepted rumor can be decreased by increasing the ρ (i.e. decreasing the rumor acceptability factor) and keeping the other parameters fixed.

3.2.2 Inoculation Strategies

The inoculation strategy is same as the site percolation problem. Each inoculated node can be seen as a site which is removed from the network. The target of the inoculation strategy is to get the percolation threshold, aimed to minimize the infecting nodes [23].

3.2.2.1 Random Inoculation in Scale Free Networks

In random inoculation (RI) strategy, randomly selected node will be inoculated. In this approach, a fraction of the nodes are inoculated randomly without any information of the network [54]. Here, the variable g $(0 \leq g \leq 1)$ defines the fraction of inoculated nodes. At the mean-field level, the presence of random inoculation will effectively reduce the spreading rate λ by a factor $(1 - g)$. In scale free networks, almost 80-90 % nodes need to be inoculated to suppress the rumor.

Table 3.1: Number of deleted edges after random, targeted and neighbor inoculation.

Removed fraction(g)	$e_{del}(RI)$	$e_{del}(TI)$	$e_{del}(NI)$
0.03	541	2833	990
0.06	981	3468	1870
0.08	1348	3980	2443
0.1	1569	4642	2924
0.2	3122	5312	4117
0.3	4502	6248	5358

3.2.2.2   Targeted Inoculation in Scale Free Networks

Scale free networks permit efficient strategies that depend upon the hierarchy of nodes. It is known that SF networks shows robustness against random inoculation [12, 54]. It implies that the high fraction of inoculation of nodes can be resisted without loosing its global connectivity. But on the other hand, SF networks are strongly affected by targeted inoculation (TI) of nodes. In targeted inoculation, the high degree nodes have been inoculated progressively as they are more likely to spread the information. In SF networks, the robustness of the network decreases even with a tiny fraction of inoculated individuals. The SF network suffers an interesting reduction of its robustness to carry information.

3.2.2.3   Neighbor Inoculation in Scale Free Networks

For random inoculation, it is necessary to inoculate almost all nodes in the network to stop the rumor. The targeted inoculation is very effective but it needs the global information of the network. At least, the knowledge of most of the nodes with higher degrees is required. Because of large, complex and time varying social networks as well as Internet, it is very difficult to determine the target nodes. Therefore, Cohen *et al.* [56] proposed an inoculation strategy known as acquaintance immunization.

In the acquaintance inoculation strategy, some nodes are being selected in the network randomly by probability p from the N nodes. Then the neighbors are selected randomly by the selected nodes for inoculation. The probability that an specific neighbor node with k degree is selected for inoculation is $kP(k)/(N\langle k \rangle)$. In this inoculation strategy, we only need the information of randomly selected nodes and neighbor nodes attached with them. In scale free networks, if a node has been selected randomly then, the probability of choosing one of its neighbor nodes with higher degree is higher than a node with lesser degree [57]. In this chapter, this inoculation strategy is referred to as neighbor inoculation (NI).

3.2.3   Simulations and Results

The numerical simulations are done to observe the complete dynamical process with and without inoculation strategies with different spreading (λ), stifling (σ) and forgetting (δ) rates with the variation of rumor acceptability factor ($1/\rho$). In each time step, all the N nodes interact with their neighbors for rumor passing. After the interaction, all the N nodes, update their states according to the proposed rumor model and time step is incremented. Scale free networks are used for the contact process. The scale free networks have been generated according to the power law, $P(k) = k^{-\gamma}$, where $2 < \gamma \leq 3$. We have taken $N = 10000$ and $\gamma = 2.5$. The random inoculation is implemented by selecting gN nodes randomly in the network.

Similarly, targeted and neighbor inoculation have been studied. In the generated SF network, random, targeted and neighbor inoculation strategies have been applied and the number of deleted edges are given in Table 3.1 and Figure 3.9. After applying inoculation, new degree distribution of scale free network has been calculated. The

---

Algorithm 1 New degree distribution after Random inoculation.

Input: A 2-d edge array after inoculation
Input: Total number of nodes in the network(N) before inoculation
Input: Number of nodes after deletion (T_nodes)
Input: Fraction of nodes deleted in the network (del_fr)
Input: Number of edges in the network (T_edges) after inoculation
Output: Degree distribution after applying random inoculation
deg, $deg\_distrib$, $norm\_deg\_distb$, count, $k = 0$, $i = 0$; $count = 0$;
while $i < T\_edges$ **do**
   Put i^{th} node in the match=Edges[i][0]
   while $i < T\_edges$ **do**
     if match(i)==Edges[i][0] **then**
       count++ ; i++ ;
     else
       break
     end if
   end while
   degree of the match(i) in degree array, deg(match) = count
end while
for $i < T\_node$ **do**
   for $j < N$ **do**
     if deg[j]==i **then**
       k++
     end if
   end for
   deg_distb[i]=k
   deg_distb will contain the node of same degree
   $norm\_deg\_distb$ [i]=k/N
end for
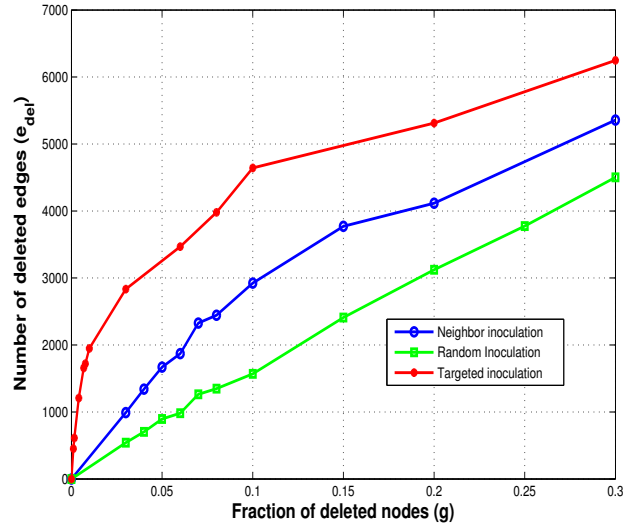
Figure 3.9: Number of deleted edges with fraction of inoculated nodes for different inoculation strategies.
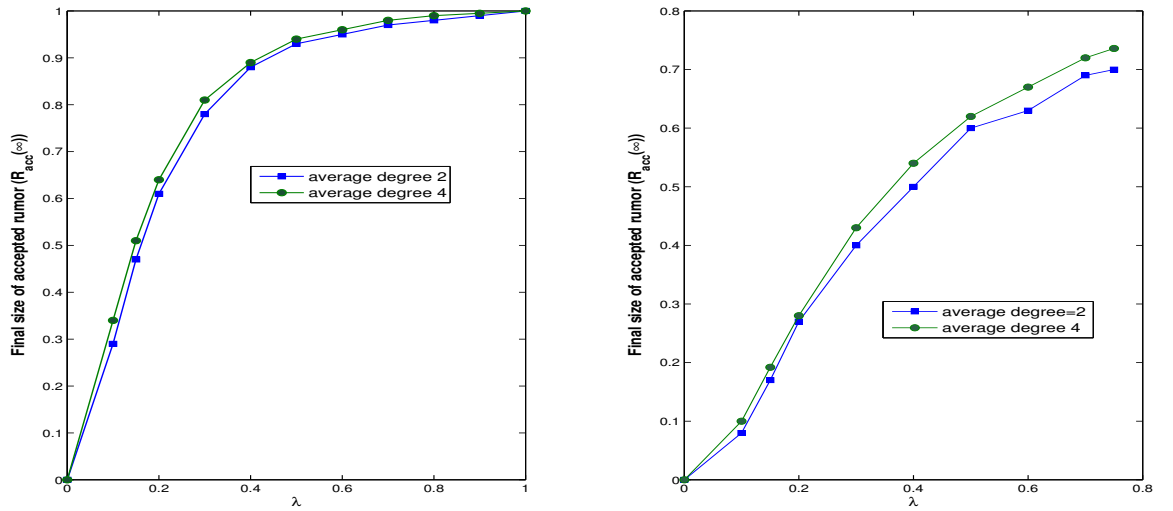
(a) $\rho = 0$ (b) $\rho = 0.25$

Figure 3.10: Final size of rumor as function of λ for $\delta = 0.2$.

algorithms used to find new degree distribution after applying inoculation strategies are similar to Algorithm 1. At the starting of each simulation, initially spreader nodes are chosen randomly, while all the other nodes are ignorants. In Figure 3.10, the threshold value of the spreading rate λ above which the rumor can spread widely, approaches almost zero with $\rho = 0.25$ and $\rho = 0$. After introducing ρ, final size of accepted rumor decreases for all the λ values in Figure 3.10(b). Figure 3.11 described the fraction of nodes for all the compartments as a function of time. If $\rho = 0$, the final size of nodes with accepted rumor is larger in comparison with the $\rho = 0.25$. In the scale free networks, initially the fraction of $R_{acc}(t)$ remains unchanged. After some time steps, the number of nodes with accepted rumor increases exponentially to a high level and finally reaches the steady state. The rumor will spread to more number of nodes in a scale free networks due to smaller average shortest path length.

(a) $\rho = 0$ (b) $\rho = 0.25$
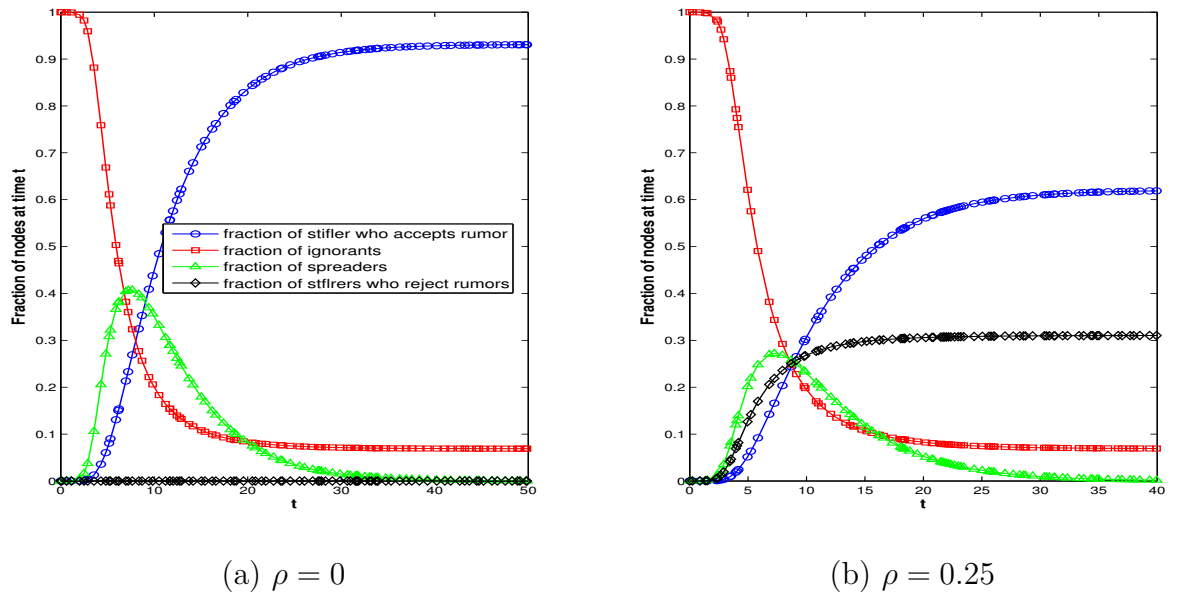
Figure 3.11: Time plots for fraction of ignorants, spreaders and stifler with accepted and rejected rumor.

In Figure 3.12, the relaxation time (time to get steady state for rumor spreading)

of the rumor dynamics has been plotted with different spreading rates. If relaxation time is smaller, then rumor can spread to other nodes within a less time. Comparable spreading and stifling rates lead to a large relaxation time, such as $\lambda = 0.1$ and $\delta = 0.3$ when the average degree is 2.
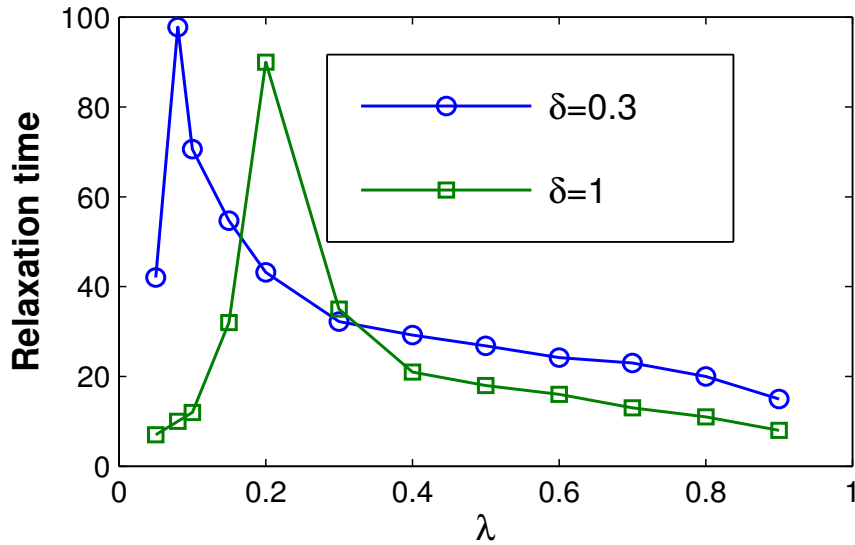
Figure 3.12: Relaxation time as a function of λ.

The final size of accepted rumor plotted against λ for random, targeted and neighbor inoculation strategies for $g = 0.1$ is given in Figure 3.13. The rumor threshold for spreading rate λ is found to be largest in targeted inoculation and smallest in random inoculation. The rumor threshold in neighbor inoculation is found between targeted and random inoculation. It shows that after applying inoculation strategies on some fraction of nodes, we can control the rumor in scale free networks. If degree of nodes are not known in scale free networks then we can apply neighbor inoculation to control rumor in scale free networks. The random inoculation strategy is not much successful in the case of scale free networks.
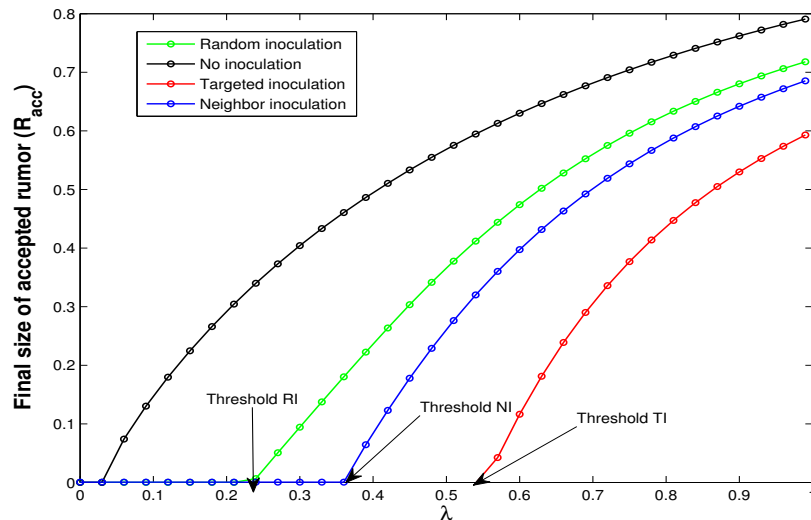
Figure 3.13: Final size rumor as a function of λ for $g = 0.1$ inoculations.

Although inoculation scheme is successful to control the rumor, but according to our investigations the rumor can be suppressed more efficiently by decreasing the rumor acceptability factor $(1/\rho)$. In Fig. 3.14, random inoculation strategy has been applied and final size of rumor has been calculated. It has been found that we need to inoculate approx. 75% of nodes to stop the rumor for $\rho = 0$. We have changed the acceptability factor to improve the random inoculation. We less number of nodes are needed to inoculate after decreasing the acceptability factor $(1/\rho)$ (50% for $\rho = 0.7$).

Similarly, for neighbor inoculation we have calculated final size of rumor with the different fraction of inoculated nodes. Here, 65% of nodes are needed to inoculate to stop the rumor for $\rho = 0$ for arresting the rumor Fig. 3.15. It can be improved after decreasing the acceptability factor (40 % for $\rho = 0.7$).

In the case of targeted inoculation, we have needed to inoculate very less number of nodes (27 %) to stop the rumor for $\rho = 0$ Fig. 3.16 . In the case of scale free networks,
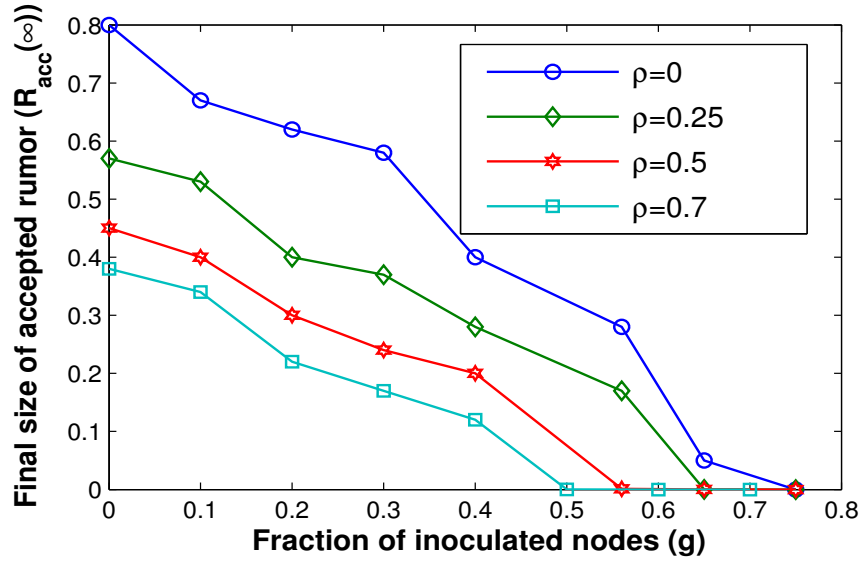
Figure 3.14: Final size of accepted rumor with the fraction of randomly inoculated nodes(g) with the variation of acceptability factor for $\lambda = 0.4$, $\sigma = 0.25$ and $\delta = 0.3$.
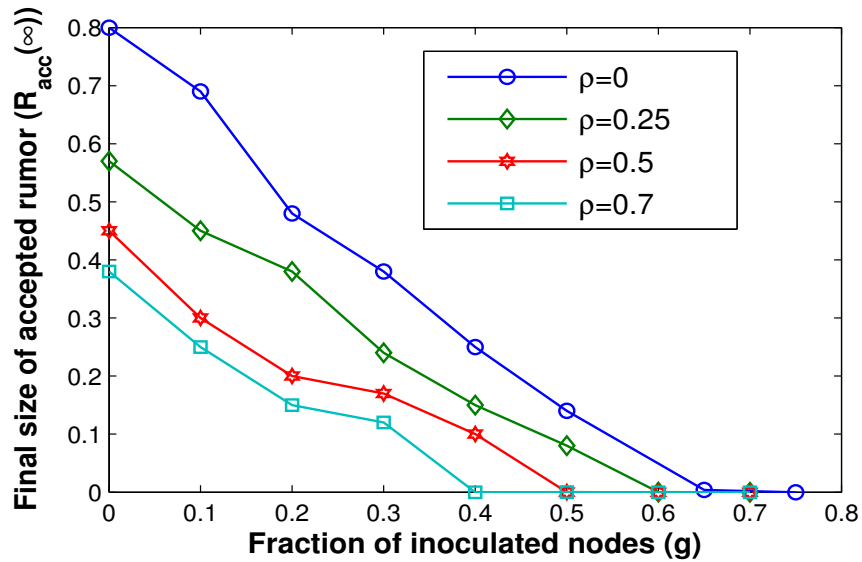
Figure 3.15: Final size of accepted rumor with the fraction of neighbor inoculated nodes(g) with the variation of acceptability factor for $\lambda = 0.4$, $\sigma = 0.25$ and $\delta = 0.4$.

targeted inoculation is very much effective itself. After decreasing the acceptability factor we can improve it slightly (15% for $\rho = 0.7$).
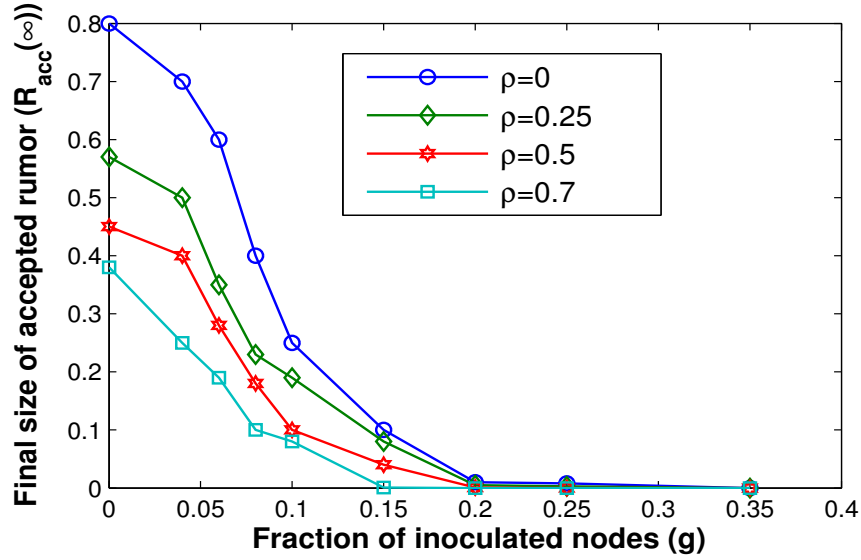
Figure 3.16: Final size of accepted rumor as a function of targeted inoculated nodes(g) with the variation of acceptability factor for $\lambda = 0.4$, $\sigma = 0.25$ and $\delta = 0.4$.

3.2.4   Conclusions

In this chapter, a new compartment of nodes viz. the stifler who rejects the rumor with rate ρ is added. In real, it is possible that a ignorant node after the meeting with spreader node can loose the interest in spreading the rumor, with or without accepting it. These nodes are similar to the inoculated nodes. The rumor acceptability factor $(1/\rho)$ has also been introduced. It is shown that the proposed model supports the small critical inoculation value g_c in the random as well as in the targeted inoculation to control rumor spreading when average degree $\langle k \rangle$ of small world network is small. It is also found that in targeted inoculation, g_c is smaller than in the case of random inoculation when degree $\langle k \rangle$ is small. It happens even when the degree distribution in

the small world networks is relatively uniform. The mean field approximation fails if $\langle k \rangle$ is high. In this case, random or targeted inoculation alone will not be effective. Therefore, one should decrease the acceptability of the rumor and apply either random or targeted inoculation method at the same time. After doing this, we got a small value of g_c to control rumor spreading, which was very high in the case of high acceptability ($\rho = 0$). The developed model does not require to inoculate large number of nodes. We have also investigated the rumor diffusion mechanism in scale free networks with this new compartment of stifling nodes who reject the rumors. After decreasing the rumor acceptability factor, the population who reject the rumor increases. Random, targeted and neighbor inoculation strategies have been applied in the proposed model to control the rumor. If the degrees of nodes are known, then targeted inoculation strategy is found to be the best for scale free networks. But, if there is no global information about the scale free networks, then neighbor inoculation strategy can be applied which is better than the random inoculation to control the rumor. It has also been observed that decrease in the rumor acceptability factor (increasing the value of ρ) makes the inoculation more effective in controlling the rumor in scale free networks.

Chapter 4

Rumor Dynamics with Inoculations for Correlated Scale Free Networks

4.1 Introduction

Rumors spread by pairwise contacts between nodes in the scale free networks with some spreading rate. Previous research [26, 64] shows that complex networks display degree-degree correlations. The connectivity of any two nodes in the real-world network is influenced by the degree-degree correlation existing in the network. When high (or low) degree vertices preferably connect to high (or low) degree vertices, it is called assortative mixing. On the other hand, when high degree vertices prefer to attach with less connected ones, it is called disassortative mixing. Recent studies show that social networks display assortative degree correlations e.g. Facebook, implying that highly connected vertices preferably connect to vertices which are also highly connected [26]. In order to study the impact of such assortative correlations on the dynamics of the rumor spreading model, the degree-degree correlation function is considered to investigate the impact of assortative degree correlations on the speed and size of the rumor spreading in scale free networks. It is interesting to note the influence of correlations on the final

size of rumor depends very much on the rumor spreading rate. This chapter describes the dependence of the rumor threshold in the scale free networks on their assortativity properties after considering the SIR model for rumor spreading [1].

4.2 Modified Rumor Spreading Model

For scale free networks, the degree distribution, $P(k) \propto k^{-\gamma}$ alone does not define the topology of the network completely. It does not tell anything about the vertices that are connected to each other. A correlated network is completely defined by its degree distribution $P(k)$ and its degree-degree correlation matrix $P(k, l)$ which defines the probability of finding an edge, emerging from a k degree node to a l degree node. The correlation matrix can defined as

$$\begin{pmatrix} P(1,1) & P(1,2) & ... & P(1, k_{max}) \\ P(2,1) & P(2,2) & ... & P(2, k_{max}) \\ . & . & . & . \\ . & . & . & . \\ P(k_{max}, 1) & P(k_{max}, 2) & ... & P(k_{max}, k_{max}) \end{pmatrix}.$$

Where, $(2 - \delta_{kl})P(k, l)$ is the probability that a randomly chosen edge connects two vertices of degree k and l respectively [33]. In case of undirected networks the probability that node i and node j connected is same as, node j and node i connected,

$$P(i, j) = P(j, i), \text{ for undirected networks}$$

According to reference [33],

$$P(l|k) = \frac{P(k, l)}{q_k}, \tag{4.2.1}$$

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[1]**Anurag Singh** and Y. N. Singh, "Rumor dynamics with inoculations for correlated scale free networks," in *Communications (NCC), 2013 National Conference on*, 2013, pp. 1–5.

where, $q_k = \frac{kP(k)}{\langle k \rangle}$ and,

$$P(k,l) = q_k[r\delta_{kl} + (1-r)q_l] \tag{4.2.2}$$

$$P(l|k) = \frac{P(k,l)}{q_k} = r\delta_{kl} + (1-r)q_l \tag{4.2.3}$$

Where, δ_{kl} is the Kronecker delta function and r is the assortativity coefficient. It is, $0 \le r \le 1$ for assortative networks, $r = 0$ for uncorrelated network and $r = 1$ for full assorted network. Therefore, for the social network r will be positive [26].

Now, rumor equations in classic rumor spreading model can be modified as,

$$
\begin{aligned}
\frac{dI(k,t)}{dt} &= -\frac{\lambda\langle k\rangle}{P(k)}I(k,t)\sum_l S(l,t)P(k,l) \\
\frac{dS(k,t)}{dt} &= \frac{\lambda\langle k\rangle}{P(k)}I(k,t)\sum_l S(l,t)P(k,l) - \frac{\sigma\langle k\rangle}{P(k)} \times \\
&\quad S(k,t)\sum_l [S(l,t) + R(l,t)]P(k,l) - \delta S(k,t) \\
\frac{dR(k,t)}{dt} &= \frac{\sigma\langle k\rangle}{P(k)}S(k,t)\sum_l [S(l,t) + R(l,t)]P(k,l) \\
&\quad +\delta S(k,t)
\end{aligned}
\tag{4.2.4}
$$

4.3   Targeted Inoculation

It is assumed that fraction g_k of nodes with degree k is successfully inoculated. All the nodes with degree $k > k_t$ get inoculated i.e. $g_k = 1$. The fraction of inoculated nodes is given by,

$$
g_k = \begin{cases} 1, k > k_t, \\ f, k = k_t, \\ 0, k < k_t. \end{cases}
$$

Where, $0 < f \leq 1$ and k_t is the cut-off degree.

Initial degree distribution is denoted by $P(k)$ with network size N. The new degree distribution after application of targeted inoculation is given as [62],

$$P'(k) = \sum_{q=k}^{\infty} \binom{q}{k} \Omega_q^{q-k} (1 - \Omega_q)^k p_q. \tag{4.3.5}$$

After applying inoculations, we can categorize all the nodes of network into two sets: one is the set of inoculated nodes and other is the set of non inoculated nodes. The probability to find a node of degree k in the non inoculated set of nodes in the network is defined by p_k. Ω_l is the probability of finding an edge from non inoculated set of nodes with the l degree to any node in the set of inoculated nodes. Both probabilities, Ω_l and p_k are defined by

$$\Omega_l = \frac{N\langle k \rangle (1 - g_l) \sum_k P(l, k) g_k}{l N P(l)(1 - g_l)} \tag{4.3.6}$$

$$p_k = \frac{(1 - g_k)P(k)}{1 - \sum_i P(i)g_i} \tag{4.3.7}$$

4.4   Random Inoculation

In random inoculation strategy, randomly selected node will be inoculated. This approach inoculates a fraction of the nodes randomly, without any information of the network. Here, the variable g ($0 \leq g \leq 1$) defines the fraction of inoculative nodes. For random inoculation put $g_k = g$ in Equation (4.3.7). The final degree distribution in the correlated network is found to be same as in the uncorrelated network.

Table 4.1: Number of deleted edges after random and targeted inoculations.

Removed Fraction(g)	r	$e_{del}(RI)$	$e_{del}(TI)$
0.02	0	6408	14077
0.02	0.1	7070	14526
0.02	0.3	6675	14342
0.02	0.5	6271	14007
0.02	0.7	6877	14074
0.03	0	1461	11194
0.03	0.1	1192	11113
0.03	0.3	1160	10628
0.03	0.5	1163	10866
0.03	0.7	928	10575
0.06	0	1915	11928
0.06	0.1	1863	11874
0.06	0.3	1623	11654
0.06	0.5	1611	11491
0.06	0.7	1576	11329
0.20	0	7527	14342
0.20	0.1	7070	14256
0.20	0.3	6746	14092
0.20	0.5	6428	14045
0.20	0.7	6271	14077

4.5   Simulations and Results

The numerical simulations are studied to observe the complete dynamical process with and without inoculation strategies for different spreading rates (λ) and with the variation of assortativity coefficient (r) for correlated networks. Stifling rate (σ) has been fixed to be 0.25 and spontaneously rumor forgetting rate (δ) is fixed to be 1. After spread of rumor in a time step, the spreader will become stifler in next time step. At the starting of each simulation, initially spreader nodes are chosen randomly for rumor spreading model, in Equation (4.2.4), while all the other nodes are ignorants. In each time step, all the N nodes interact with each other for rumor passing. After N nodes update their states according to the proposed rumor model, time step is incremented.

Scale free networks are used for the contact process. The scale free networks have been generated according to the power law, $P(k) = k^{-\gamma}$, where $2 < \gamma \leq 3$. We have taken $N = 10000$ and $\gamma = 2.1$. The random inoculation is implemented by randomly selecting gN nodes in the network.

The degree distribution of the correlated scale free networks before and after application of the targeted (Figure 4.1) and random inoculation are shown in Figures 4.1-4.2, respectively. The degree distributions before and after the inoculation schemes has been plotted from the theory and simulation. Theoretical and the simulation results have been found to be similar. Table 4.1 shows the number of removed edges from the scale free network after applying random and targeted inoculations for different fraction of removed nodes and assortativity coefficients (r). Initially total number of edges are 16500 without any inoculation.

In Figures 4.3-4.5, final size of the rumor spread has been plotted against the rumor transmission rate, λ for no inoculations, random and targeted inoculation schemes for various assortativity coefficients $(r = 0, 0.1, 0.3, 0.5, 0.7)$.

The final size of rumor has been observed to be large with high values of r in correlated networks for spreading rate, $\lambda \geq 0.5$ without inoculations. Before this rate, the rumor size is higher in uncorrelated network $(r=0)$. Similar patterns are observed in the random (Figure 4.4) and targeted inoculations (Figure 4.5). It is interesting that the rumor threshold is very less for random inoculation scheme shown in Figure 4.4 for both correlated and uncorrelated networks and it is same for all the values of assortative coefficients (r). On the other hand for targeted inoculation in Figure 4.5, the rumor threshold is higher than the random inoculation and it increases with the
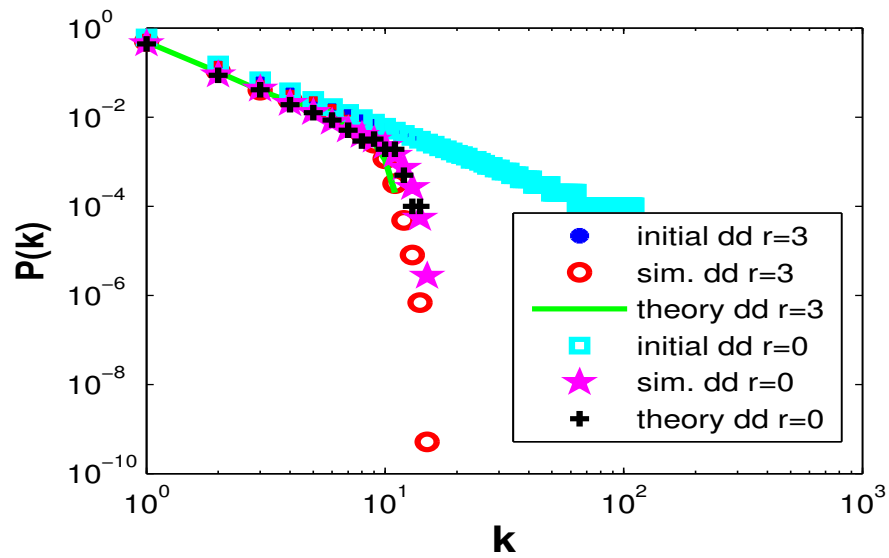
Figure 4.1: Degree distribution of correlated scale free network before and after applying targeted inoculation with cutoff degree=60.
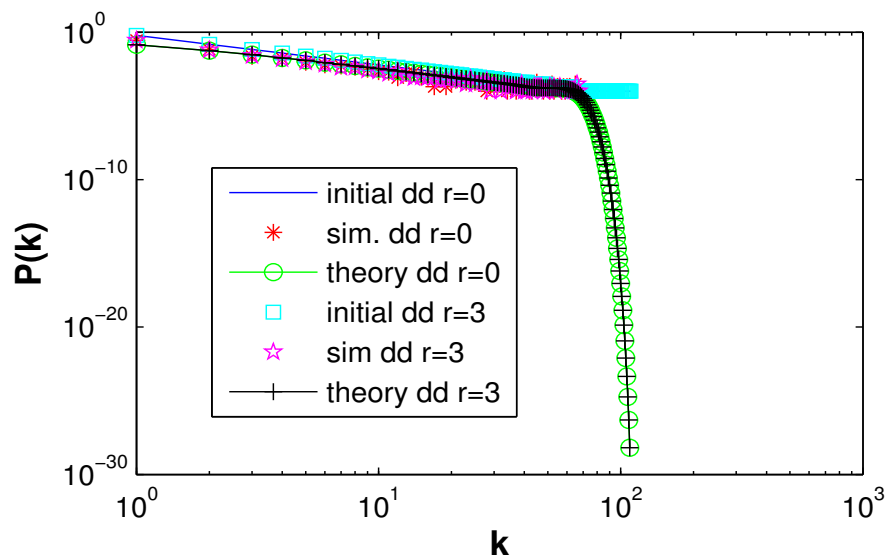
Figure 4.2: Degree distribution of correlated scale free network before and after applying random inoculation $g = 40 \%$.
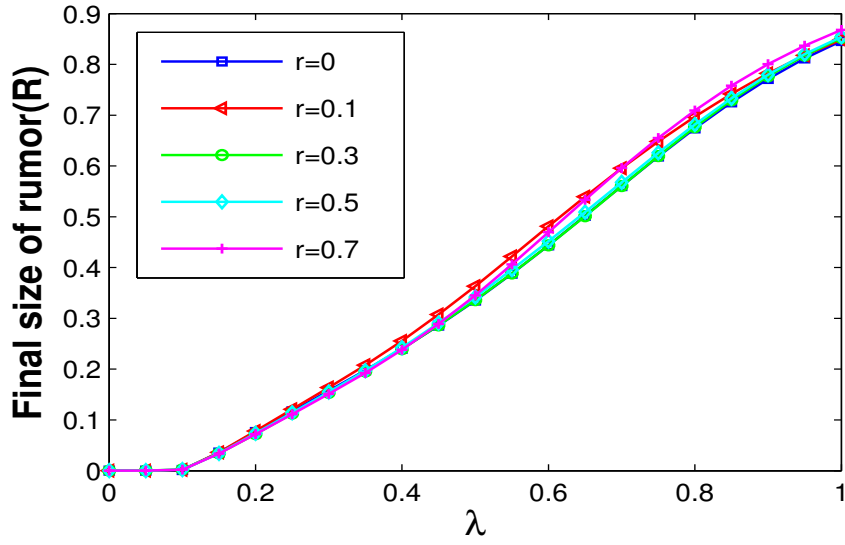
Figure 4.3: Final rumor size R vs λ plot for correlated and uncorrelated scale free networks without inoculations.

decrease of assortative coefficient (r). The maximum rumor threshold has been found in the uncorrelated scale free network.
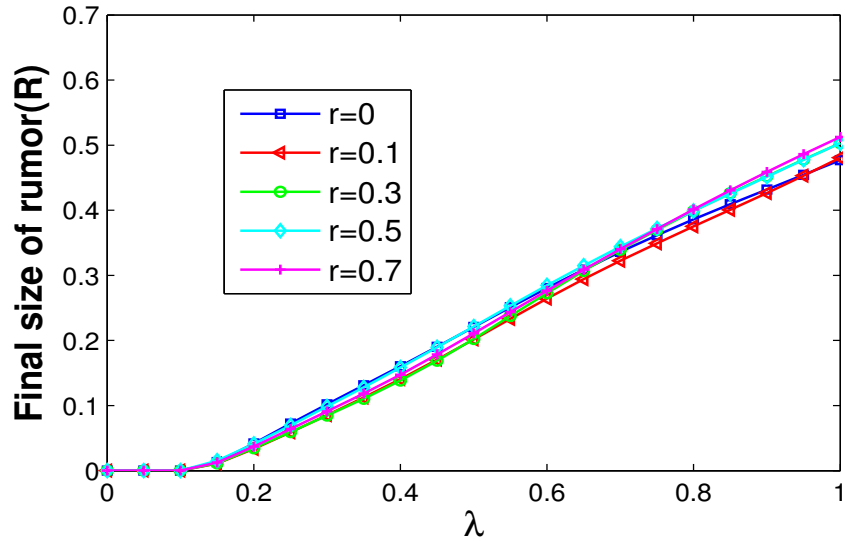
Figure 4.4: Final rumor size R vs λ plot for 25% random inoculation of nodes in correlated and uncorrelated scale free networks.
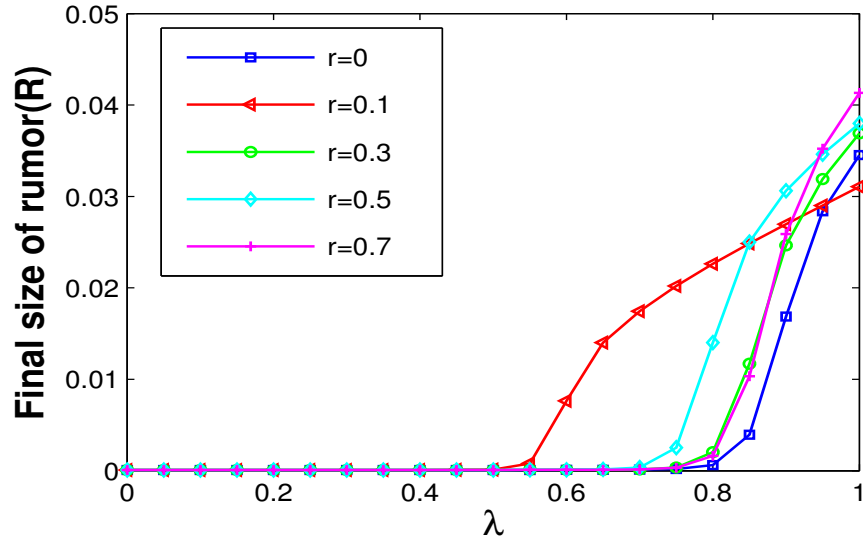
Figure 4.5: Final rumor size R vs λ plot for targeted inoculation with cut off degree $=10$ in correlated and uncorrelated scale free networks.

The sizes of informed nodes and spreader nodes observed with the time are shown in Figures 4.6-4.9. It is observed from Figures 4.6-4.9 that size of the informed nodes initially increases exponentially with the time. After some time they achieve steady state and remain forever in this state. The size of spreader initially increases with the increase in time. But after some time, when spreaders turn into stifler then the number of spreaders decreases with the increase in time. Therefore, after some time the size of spreaders will be zero as the system will attain steady state. $R(t)$ and $S(t)$ are plotted against time for scale free network with different assortativity coefficient (r) and 25% random inoculation in Figure 4.6 and 25% targeted inoculation in Figure 4.7, respectively. It can be easily observed that final size of rumor increases with the increase of r. Initially, spreading rate of rumor increases with the increment of r value and dies out early when correlations are stronger and rumor spreading is found almost zero in the targeted inoculations (Figure 4.6). Similar patterns have been found for the 10 % inoculation of nodes in random (Figure 4.8) and targeted (Figure 4.9).
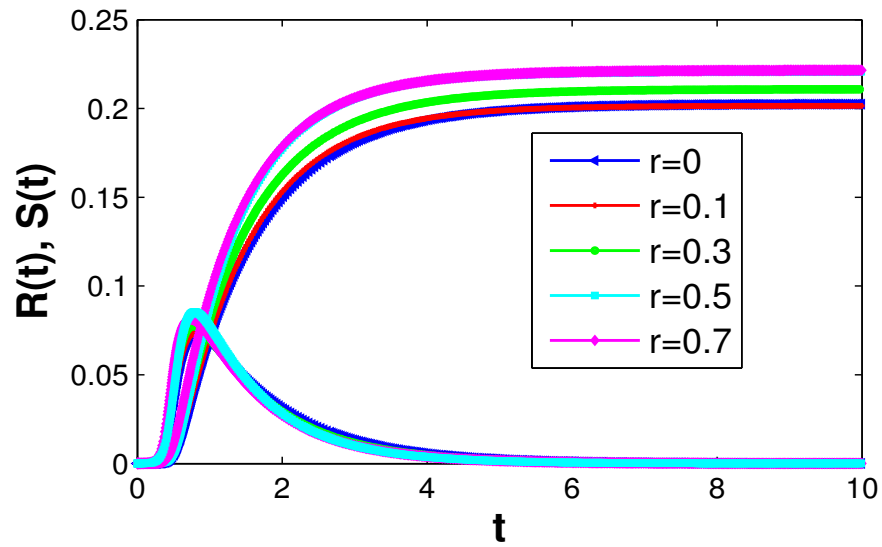
Figure 4.6: Size of rumor and spreaders with time for correlated networks with $g = 25\%$ random inoculations for $\lambda = 0.5$.
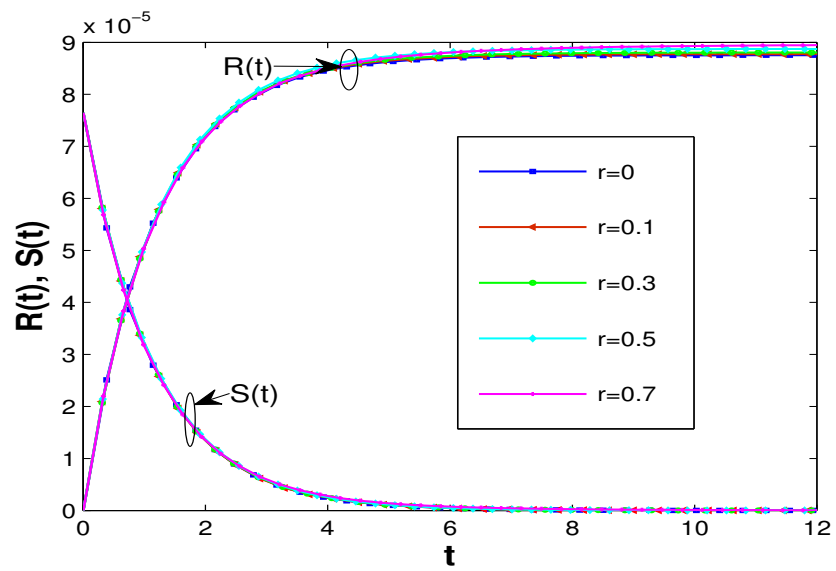
Figure 4.7: Size of rumor and spreaders with time for correlated networks with $g = 25\%$ targeted inoculations for $\lambda = 0.5$.
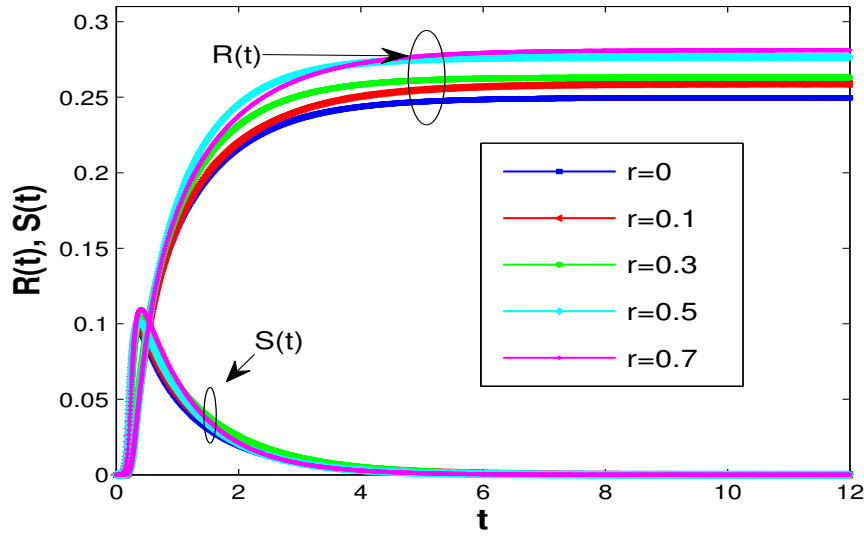
Figure 4.8: Size of rumor and spreaders with time for correlated networks with 10% random inoculations and $\lambda = 0.5$.

Here, in the size of rumor and spreaders, some perturbations are observed. When, g_k is less in targeted inoculations, the rumor spreads very fast initially because of hub neighbors. After some time steps, if spreaders will become more then the number of stifler will increase.

At that time, the number of spreaders will be less. But again in next time steps, hub ignorants neighbors can be found and same process of increasing and decreasing of spreaders happens. Therefore, we will get perturbations in the spreaders and stifler sizes with time. For the higher g_k in targeted inoculation scheme, correlated scale free networks will loose the heterogeneity and this scheme will be similar as random inoculation. Therefore, no perturbations is found in the size of spreaders and stifler.
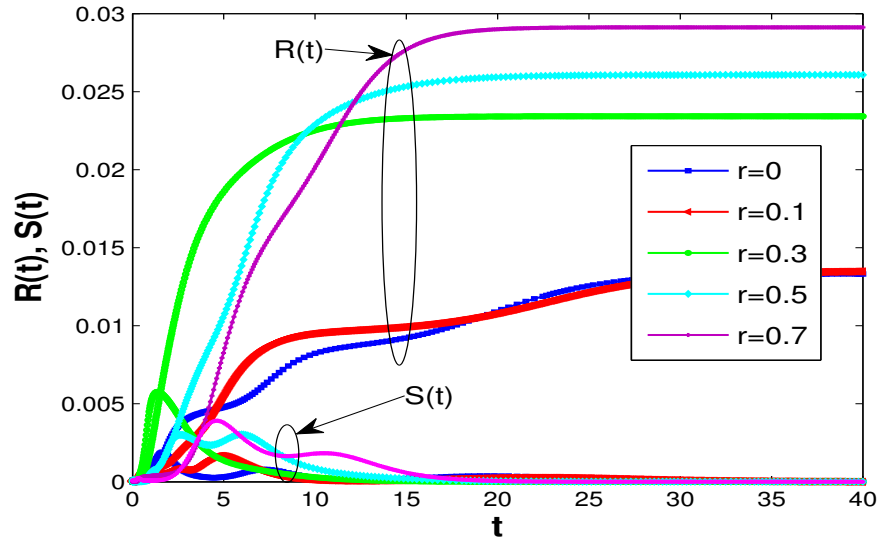
Figure 4.9: Size of rumor and spreaders with time for correlated networks with 10% targeted inoculations and $\lambda = 0.5$.

4.6 Conclusions

The rumor spreading model has been proposed for correlated networks. It has been observed that the rumors can be stopped more effectively in correlated networks using targeted inoculation. The degree distribution in the reduced network after random inoculation is found to be independent even if the degree-degree correlation is present or not. However, it does not hold true for targeted inoculations. The new degree distribution is generated after targeted and random inoculation schemes. It is interesting to observe that for small values of rumor transmission rate (λ), final size of rumor in uncorrelated networks is larger than in correlated one with or without inoculation. On the other hand, for higher values of transmission rate, rumor size is lower in uncorrelated scale free networks. It is concluded that removal of connections among hub nodes can decrease the rumor spreading.

Chapter 5

Nonlinear Rumor Spread with Degree dependent Tie Strength in Complex Networks

Yan *et al.* [65] have demonstrated that the asymmetry of infection plays an important role. They redistributed the asymmetry to balance the degree heterogeneity of the network and found the finite value of epidemic threshold. Zhou *et al.* [31, 66] have developed a susceptible-infected model with identical infectivity, where each node can contact with a constant number of neighbors at each time step. They concluded that this hypothesis is not always correct. In rumor spreading, the hub nodes have many acquaintances; however they can not contact all their acquaintances in single time step. They assumed that the rumor spreadness is not equal to the degree but identical for all nodes of the scale free networks and obtained the threshold, $\lambda_c = \frac{1}{A}$, where A is the constant infectivity of each node and is not equal to the degree of node. Recently, Fu *et al.* [67] have defined piecewise linear infectivity. They suggested if the degree k, of a node is small, its infectivity is $\alpha' k$, otherwise its infectivity is a saturated value A when k is beyond a constant A/α'. In both constant and piecewise linear infectivity, the heterogeneous infectivity of the nodes due to different degrees has not been considered.

While in scale free networks heterogeneity in nodal degree is very common. There may be nodes with different degrees, having the same infectivity, and there will be a large number of such nodes if infectivity does not saturate, or the size of network is infinite. In this chapter, we have investigated rumor spread in the scale free network considering the varying tie strengths between the nodes. Therefore, rumor spreading and stifling rate vary with tie strength of edges. Further, we have assumed that a non linearly varying number of neighbors are infected with the rumor in each time step. While in the earlier models [45, 3], tie strength has been considered to be uniform, and a constant number of neighbors has been assumed to be infected in each time step by each node. In the earlier models if a node has k neighbors, in each time step, all the k neighbors will be infected. We have modified the earlier SIR model given by Nevokee *et al.* [3] and included a rumor spreading exponent α . In this work, k^{α} neighboring nodes will be infected in each time step. Here, α is the rumor spreading exponent where, $0 < \alpha \le 1$. The tie strength between two nodes is $(k_i k_j)^{\beta}$ where, k_i and k_j are the degrees of nodes i and j, and β is tie strength exponent [1]. We have used Barabasi-Albert (BA) model [21] to create scale free networks with power law distribution of nodal degree, and then used the proposed strategy in them to study the rumor spread. Scale free networks have been specifically chosen as they are much more heterogeneous than the small world or the random network models, and thus a good candidate for testing our proposition.

5.1   Modified Rumor Spreading Model

In classic rumor spreading model, a node spreads rumors to all of its neighbor nodes in a single time step. But, it may be possible that a node can spread only some

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[1]**Anurag Singh** and Y. N. Singh, "Nonlinear spread of rumor and inoculation strategies in the nodes with degree dependent tie stregth in complex networks," *Acta Physica Polonica B*, vol. 44, no. 1, pp. 5–28, Jan 2013.

fraction of the neighbor nodes in single time step. The real world networks can have the intimacy, confidence etc. between the nodes. In the real network, a person can send the information only to some of his/her friends. Unlike previous studies where each node can spread the rumor with constant transmission rate λ, in this study, we have considered a rumor spreading model with varying rumor spreading rates. In a social network a person has many friends but he may have different intimacy with all his friends. Some of his friends may be very close to him, and some may be occasional friends. Therefore, the chances are greater to spread the rumor to his very close friends than occasional ones. Similar things can happen in call networks where a person may call higher number of duration than other friends. The transmission rate between two connected nodes has been considered as a function of their degrees. If $\Phi(l)$ represents the rumor spreadness, the number of neighbors contacted to spread the rumor in a single time step, for a node with degree l, λ_{lk} and σ_{lk} represents the rumor spreading rate and stifling rate from nodes of degree l to nodes with degree k, respectively. These spreading rates depend on the degrees of contacted nodes. Therefore, in the classic rumor spreading model, $P(l|k)$ can be replaced by $\frac{\Phi(l)P(l|k)}{l}$. Based on this assumption, we can write the rate equations as follows,

$$\frac{dI(k,t)}{dt} = -kI(k,t)\sum_l P(l|k)S(l,t)\frac{\Phi(l)}{l}\lambda_{lk}, \tag{5.1.1}$$

$$\frac{dS(k,t)}{dt} = kI(k,t)\sum_l P(l|k)S(l,t)\frac{\Phi(l)}{l}\lambda_{lk} - kS(k,t)\sum_l (S(l,t)$$
$$+R(l,t))P(l|k)\frac{\Phi(l)}{l}\sigma_{lk} - \delta S(k,t), \tag{5.1.2}$$

$$\frac{dR(k,t)}{dt} = kS(k,t)\sum_l (S(l,t)+R(l,t))P(l|k)\frac{\Phi(l)}{l}\sigma_{lk} + \delta S(k,t). \tag{5.1.3}$$

Where, $\Phi(l)$ represents the rumor spreadness of a node with degree l, λ_{lk} and σ_{lk} represents the rumor spreading rate and stifling rate from nodes of degree l to nodes

with degree k, respectively and $P(l|k)$ replaced by $\frac{\Phi(l)P(l|k)}{l}$.

5.1.1   Tie Strength in Complex Networks

The topological properties of a graph are fully encoded in its adjacency matrix **A** , whose elements a_{ij} $(i \neq j)$ are 1 if a link connects node i to node j, and 0 otherwise. The indices i, j run from 1 to N, where N is the size of the network. Similarly, a weighted network is entirely described by a matrix **W** whose entry w_{ij} gives the weight on the edge connecting the vertices i and j ($w_{ij} = 0$, if the nodes i and j are not connected). In this study, we will consider only the case of symmetric weights ($w_{ij} = w_{ji}$) while the undirected case of the network is considered [68]. In a call network, if two nodes call each other for a long duration then weight of the connecting edge will be high and it shows high tie strength between them [69]. Here, weight of the edge in terms of total call duration defines the tie strength between the nodes. It has also been observed in the dependence of the edge weight w_{ij} to define strength between nodes with end point degrees k_i and k_j. Weight as a function of the end-point degrees can be well approximated by a power-law dependence,

$$w_{ij} = b(k_i k_j)^{\beta}.$$

Where, β is the degree influenced real exponent which depends on the type of complex networks and b is a positive quantity. When $\beta > 0$ then rumor transmits to high degree nodes and when $\beta < 0$ then rumor will transmit to low degree nodes. Further, if $\beta = 0$ there will be degree independent transmission.

It has been observed that the individual edge weight does not provide clear view of network's complexity. A detailed measurement if tie strength using the actual weights is obtained by enhancing the property of a vertex degree $k_i = \sum_j a_{ij}$ in terms of

the vertex strength $S_i = \sum_{j=1}^{N} a_{ij} w_{ij}$ (total weights of their neighbors). Therefore, there is a coupling between interaction strengths of the nodes with the counterintuitive consequence that social networks are robust enough for the removal of the strong ties but fall apart after a phase transition if the weak ties are removed [68]. Therefore, we can measure the strength of a node of degree k for scale free network,

$$
\begin{aligned}
S_k &= k \sum_l P(l|k) w_{kl}, \\
&= k \sum_l \frac{l P(l)}{\langle k \rangle} w_{kl}, \\
&= b \frac{k^{1+\beta}}{\langle k \rangle} \langle k^{1+\beta} \rangle.
\end{aligned}
\tag{5.1.4}
$$

Here, the rumor spreading model has been considered, where the rumor transmission rate in contact process between a spreader node and an ignorant node is influenced by their degrees. If w_{kl} is the tie strength between k-degree node and l-degree node for (k,l) edge, S_k is the node strength with degree k. In scale free network for each node of degree k, there is a constant rumor transmission rate λk. Therefore, the rumor transmission rate from k-degree node to l-degree node is given by the proportion of w_{kl} to S_k. Hence, λ_{kl} can be defined as,

$$
\lambda_{kl} = \lambda k \frac{w_{kl}}{S_k}.
\tag{5.1.5}
$$

We can see in Equation (5.1.5) that by increasing the proportion of w_{kl}/S_k, the possibility of rumor transmission rate can be increased through the edge. In the present work, uncorrelated networks have been considered, hence $\lambda_{kl} = \lambda l^{\beta} \langle k \rangle / \langle k^{1+\beta} \rangle$. In this model, rumor spreadness, $\Phi(k) = k^{\alpha}$ where $0 < \alpha \leq 1$, it defines that each spreader node may contact with k^{α} neighbors within one time step. Therefore, spreadness of a rumor will vary nonlinearly with the growing degree k. In Equations (5.1.1)-(5.1.3), we

can explain the rumor equations for $\Phi(k)$ and λ_{lk} as,

$$\frac{dI(k,t)}{dt} = -\frac{\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l) S(l,t), \tag{5.1.6}$$

$$\frac{dS(k,t)}{dt} = \frac{\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l) S(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l [S(l,t)$$
$$+ R(l,t)] l^\alpha P(l) - \delta S(k,t), \tag{5.1.7}$$

$$\frac{dR(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l [S(l,t) + R(l,t)] l^\alpha P(l) + \delta S(k,t). \tag{5.1.8}$$

After solving rumor Equations (5.1.6)-(5.1.8) with initial conditions $I(k,0) \simeq 1, S(k,0) \simeq 0, R(k,0) \simeq 0$ we get,

$$I(k,t) = e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} \Theta(t)}. \tag{5.1.9}$$

Where, $\Theta(t)$ is an auxiliary function defined as,

$$\Theta(t) = \sum_k k^\alpha P(k) \int_0^t S(k,t') dt'. \tag{5.1.10}$$

5.2   Rumor Threshold of the Modified Model

In the infinite time limit, i.e., at the end of rumor spreading, we will have $S(k,\infty)=0$, $lim_{t\to\infty}\Theta(t) \to \Theta$ and $lim_{t\to\infty}d\Theta/dt=0$. Near the critical threshold, the value of Θ will be small, as $S(k,\infty) = 0$. After solving Equations (5.1.6) - (5.1.8) and Equation (5.1.10) we get,

$$\Theta = \frac{(\lambda \frac{\langle k^{\alpha+\beta+1} \rangle}{\langle k^{1+\beta} \rangle} - \delta)}{\lambda^2 \frac{\langle k^{\alpha+2\beta+2} \rangle}{\langle k^{1+\beta} \rangle^2} (1/2 + \sigma \delta \frac{\langle k^{\alpha+\beta+1} \rangle}{\langle k^{1+\beta} \rangle} I)}. \tag{5.2.11}$$

Where, $I = \int_0^t e^\lambda (t - t\prime) f(t') dt\prime$ is the finite and positive integral and $\Theta(t) = \Theta f(t)$ where, $f(t)$ is a finite function. Equation (5.2.11) will give positive value for Θ, when,

$$\lambda \frac{\langle k^{\alpha+\beta+1}\rangle}{\langle k^{1+\beta}\rangle} - \delta \geq 0,$$
$$\frac{\lambda}{\delta} \geq \frac{\langle k^{1+\beta}\rangle}{\langle k^{\alpha+\beta+1}\rangle}. \tag{5.2.12}$$

Therefore, to leading order in σ, the critical threshold is independent of the stifling mechanism, for $\delta = 1$ the critical rumor spreading threshold is given by,

$$\lambda_c = \frac{\langle k^{1+\beta}\rangle}{\langle k^{\alpha+\beta+1}\rangle}. \tag{5.2.13}$$

It is interesting to note that by putting, $\alpha = 1$ and $\beta = 0$ in Equation (5.2.13), the threshold for this model reduces to $\langle k\rangle/\langle k^2\rangle$ for classical rumor spreading model [7].

When, $t \to \infty$ spreader nodes will be 0, $(S(k, \infty) = 0)$ and from Equation (5.1.9), $I(k, \infty) = e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta}$. Therefore, final size of rumor R at $t \to \infty$ $(lim_{t\to\infty} R(k, t) = R)$ will be as,

$$
\begin{aligned}
R &= \sum_k P(k) R(k, \infty) & \text{(5.2.14)}\\
&= \sum_k P(k)(1 - S(k, \infty))\\
&= \sum_k P(k)(1 - e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta})\\
&= 1 - \sum_k P(k) e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta} & \text{(5.2.15)}
\end{aligned}
$$

In the real world complex networks rumor spreads on a finite size complex networks. It may be possible that size of scale free network is very large. The maximum or

minimum degrees of scale free network is mentioned by k_{max} or k_{min}. Pastor *et al.* [9] found that the epidemic threshold λ_c for k_{max} for SIS model on bounded scale free networks with $P(k) \sim k^{-2-\gamma'}$, $0 < \gamma' \le 1$. They assumed that with the soft and hard cut-off k_{min} and k_{max}, when $\alpha = 1$. The hard cut-off denotes that, a network does not possess any node with degree $k > k_{max}$. As k_{max} of a node is network age, defined in terms of number of nodes N,

$$k_{max} = k_{min} N^{\frac{1}{\gamma'+1}}. \tag{5.2.16}$$

The normalized degree distribution is defined by,

$$P(k) = \frac{(1+\gamma')k_{min}^{1+\gamma'}}{1 - (k_{max}/k_{min})^{-1-\gamma'}} k^{-2-\gamma'} \theta(k_{max} - k). \tag{5.2.17}$$

Here, $\theta(x)$ is a heaviside step function [9] .

In modified rumor spreading model if $\alpha = 1$ and $\beta = 0$ then, it converges to classic rumor spreading model. As the degree distribution in scale free networks $P(k) = k^{-\gamma}$ where, $2 < \gamma \le 3$, therefore (Appendix B.1)

$$\lambda_c{}'(k_{max}) = \frac{\langle k \rangle}{\langle k^2 \rangle} \tag{5.2.18}$$

$$= \frac{\int_{k_{min}}^{k_{max}} k^{1-\gamma} dk}{\int_{k_{min}}^{k_{max}} k^{2-\gamma} dk} \tag{5.2.19}$$

$$\simeq \frac{3-\gamma}{(\gamma-2)k_{min}} (k_{max}/k_{min})^{\gamma-3} \tag{5.2.20}$$

Equation (5.2.16) is modified for the given scale free network as,

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}. \tag{5.2.21}$$

Therefore, rumor threshold for modified model for $2 < \gamma < 3$ is,

$$\lambda_c{}'(N) \simeq \frac{3-\gamma}{(\gamma-2)k_{min}} (N)^{(\gamma-3)/(\gamma-1)}. \tag{5.2.22}$$

Rumor threshold for $\gamma = 3$ (Appendix B.2),

$$\gamma_c'(N) \simeq 2[k_{min} ln(N)]^{-1}. \tag{5.2.23}$$

Equations (5.2.22)-(5.2.23) show that $\lambda_c' \to 0$ if $N \to \infty$

In modified rumor spreading model, nonlinear rumor spread is considered using $\Phi(k) = k^\alpha$ and degree dependent spreading and stifling rates are considered with $\lambda_{kl} = \lambda l^\beta \langle k \rangle / \langle k^{1+\beta} \rangle$, $\sigma_{kl} = \sigma l^\beta \langle k \rangle / \langle k^{1+\beta} \rangle$, respectively. The rumor threshold for the modified model is given by,

$$
\begin{aligned}
\lambda_c{}^\#(k_{max}) &= \frac{\int_{k_{min}}^{k_{max}} k^{\beta+1-\gamma} dk}{\int_{k_{min}}^{k_{max}} k^{\alpha+\beta+1-\gamma} dk} \\
&= k_{min}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\beta-\gamma+2} \frac{[(k_{max}/k_{min})^{\beta-\gamma+2}-1]}{[(k_{max}/k_{min})^{\alpha+\beta-\gamma+2}-1]}.
\end{aligned} \tag{5.2.24}
$$

Theorem 5.2.1 *In classic rumor spread model ($\alpha = 1, \beta = 0$) threshold is smaller than the modified rumor spread model ($0 < \alpha < 1$ and $\beta \neq 0$).*

The proof is given in the next section after lemmas.

Lemma 5.2.2 *When the size of network (N) increases, the value of critical threshold $\lambda_c^\# > 0$ for $\alpha + \beta + 2 < \gamma$, otherwise it approaches to 0.*

Proof Since $k_{max}/k_{min} = N^{\frac{1}{\gamma-1}}$, therefore k_{max}/k_{min} increases when N increases, it becomes infinity when $N \to \infty$. When, $\alpha+\beta+2 < \gamma$, $(\frac{k_{max}}{k_{min}})^{\beta-\gamma+2} = (\frac{k_{max}}{k_{min}})^{\alpha+\beta-\gamma+2} = 0$. The value of $\lambda_c^\#$ will be positive. Here, $\beta < 0$ (rumor transmission influenced to low degree nodes) is considered. Now from Equation (5.2.24), we can conclude that $\lambda_c^\#$ will

be positive . For $\alpha + \beta + 2 \geq \gamma$, $\lambda_c^\# \to 0$ when N increases. It can be summarized as,

$$\lambda_c{}^\#(k_{max}) = \begin{cases} k_{min}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\gamma-\beta-2}(k_{max}/k_{min})^{\gamma-\alpha-\beta-2}, & \alpha+\beta+2 > \gamma \\ k_{min}^{(-\alpha)} \frac{\gamma-\alpha-\beta-2}{\gamma-\beta-2}, & \alpha+\beta+2 < \gamma \quad (5.2.25) \\ k_{min}^{(-\alpha)} \frac{1}{\alpha ln(k_{max}/k_{min})}, & \alpha+\beta+2 = \gamma \end{cases}$$

Lemma 5.2.3 *In given rumor spreading model when $\alpha+\beta+2 < \gamma$ then rumor spreading threshold $\lambda^\#$ is independent from the size of scale free network (N).*

Proof It may also be defined using Equations (5.2.21)-(5.2.25) in the term of the number of nodes N,

$$\lambda_c{}^\#(N) = \begin{cases} k_{min}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\gamma-\beta-2}(N)^{(\gamma-\alpha-\beta-2)/(\gamma-1)}, & \alpha+\beta+2 > \gamma \\ k_{min}^{(-\alpha)} \frac{\gamma-\alpha-\beta-2}{\gamma-\beta-2}, & \alpha+\beta+2 < \gamma \quad (5.2.26) \\ k_{min}^{(-\alpha)} \frac{\gamma-1}{\alpha ln(N)}, & \alpha+\beta = \gamma \end{cases}$$

Here, it is observed that for $\alpha + \beta + 2 < \gamma$, $\lambda_c^\#$ is independent of N.

Proof Now using lemmas 5.2.2 and 5.2.3, the theorem can be proved for $\alpha+\beta+2 > \gamma$. The ratio of rumor threshold in classic model and given model is given as,

$$\frac{\lambda_c'(N)}{\lambda_c{}^\#(N)} = \frac{(2-\gamma)(\gamma-\beta-2)}{(\gamma-2)k_{max}^{(1-\alpha)}(\alpha+\beta-\gamma+2)N^{(1-\gamma-\beta+2)/(\gamma-3)}} \quad (5.2.27)$$

It has been found from Equation (5.2.27) that $\frac{\lambda_c'(N)}{\lambda_c{}^\#(N)} < 1$ for finite scale free networks. Therefore, it has been justified that rumor threshold $\lambda_c^\#(N)$ is greater than the $\lambda_c'(N)$ in finite size scale free networks. In finite size scale free networks, when $0 < \alpha < 1$, $\beta \neq 0$ and $\alpha + \beta + 2 > \gamma$ then, it is hard to spread rumor in comparison to networks having $\alpha = 1$ and $\beta = 0$. Finite rumor threshold is possible for any size of networks as seen in Equation (5.2.26). However, it will be 0 when N approaches infinity.

5.3   Random Inoculation

In random inoculation strategy, randomly selected nodes will be inoculated. This approach inoculates a fraction of the nodes randomly, without any information of the network. Here, variable g $(0 \leq g \leq 1)$ defines the fraction of inoculative nodes. In the presence of random inoculation, rumor spreading rate λ is reduced by a factor $(1-g)$. In mean field level, for the scale free networks in the case of random inoculation, the rumor equations are modified using initial conditions as,

$$
\frac{dI(k,t)}{dt} = -\frac{(1-g)\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l)S(l,t), \tag{5.3.28}
$$

$$
\frac{dS(k,t)}{dt} = \frac{(1-g)\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l)S(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l (S(l,t)
$$
$$
+ R(l,t))l^\alpha P(l) - \delta S(k,t), \tag{5.3.29}
$$

$$
\frac{dR(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l (S(l,t) + R(l,t))l^\alpha P(l) + \delta S(k,t). \tag{5.3.30}
$$

Therefore, final size of the informed nodes (R) is,

$$
R = 1 - \sum_k P(k)(1-g)e^{\frac{-\lambda(1-g)k^{1+\beta}}{\langle k^{1+\beta} \rangle}\Theta} - g. \tag{5.3.31}
$$

The rumor spreading threshold in the case of random inoculation is obtained from Equations (5.3.28)-(5.3.30) as,

$$
\hat{\lambda}_c = \frac{\langle k^{\beta+1} \rangle}{(\langle k^{\alpha+\beta+1} \rangle)(1-g)}. \tag{5.3.32}
$$

The relation between rumor spreading threshold, with inoculation $(\hat{\lambda}_c)$ and without

inoculation(λ_c) can be defined as,

$$\hat{\lambda}_c = \frac{\lambda_c}{1-g}. \tag{5.3.33}$$

It is to noted that by applying random inoculation, the rumor spreading threshold ($\hat{\lambda}_c$) can be increased as seen in Equation (5.3.33) (i.e., $\hat{\lambda}_c > \lambda_c$).

5.4   Targeted Inoculation

Scale free networks permit efficient strategies and depend upon the hierarchy of nodes. It has been shown that scale free networks show robustness against random inoculation. It shows that the high fraction of inoculation of nodes can be resisted without loosing its global connectivity. But on the other hand, scale free networks are strongly affected by targeted inoculation of nodes. The scale free network suffers an interesting reduction of its robustness to carry information. In targeted inoculation, the high degree nodes have been inoculated progressively, i.e more likely to spread the information. In scale free networks, the robustness of the network decreases at the effect of a tiny fraction of inoculated individuals.

Let us assume that fraction g_k of nodes with degree k are successfully inoculated. An upper threshold of degree k_t , such that all nodes with degree $k > k_t$ get inoculated. Fraction g_k of nodes with the degree k are successfully inoculated. The fraction of inoculated nodes is given by,

$$g_k = \begin{cases} 1, & k > k_t, \\ f, & k = k_t, \\ 0, & k < k_t. \end{cases} \tag{5.4.34}$$

Where $0 < f \leq 1$, and $\sum_k g_k P(k) = \bar{g}$, here $\bar{g}$ is the average inoculation fraction. Now rumor spreading equation is defined for targeted inoculation as,

$$\frac{dI(k,t)}{dt} = -\frac{(1-g_k)\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l)S(l,t), \tag{5.4.35}$$

$$\frac{dS(k,t)}{dt} = \frac{(1-g_k)\lambda k^{1+\beta}}{\langle k^{1+\beta} \rangle} I(k,t) \sum_l l^\alpha P(l)S(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l (S(l,t)$$
$$+R(l,t))l^\alpha P(l) - \delta S(k,t), \tag{5.4.36}$$

$$\frac{dR(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta} \rangle} \sum_l (S(l,t) + R(l,t))l^\alpha P(l) + \delta S(k,t). \tag{5.4.37}$$

Next, rumor spreading threshold in the case of targeted inoculation is obtained from Equations (5.4.35)-(5.4.37) as,

$$\tilde{\lambda}_c = \frac{\langle k^{\beta+1} \rangle}{\langle k^{\alpha+\beta+1} \rangle - \langle g_k k^{\alpha+\beta+1} \rangle}. \tag{5.4.38}$$

Here, $\langle g_k k^{\alpha+\beta+1} \rangle = \bar{g} \langle k^{\alpha+\beta+1} \rangle + \eta\prime$, where $\eta\prime = \langle (g_k - \bar{g})[\langle k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1} \rangle \rangle] \rangle$ is the co-variance of g_k and $k^{\alpha+\beta+1}$. The cut-off degree k_t is large enough where, $\eta\prime < 0$, but for small k_t, $g_k - \bar{g}$ and $k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1} \rangle$ have the same signs except for k's where $g_k - \bar{g}$ and/or $k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1} \rangle$ is 0.

Hence, $\eta\prime > 0$ for appropriate k_t,

$$\tilde{\lambda}_c > \frac{1-g}{1-\bar{g}} \hat{\lambda}_c. \tag{5.4.39}$$

If average inoculation fraction of nodes in targeted inoculations is same as fraction of nodes in the random inoculations then $g = \bar{g}$,

$$\tilde{\lambda}_c > \hat{\lambda}_c. \tag{5.4.40}$$

The above relation shows that in scale free networks, the targeted inoculation is more effective than the random inoculation.

5.5   Numerical Simulations: Results and Discussion

The studies of uncorrelated networks have been performed using the degree distribution of scale free network. The size of the network is considered to be $N = 10^5$, the degree exponent (γ)=2.4, $\delta = 1$ and $\sigma = 0.2$. At the starting of rumor spreading, the spreaders are randomly chosen. In Figures 5.1 and 5.2, the final size of rumor R is plotted against rumor transmission rate for N=100000, 1000 and 100 by tuning α and β as,

- $\alpha + \beta = 0$: Finite rumor threshold has been found. It has been observed that for different size of networks, constant threshold is there (after fixing the value of α and β). For the case $\alpha + \beta + 2 < \gamma$, since $\gamma = 2.4$. Therefore it seems interesting that finite threshold has been found which is independent from the size of network same as obtained from Equations (5.2.25) and (5.2.26).

- $\alpha + \beta = -1$: The simulation results are found same as above since $\alpha + \beta + 2 < \gamma$ with finite threshold and constant for any network size (for fix values of α and β).

- $\alpha + \beta = 1$: In this case the rumor threshold has nonzero value but it tends to be 0 when the network size increases. For this case $\alpha + \beta + 2 > \gamma$, since $\gamma = 2.4$. Therefore, the threshold approaches to 0 as network size increases. Similar results have been obtained by Equations (5.2.25)-(5.2.26).

- $\alpha + \beta = 2$: The simulation results are found same as above since $\alpha + \beta + 2 > \gamma$ with threshold approaches to 0 as size of network increases.

Final size of rumor (R) obtained in numerical simulation is plotted against time (t) in Figures 5.3-5.6. It has been observed that rumor size increases exponentially as time
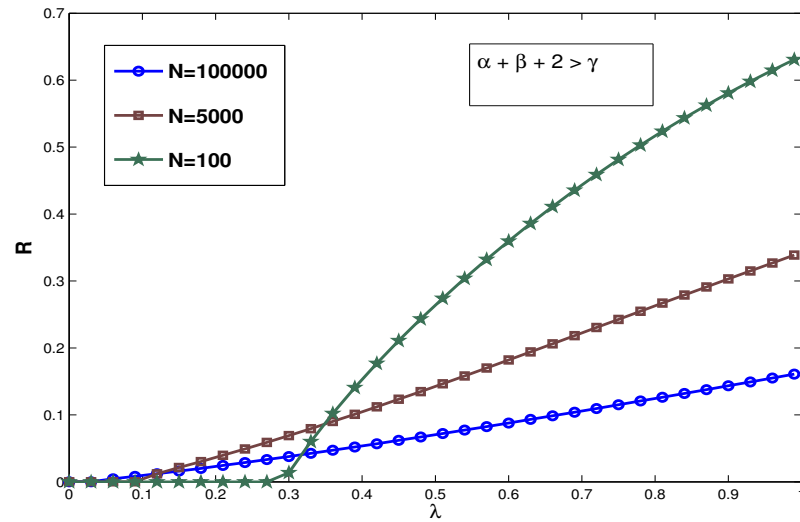
Figure 5.1: R vs λ with $\alpha + \beta + 2 > \gamma$ for different size of scale free networks.
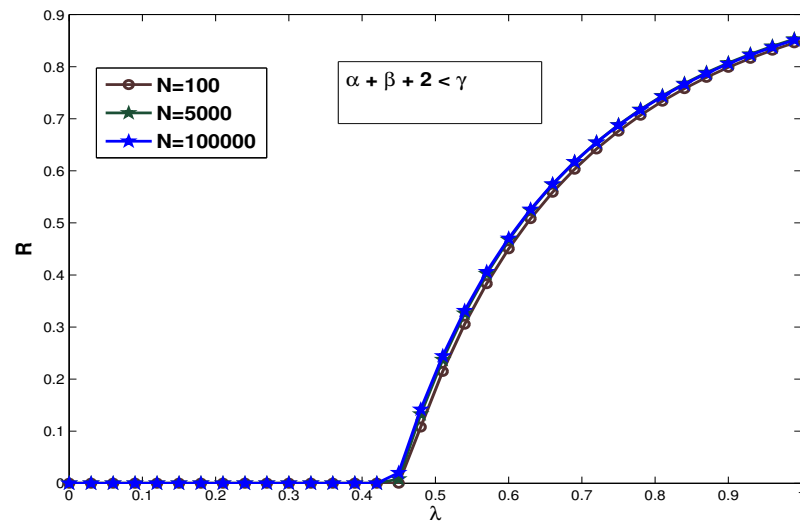

Figure 5.2: R vs λ with $\alpha + \beta + 2 < \gamma$ for different size of scale free networks.

increases and after some time it approaches a steady state, that will remain constant, since spreader density is 0 at that time. It has also been observed that when $\alpha + \beta$ is low, then rumor size initially increases slowly but when, $\alpha + \beta$ increases the rumor size increases rapidly with time. While tuning the parameter α and β from $\alpha + \beta = -1$ to $\alpha + \beta = 2$ rumor increments are faster, initially (Figures 5.3-5.6). When ratio of α and β is high than the rumor size is also high. This observation justifies that the α affects more the final size of rumor R than β. It is seen from Equation (5.2.13) that when α is very small (0.1- 0.3), then the rumor threshold will be high. The final size of rumor will be too small when rumor transmission rate (λ) is less than the rumor threshold (λ_c).
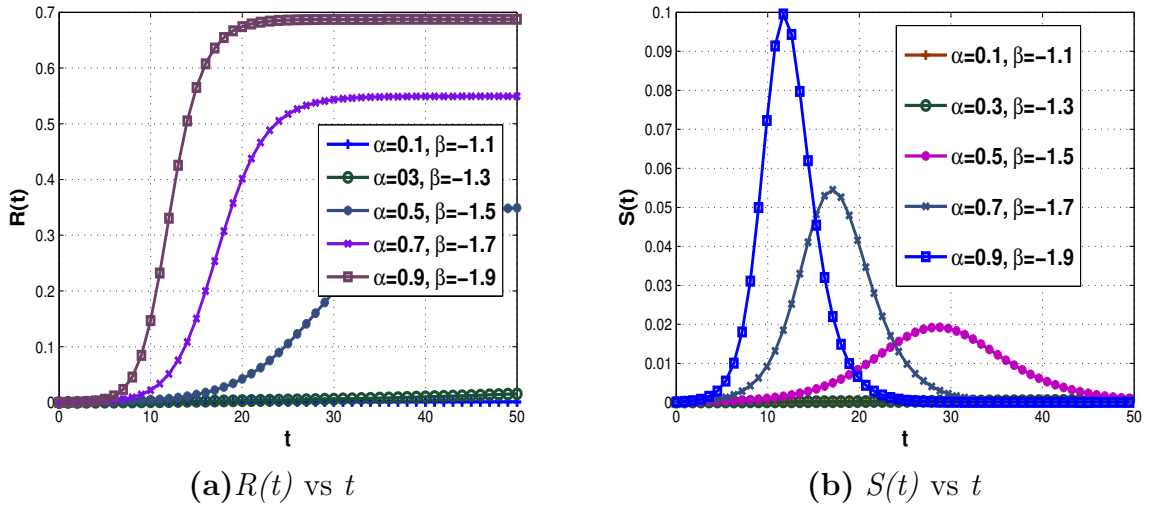
(a) $R(t)$ vs t
(b) $S(t)$ vs t

Figure 5.3: $R(t)$ vs t and $S(t)$ vs t for $\lambda = 0.8$ and $\alpha + \beta = -1$.

Critical rumor threshold is plotted against α in Figures 5.7-5.8 while considering $\beta = 0$. Here, λ_c decreases exponentially with the increase of α. It is maximum for $\alpha = 0.1$ and almost 0 at $\alpha = 1$ for $N = 100000$. Interestingly, this also happens in real life situation, when an informed node passes information to its maximum number of neighbors then rumor spreading will get outbreak in the network. However, the outbreak is hard to achieve when it passes rumor to less number of neighbors ($\simeq$ 10-
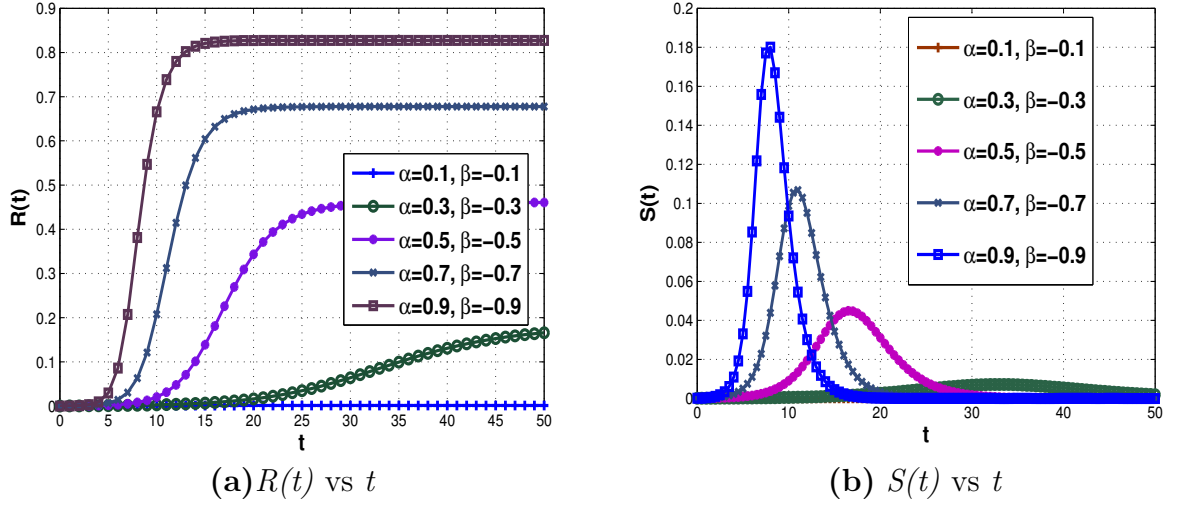
(a) $R(t)$ vs t

(b) $S(t)$ vs t

Figure 5.4: $R(t)$ vs t and $S(t)$ vs t for $\lambda = 0.8$ and $\alpha + \beta = 0$.
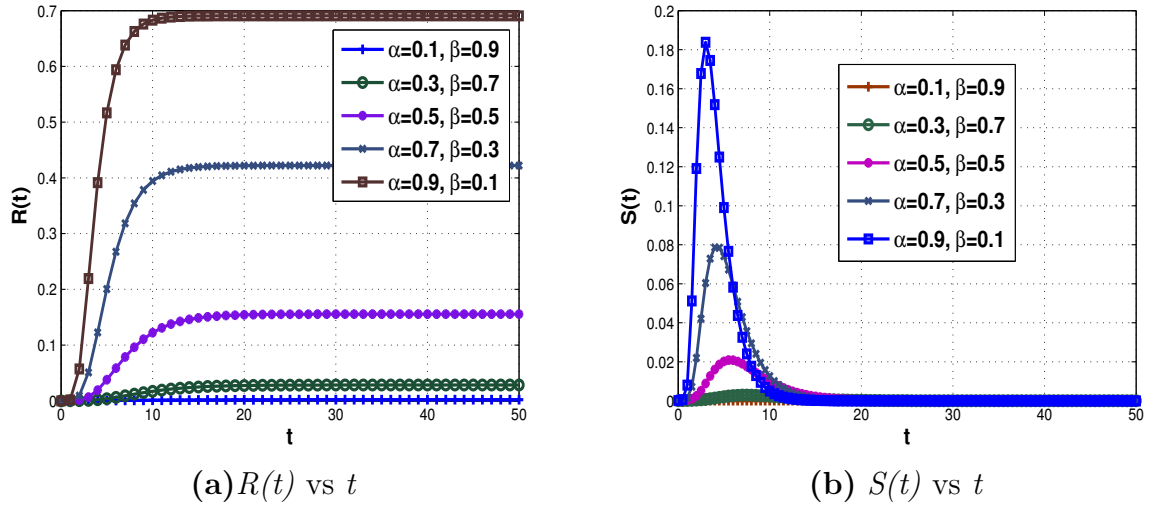
(a) $R(t)$ vs t

(b) $S(t)$ vs t

Figure 5.5: $R(t)$ vs t and $S(t)$ vs t for $\lambda = 0.8$ and $\alpha + \beta = 1$.

(a) $R(t)$ vs t **(b)** $S(t)$ vs t
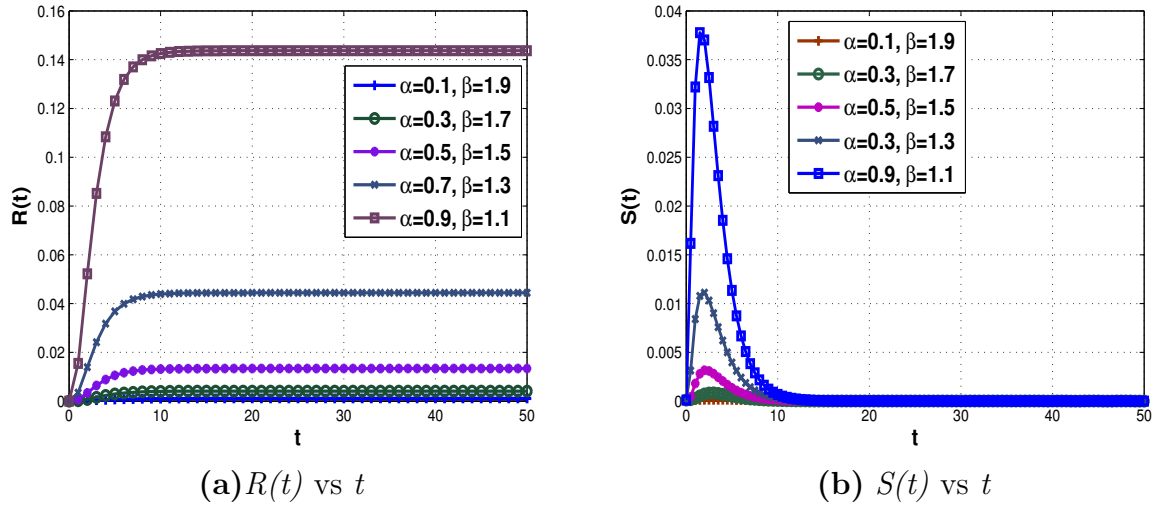
Figure 5.6: $R(t)$ vs t and $S(t)$ vs t for $\lambda = 0.8$ and $\alpha + \beta = 2$.

30%). Similarly, λ_c has been studied against β at $\alpha = 1$ in Figures 5.7-5.8. It is found that β has less effect on the rumor threshold for entire range except when $\beta >$ 0. Further, it approaches to 0. When size of the network (N) increases then rumor threshold is decreased as shown in Figure 5.7 and Figure 5.8.

Figure 5.7: Threshold (λ_c) vs α for $\gamma = 2.4$ for different size of scale free networks.

Figure 5.8: Threshold (λ_c) vs β for $\gamma = 2.4$ for different size of scale free networks.

In Figures 5.9-5.10, final size of rumor has been plotted against β for $\alpha = 1$, $\gamma = 2.4$, 3, and $\lambda = 1$. It can be seen that that R is maximum when $\beta = 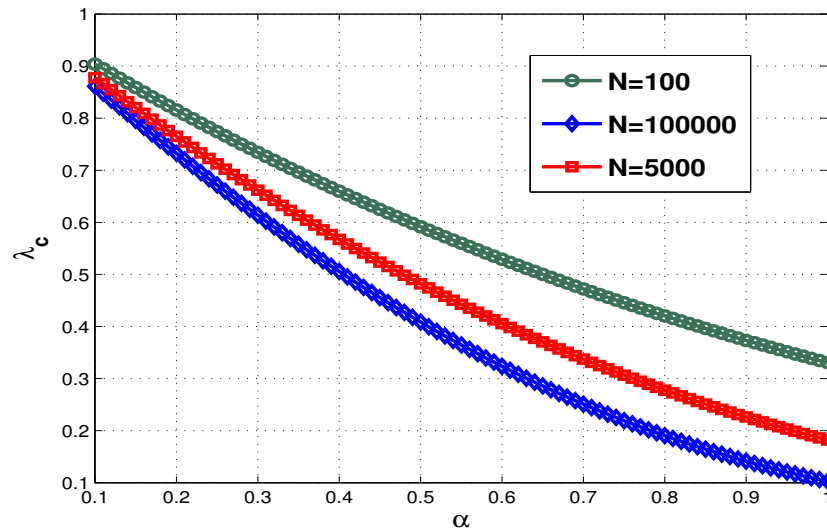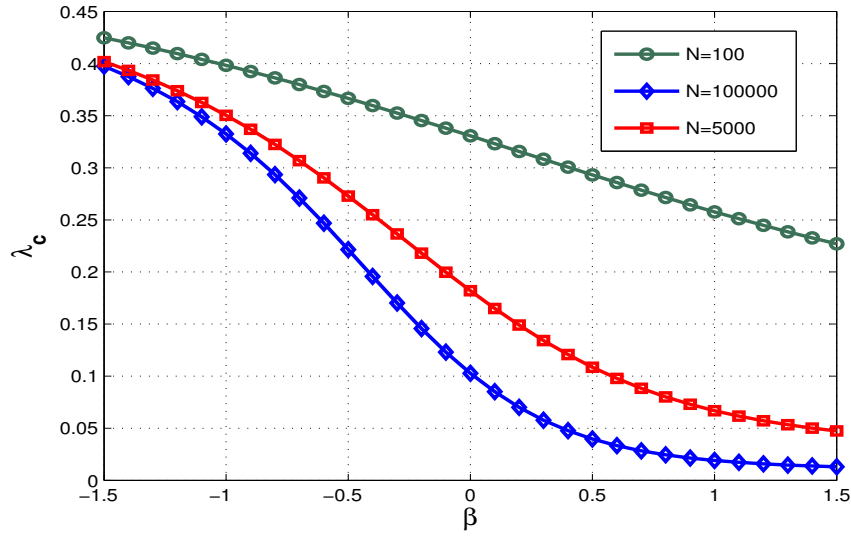-1$. Initially rumor size R increases with β but after achieving a maximum value for $\beta = -1$, it decays exponentially. Further, for $\alpha = 1$, $\gamma = 2.4$, the final rumor size R approaches to 0 (beyond $\beta = 1.5$).

Furthermore, it is interesting to note that the final rumor size R increases with increase of α, see Fig. 5.11.

For random inoculation $g = 0.1, 0.3, 0.5, 0.7, 0.9$, the final rumor size R has been plotted against β (Figures 5.12-5.14). It is observed that to get maximum value of R, β increases when g increases. Also, maximum size of rumor decreases with increase of g. It is because in random inoculation rumor, the threshold value is larger than the threshold in model without inoculation as inferred from from Equation (5.3.33) ($\hat{\lambda}_c > \lambda_c$). A sharp decrease in the value of R is seen when rumor transmission rate (λ)
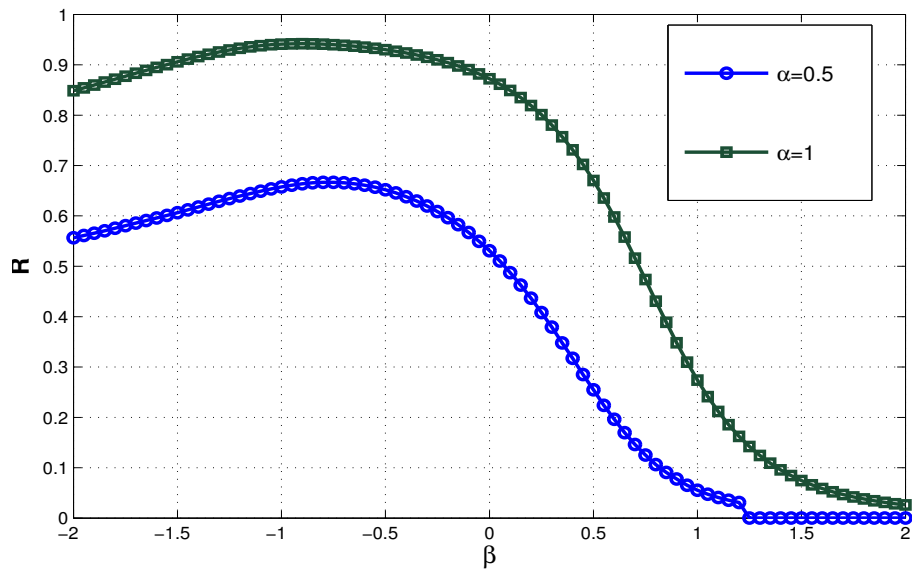
Figure 5.9: Final size of rumor R vs β ($N = 10^5$ nodes, $\alpha = 0.5$ and $\alpha = 1$) for $\gamma = 2.4$.
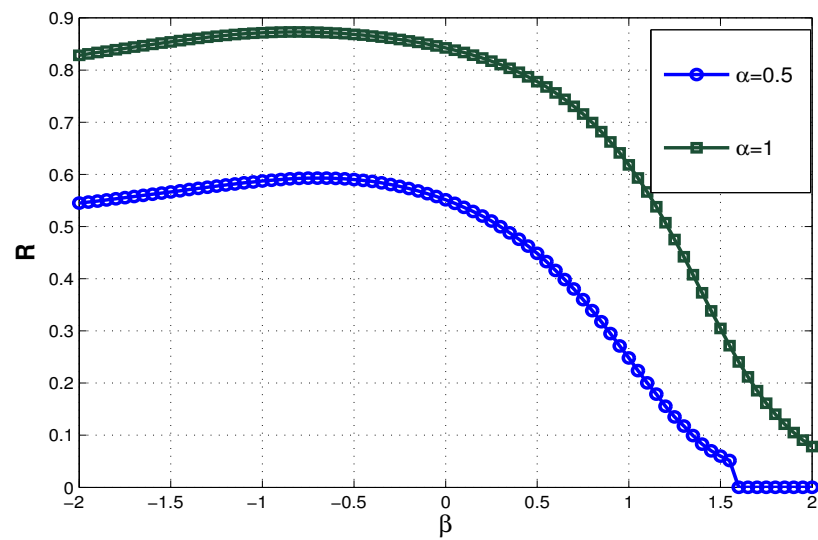

Figure 5.10: Final size of rumor R vs β ($N = 10^5$ nodes, $\alpha = 0.5$ and $\alpha = 1$) for $\gamma = 3$.
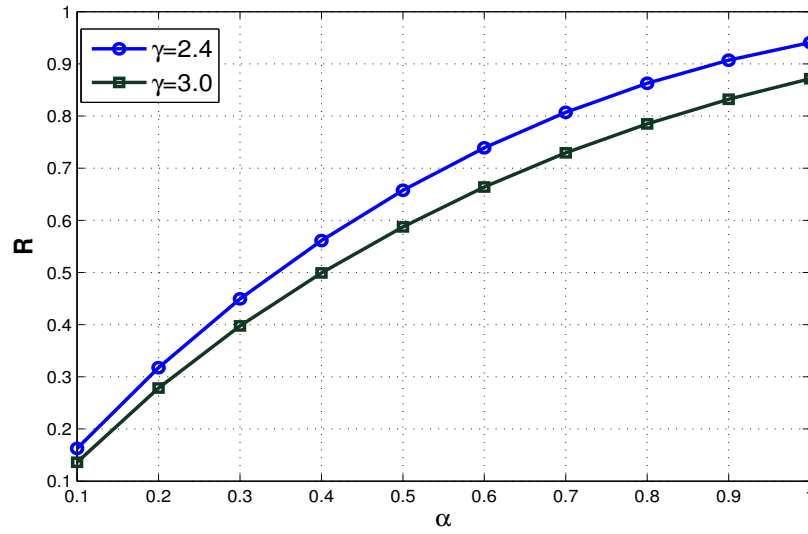
Figure 5.11: Final size of rumor R vs α ($N = 10^5$ nodes, $\beta = -1$ and $\lambda = 1$) for $\gamma = 2.4, 3$.

is decreased by 0.5 in comparison to the case where decrease of R is shallow and α is decreased by 0.5. For $\lambda \simeq 1$ there may be a chance that rumor spreads to some extent at any value of $g < 1$ for very large N.

Similar results have been observed in the case of targeted inoculation in Figures 5.15-5.17. Here, maximum rumor size is much smaller with the inoculation of very less fraction of nodes (e.g. for g=0.25 the final rumor size R is almost suppressed), as the rumor threshold in targeted inoculation is larger than the random inoculation.

For random inoculation strategy, the rumor spreading is plotted against time evolution using modified model through simulation results. For $g = 0.1, 0.3, 0.5, 0.7$, the Figures 5.18-5.21 show, if $\alpha + \beta$ increases from -1 to 2, R will increase since the rumor threshold decreases. Further, it can be observed from Equation (5.2.26) that for $\alpha + \beta = -1$ and 0, λ_c is finite and higher than the case where $\alpha + \beta = 1$ and 2. There-

Figure 5.12: Final size of rumor R vs β with $\alpha = 1$ and $\lambda = 1$ in random inoculation scheme for different fractions of inoculation (g).
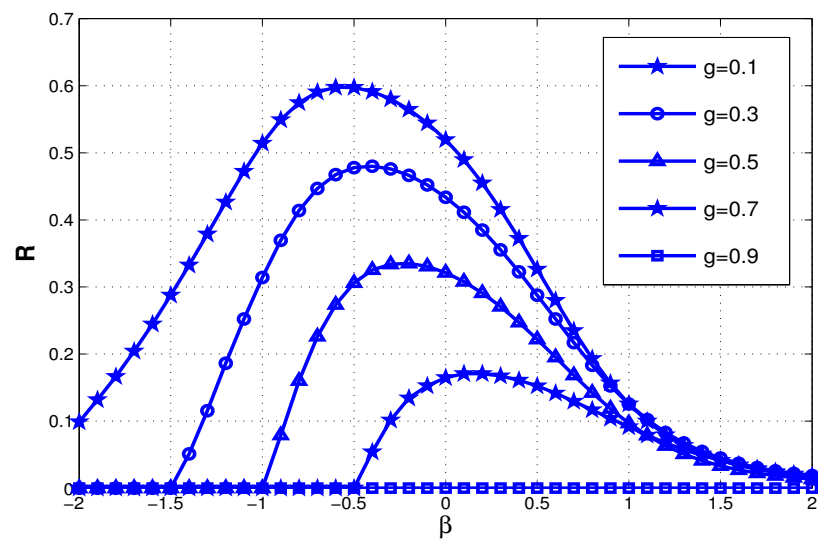
Figure 5.13: Final size of rumor R vs β with $\alpha = 1$ and $\lambda = 0.5$ in random inoculation scheme for different fractions of inoculation (g).
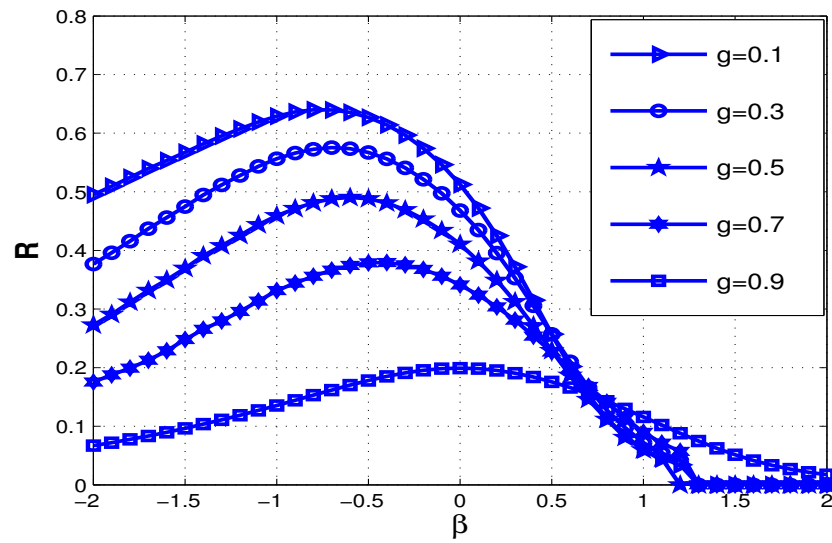
Figure 5.14: Final size of rumor R vs β with $\alpha = 0.5$ and $\lambda = 1$ in random inoculation scheme for different fractions of inoculation (g).
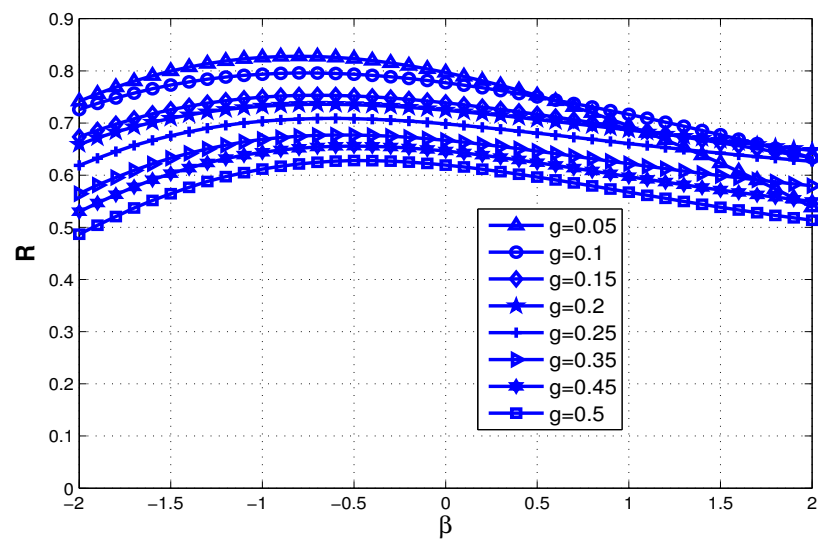
Figure 5.15: Final size of rumor R vs β with $\alpha = 1$ and $\lambda = 1$ in targeted inoculation scheme for different fractions of inoculation (g).
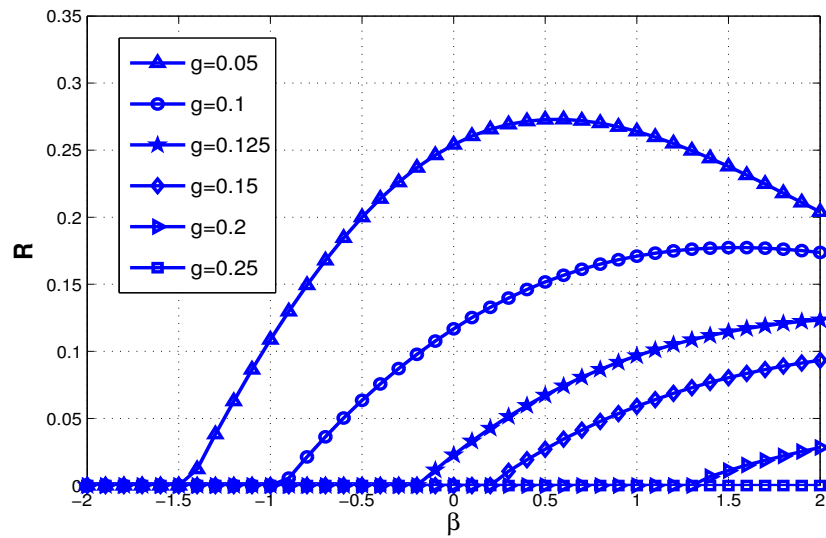
Figure 5.16: Final size of rumor R vs β with $\alpha = 1$ and $\lambda = 0.5$ in targeted inoculation scheme for different fractions of inoculation (g).
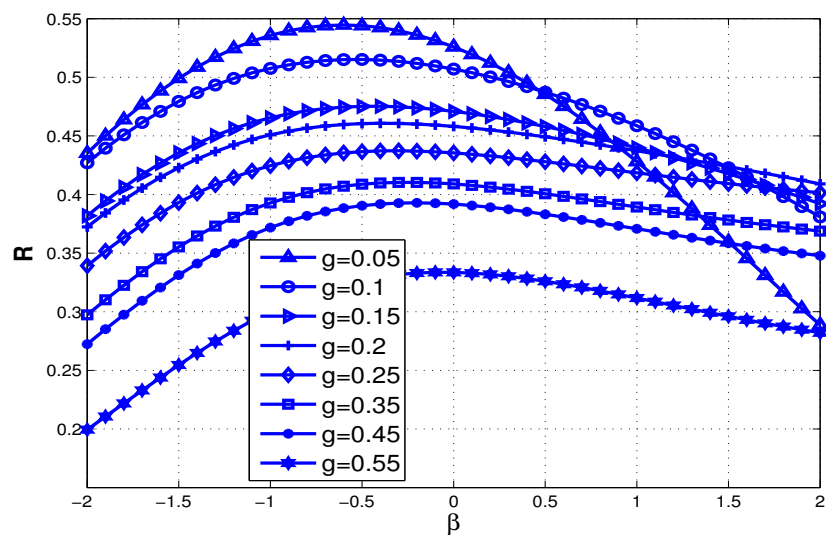
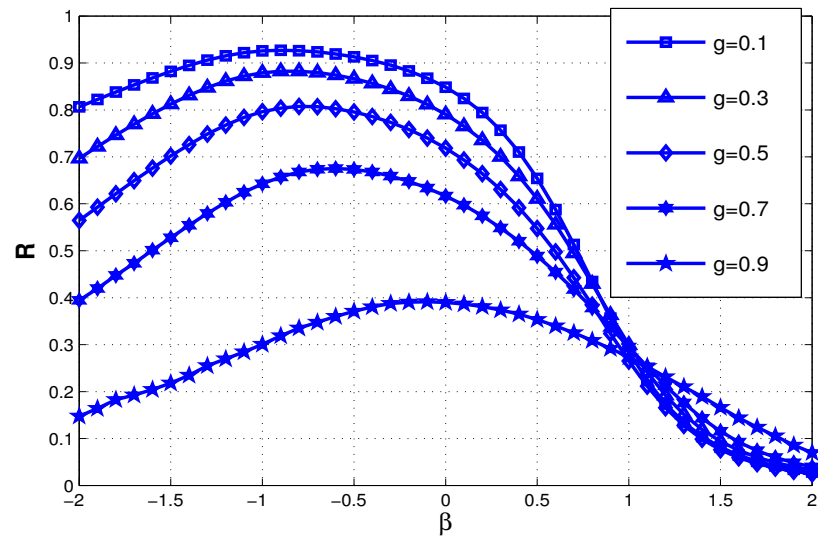Figure 5.17: Final size of rumor R vs β with $\alpha = 0.5$ and $\lambda = 1$ in targeted inoculation scheme for different fractions of inoculation (g).

fore, $R(t)$ is almost 0 and grows slowly with time when, $\alpha + \beta = -1$. The growth in $R(t)$ is higher for lower values of g, but the case is reversed for the higher values of g.

Figure 5.18: $R(t)$ vs t with $\alpha + \beta = -1$, $\lambda = 0.6$ in random inoculation for $g = 0.1, 0.3$ (upper) 0.5, 0.7 (lower).

Similarly, in the case of targeted inoculation scheme using lower values of $g = 0.05, 0.1, 0.15, 0.2$ for $\alpha + \beta = -1$ to 2, the rumor threshold is found more than the random inoculation scheme (Equation (5.4.40)) and rumor spreading is suppressed after inoculating less number of nodes than the random inoculation scheme (Fig. 5.22-Fig. 5.25).

5.6   Conclusions

In present investigations, the modified SIR model has been proposed by considering standard SIR rumor spreading model with degree dependent tie strength of nodes and nonlinear spread of rumor. The two parameters - nonlinear exponent α and degree de-
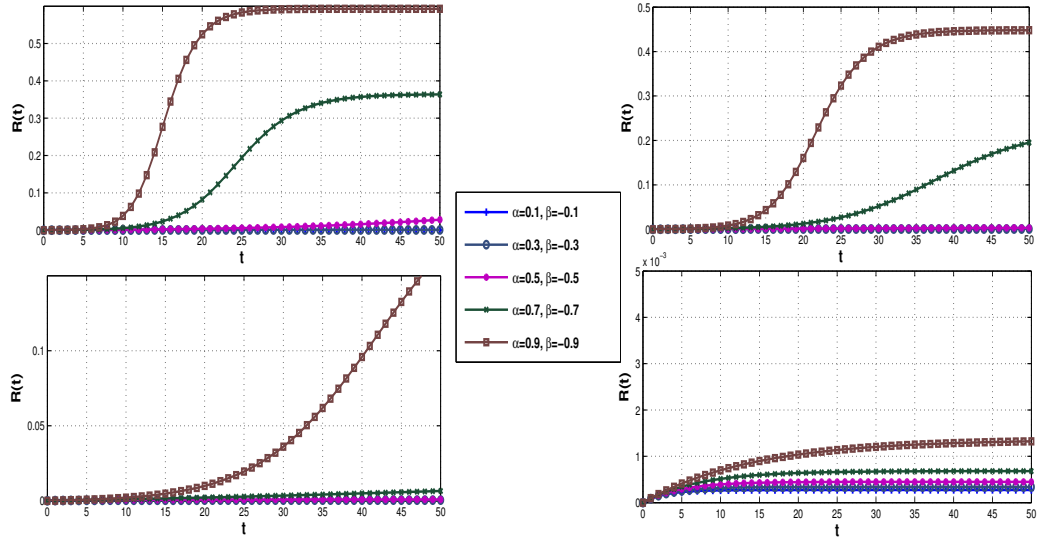
Figure 5.19: $R(t)$ vs t with $\alpha + \beta = 0$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).
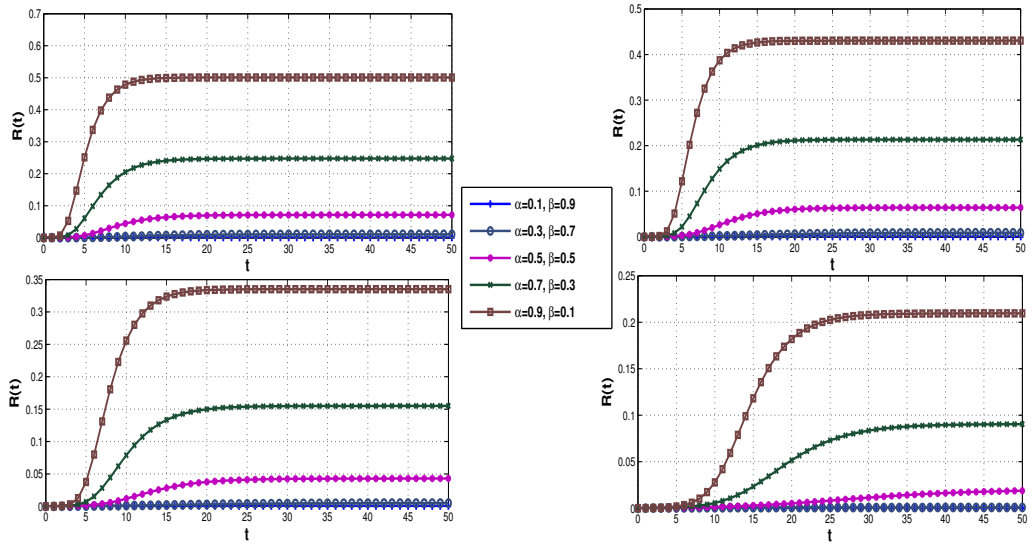
Figure 5.20: $R(t)$ vs t with $\alpha + \beta = 1$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).
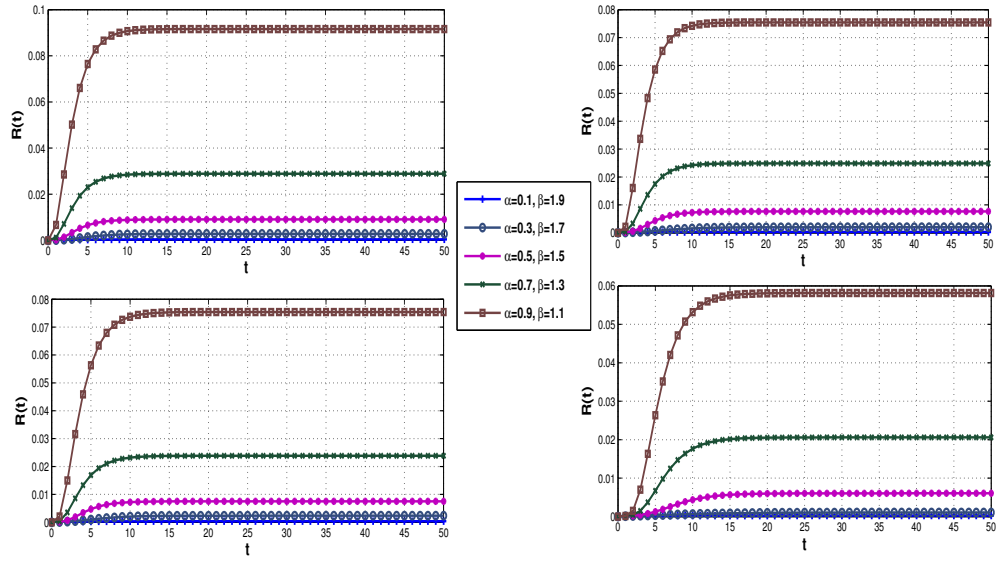
Figure 5.21: $R(t)$ vs t with $\alpha + \beta = 2$, $\lambda = 0.6$ in random inoculation for g= 0.1, 0.3 (upper) 0.5, 0.7 (lower).
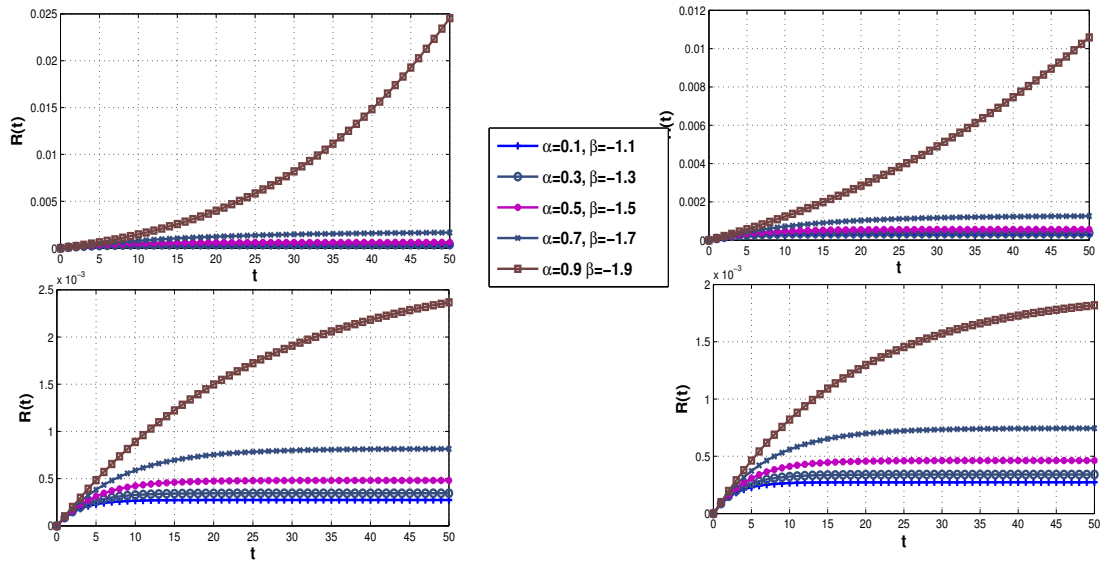
Figure 5.22: $R(t)$ vs t with $\alpha + \beta = -1$, $\lambda = 0.6$ in targeted inoculation for g= 0.05, 0.1 (upper) 0.15, 0.2 (lower).
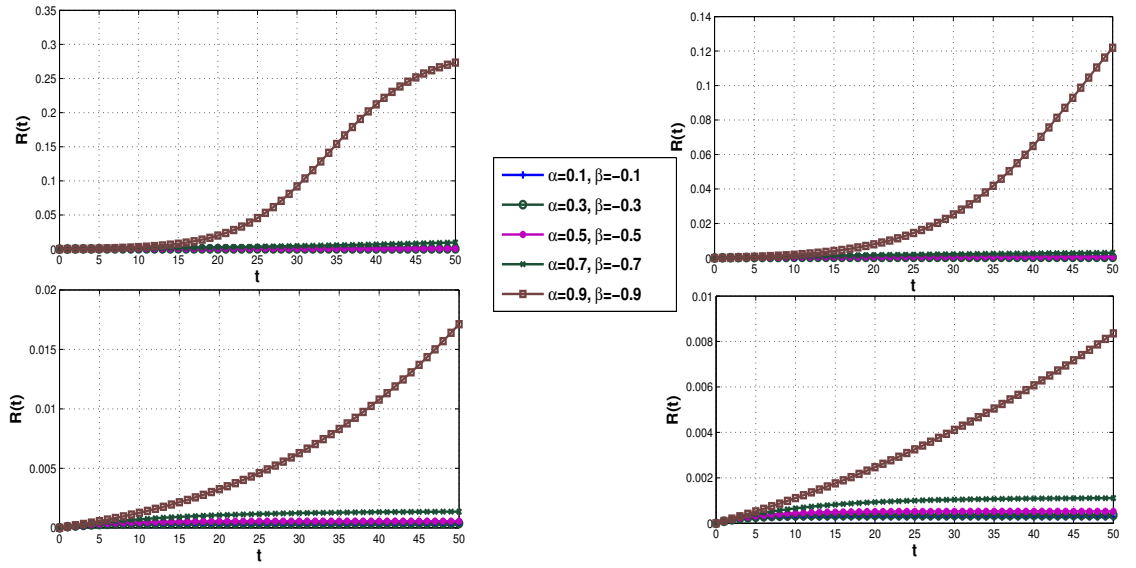
Figure 5.23: $R(t)$ vs t with $\alpha + \beta = 0$, $\lambda = 0.6$ in targeted inoculation for $g= 0.05,\ 0.1$ (upper) 0.15, 0.2 (lower).
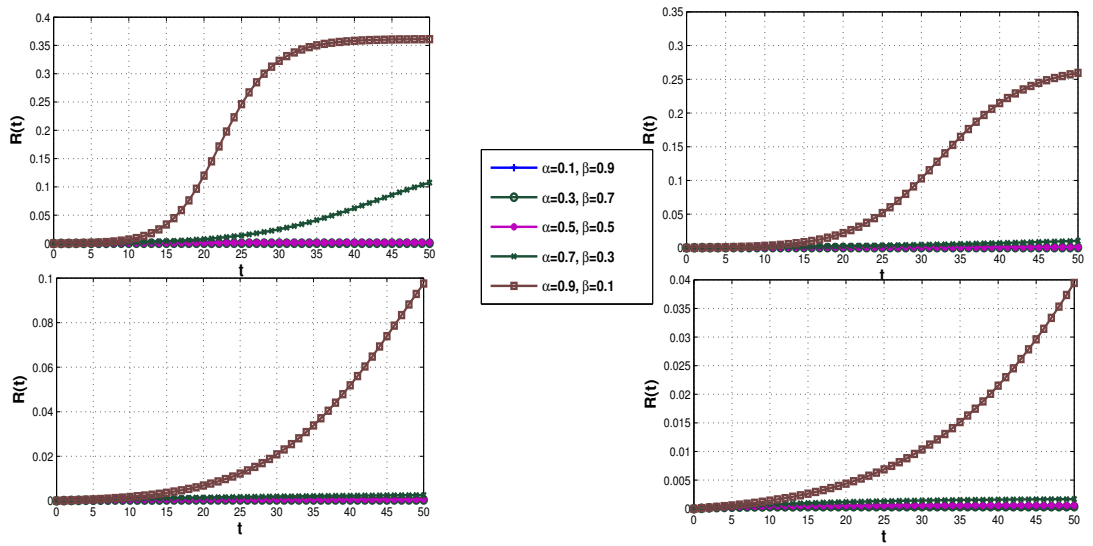
Figure 5.24: $R(t)$ vs t with $\alpha + \beta = 1$, $\lambda = 0.6$ in targeted inoculation for $g= 0.05,\ 0.1$ (upper) 0.15, 0.2 (lower).
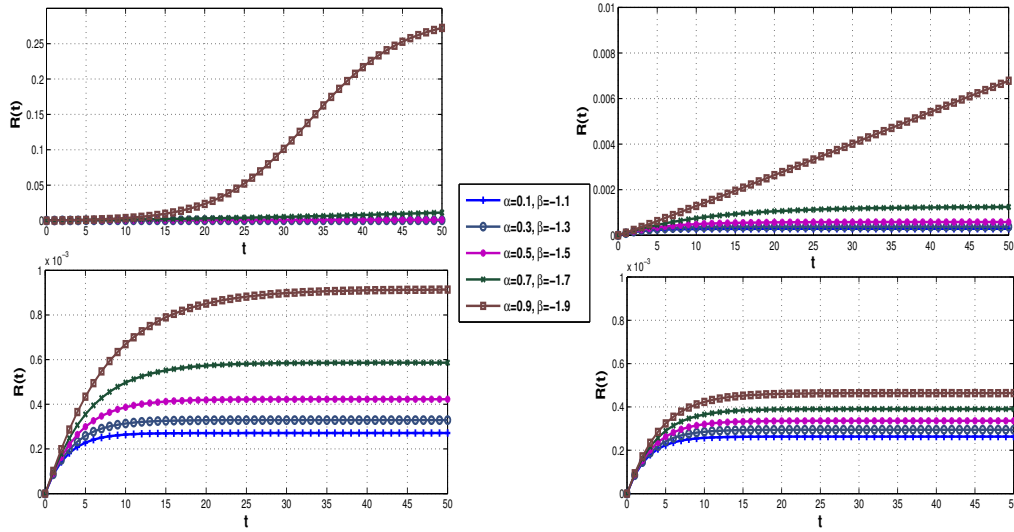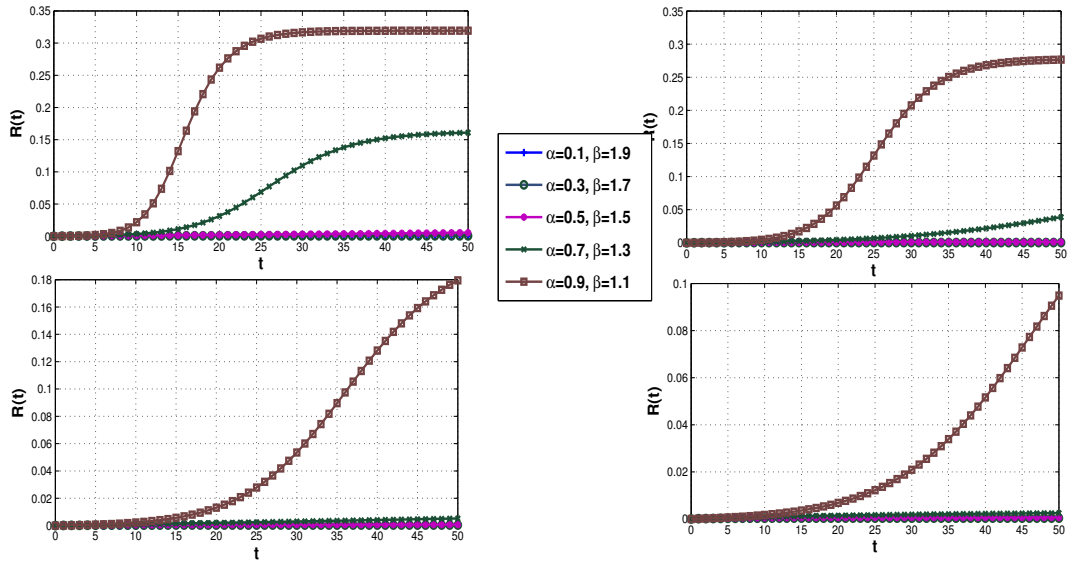
Figure 5.25: $R(t)$ vs t with $\alpha + \beta = 2$, $\lambda = 0.6$ in targeted inoculation for $g= 0.05, 0.1$ (upper) $0.15, 0.2$ (lower).

pendent tie strength exponent β have been introduced for this purpose. In the modified rumor spreading model, finite rumor spreading threshold has been found for finite scale free networks. Further, the rumor threshold has been found to be fixed for any size of network when $\alpha + \beta + 2 < \gamma$. Random and targeted inoculation schemes have been introduced in the proposed modified model. Rumor threshold in targeted inoculation scheme is found to be highest and without inoculation lowest in the modified model. For random inoculation, the rumor threshold will be in between the two extremes. It has also been observed that for scale free networks targeted inoculation scheme is successful in suppressing the rumor spreading in the network, to inoculate less number of nodes than in random inoculation. Further, the rumor threshold is found to be more sensitive against α than β as it affects more the rumor threshold. Finally, it is seen that in real world networks finite rumor threshold can be achieved by considering more realistic parameters (degree dependent tie strength of nodes and nonlinear spread of

rumor). The targeted inoculation scheme can be successfully applied to suppress the rumor spreading over scale free networks.

Chapter 6

Structural Centrality Based Inoculation on Rumor Dynamics in Complex Networks

6.1   Introduction

In social networks, the mechanism to suppress harmful rumors is of great importance. A rumor spreading model has been defined using the susceptible-infected-refractory (SIR) model to characterize rumor propagation in social networks. In this chapter, a new inoculation strategy based on structural centrality has been applied on rumor spreading model for heterogeneous networks. Using the proposed method, we can find the most influential node in the context of centrality measures in the graph [1]. It is compared with the targeted and random inoculations. The structural centrality of each node has been ranked in the topology of social networks which is modeled as scale free network. The nodes with higher structural centrality are chosen for inoculation in the proposed strategy. The structural centrality based inoculation strategy is more

[1]**Anurag Singh**, R. Kumar, and Y. N. Singh, "Effects of inoculation based on structural centrality on rumor dynamics in social networks," in *Computing and Combinatorics*.   Springer, 2013, pp. 831–840.

efficient in comparison with the random and targeted inoculation strategies. One of the bottleneck is the high complexity to calculate the structural centrality of the nodes for very large number of nodes in the complex networks. The proposed hypothesis has been verified using simulation results for email network data [70] and the generated scale free networks.

In this work, for all the simulations of the complex networks, the scale free property has been considered with power law degree distribution. The scale free properties are found in the real world networks e.g., email networks, Internet networks, telephone call graphs etc. [24].

6.2   Complex Network Topology Using Graph Spectra

The complex network topology can be understood by the graph structure [71, 72]. A graph is defined by $G = (V, E)$, where V is the set of vertices or nodes and E is the set of edges or links. $A = [a_{ij}]$ is an adjacency matrix of $|n \times n|$ size, where $n = |V|$, a_{ij} will be 1 if edge exists between i and j vertices otherwise 0 and $a_{ij} = a_{ji}$ for undirected (symmetric) graphs. The degree of the ith vertex, $d_i = \sum_j a_{ij}$ and $D = [d_i]$ is the degree matrix which is a diagonal matrix.

Spectral graph theory using eigenvalues and eigenvectors can be applied in the graphs to find out the structural centrality of the graphs. If a matrix is square, symmetric and positive semidefinite [73] then, eigenvectors and eigenvalues will exist for the matrix. Eigenvectors and eigenvalues exist for A, since the adjacency matrix A of a graph is symmetric, and it is positive semidefinite.

The Laplace matrix, L of the adjacency matrix A for graph G is given by $L = D - A$. The Laplace matrix of the graph is a positive semidefinite and symmetric, therefore it has all eigenvalues, i.e. $\lambda_i \geq 0$, $\forall i$. Hence, these eigenvalues (λ_i) ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0^2$, have eigenvectors $\vec{z_i}$ respectively such that $||\vec{z_i}||^2 = \vec{z_i}^T \vec{z_i} = 1$. The set of eigenvectors of L, $\vec{Z} = [\vec{z_1}, \ldots \vec{z_i} \ldots, \vec{z_n}]$, will be orthonormal i.e., $Z^T Z = I$. If Λ is a diagonal matrix, $\Lambda = [\lambda_{ii}]$ of eigenvalues then L follows the eigen decomposition as $L = \vec{Z} \Lambda \vec{Z}^T$.

6.3 Structural Centrality

From Laplace matrix L, Moore-Penrose pseudo inverse matrix L^+ can be defined. It follows all the properties (square, symmetric, doubly-centered, positive semidefinite) of L. The eigen decomposition of L^+ will be $\vec{Z}^T \Lambda^{-1} \vec{Z}$. $\vec{Z}$ is an orthonormal matrix made of the eigenvectors of L^+. If Λ has an eigenvalue value, $\lambda_i = 0$ then, corresponding eigenvalue λ^{-1} in Λ^{-1} will also be 0. As L^+ has the doubly centered (all rows and columns sum will be zero) property therefore, centroid of the nodes (having position vectors) lies on the origin of the space [73]. The graph matrix maps into the new euclidean space. We can represent each node by a unit vector $\vec{v}$ as,

$$\vec{v_i} = [0 \ - \ -- \quad 1 \ - \ - \ - \ 0]^T$$
$$i$$

$$\vec{v_j} = [0 \ - \ -- \quad 1 \ - \ - \ - \ 0]^T$$
$$j$$

Now, we can calculate distance between the nodes i and j in terms of number of

hops required to reach j from i and vice versa, is defined by average commute distance $n(j|i)$. Average commute distance measure is,

$$n(i,j) = m(j|i) + m(i|j) \qquad (6.3.1)$$

where, $m(i,j)$ is the first passage time, when a random walker starts from node $i \neq k$ and enter into state k first time. $n(i,j)$ will follow the distance measure for any node i,j and k,

1. $n(i,j) \geq 0$

2. $n(i,j) = 0$ iff $i = j$

3. $n(i,j) = n(j,i)$

4. $n(i,j) \leq n(i,k) + n(k,j)$

Therefore, using L^+ matrix and graph volume, V_G $(=\sum_{k=1}^{n} d_{kk})$, $n(i,j)$ can be expressed as [73],

$$n(i,j) = V_G(l_{ii}^+ + l_{jj}^+ - 2l_{ij}^+) \qquad (6.3.2)$$

Now ,the node vector $\vec{v_i}$ can be mapped into the new euclidean space by using the following transformations,

$$\vec{v_i} = \vec{Z}\vec{y_i}, \qquad (6.3.3)$$

$$\vec{y_i}' = \vec{\Lambda_i}^{1/2}\vec{y_i} \qquad (6.3.4)$$

Where, $\vec{y_i}$ is the transformation node vector. Now, Equation (6.3.2) can be decomposed as,

$$\bar{n}(i,j) = V_G(\vec{y_i}' - \vec{y_j}')^T(\vec{y_i}' - \vec{y_j}') \qquad (6.3.5)$$

Hence, in the new euclidean space the node vectors $\vec{y_i}$ and $\vec{y_j}$ are separated by average commute euclidean distance measure $(\bar{n}(i,j))$.

Therefore, euclidean distance measure for the node i from the origin can be found as the diagonal entry of the L^+,

$$||\vec{y_i'}||_2^2 = l_{ii}^+. \tag{6.3.6}$$

Definition If L_e be the Laplacian of the graph on n vertices consisting of just the edge e and $\vec{w} \in \Re^n$ then,

$$\vec{w}^T L \vec{w} = \sum_{e \in E} \vec{w}^T L_e \vec{w} = \sum_{(i,j) \in E} (\vec{w_i} - \vec{w_j})^2. \tag{6.3.7}$$

Definition Structural centrality is able to make the hierarchy from the most influential nodes to least influential nodes.

The structural centrality of node i for graph G is

$$SC(i) = \frac{1}{l_{ii}^+}. \tag{6.3.8}$$

From Equation (6.3.8), for the lower value of l_{ii}^+ the structural centrality (SC) will be high and vice versa . Therefore, the value of l_{ii}^+ determines the influential nodes.

If a node i is closer to origin in n- dimensional space then it will have lower value of l_{ii}^+, i.e., more centrally located in the network. Therefore, the value of l_{ii}^+ in pseudo inverse matrix L^+ can be defined as,

$$l_{ii}^+ = \sum_{k=1}^{n-1} \frac{\vec{z_{ki}}^2}{\lambda_k}. \tag{6.3.9}$$
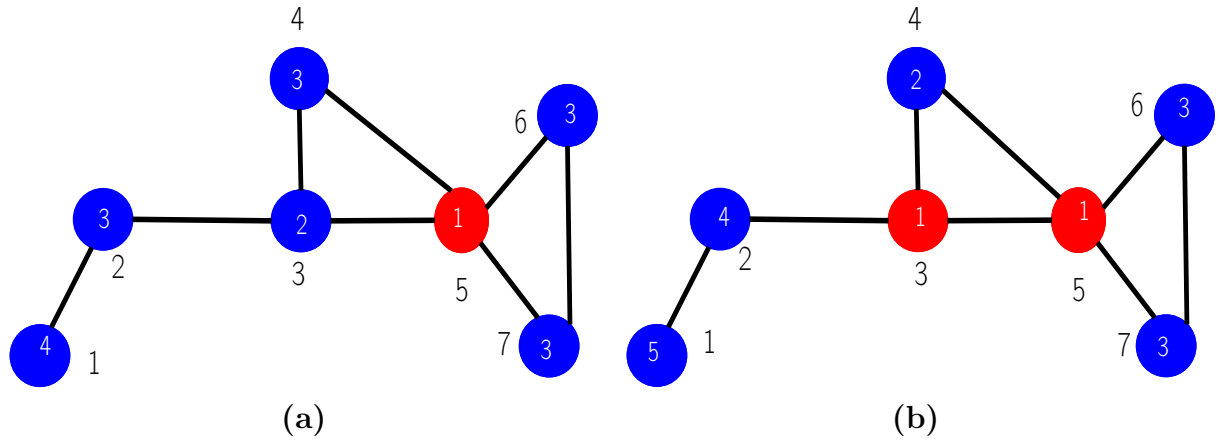
Figure 6.1: The node ranks in graph with (a) degree centralities (b) structural centralities mentioned inside the nodes.

It is observed from Equation (6.3.9) that the structural centrality of a node is defined by the eigenvectors and eigenvalues of the Laplace matrix, L of the graph.

The concept of the structural centrality can be understood with the help of an example given in Fig. 6.1. There are seven nodes in the graph and the hierarchy of their degrees is given in the center of the nodes. Hence, node 5 is the most influential in the case of targeted inoculation based on nodal degree as shown in Fig. 6.1(a). After defining the adjacency matrix A and degree matrix D of the given graph, we can calculate the Laplace matrix $L = D - A$ as

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

Laplace matrix, L holds following desirable properties to calculate the structural centrality,

1. Symmetric: $a_{ij} = a_{ji}$, in L

2. Square matrix : L is 7×7

3. Doubly centered: Summation of all rows and columns in L is 0

4. Positive semidefinite: Let $\vec{w}$ be any vector, i.e., $\vec{w} = \begin{bmatrix} -0.8507 \\ -0.5257 \end{bmatrix}$, then $\vec{w^T} = \begin{bmatrix} -0.8507 & -0.5257 \end{bmatrix}$, for edge between node 1 and 2, $L_{12} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, and

$$\vec{w}^T L_{12} \vec{w} = 0.3820. \tag{6.3.10}$$

Therefore, L will be positive semidefinite.

Using the L, pseudo inverse matrix L^+ can be generated as,

$$L^+ = \begin{pmatrix} 1.4626 & 0.6054 & -0.1088 & -0.3469 & -0.4422 & -0.5850 & -0.5850 \\ 0.6054 & 0.7483 & 0.0340 & -0.2041 & -0.2993 & -0.4422 & -0.4422 \\ -0.1088 & 0.0340 & 0.3197 & 0.0816 & -0.0136 & -0.1565 & -0.1565 \\ -0.3469 & -0.2041 & 0.0816 & 0.5102 & 0.0816 & -0.0612 & -0.0612 \\ -0.4422 & -0.2993 & -0.0136 & 0.0816 & 0.3197 & 0.1769 & 0.1769 \\ -0.5850 & -0.4422 & -0.1565 & -0.0612 & 0.1769 & 0.7007 & 0.3673 \\ -0.5850 & -0.4422 & -0.1565 & -0.0612 & 0.1769 & 0.3673 & 0.7007 \end{pmatrix}$$

From the above matrix diagonal values l_{ii}^+ are defined for i^{th} node respectively. Thus vector for l_{ii}^+ $\forall i$ is,

$$l_{ii}^+ = \begin{bmatrix} 1.462 & 0.7483 & 0.3197 & 0.5102 & 0.3197 & 0.7007 & 0.7007. \end{bmatrix}$$

After observing the above values of l_{ii}^+, it is found that nodes 3 and 5 have the most structural centrality in the network. Therefore, node 3 can also be most influential like node 5 (i.e. most influential in degree centrality).

6.4 Structural Centrality Inoculation

The diagonal elements, l_{ii}^+ can be sorted from low to high with their node numbers. Now, we will be able to get the list of the nodes sorted according to their degree centralities from high to low from Equation (6.3.8). Then, we can select fraction, g of inoculated nodes from the sorted array. Therefore, we will be able to inoculate most structurally central nodes first.

Random Inoculation

In random inoculation strategy, randomly selected node will be inoculated. This approach inoculates a fraction of nodes randomly, without any information about the network. Here, variable g $(0 \leq g \leq 1)$ defines the fraction of inoculated nodes. In the presence of random inoculation, rumor spreading rate λ is reduced by a factor $(1 - g)$.

Targeted inoculation

Scale free networks permit efficient strategies which depend upon the hierarchy of the degrees of nodes (degree centrality). The scale free networks are strongly affected by targeted inoculation of nodes [17]. In targeted inoculation, the high degree nodes have been inoculated as they are more likely to spread the information. In scale free networks, the robustness of the network decreases with a tiny fraction of inoculated individuals.

Let us assume that fraction g_k of nodes with degree k are successfully inoculated. An upper threshold of degree is k_t , so that all nodes with degree $k > k_t$ get inoculated

$(g_k = 1)$. Fraction g_k of the nodes with degree k are successfully inoculated.
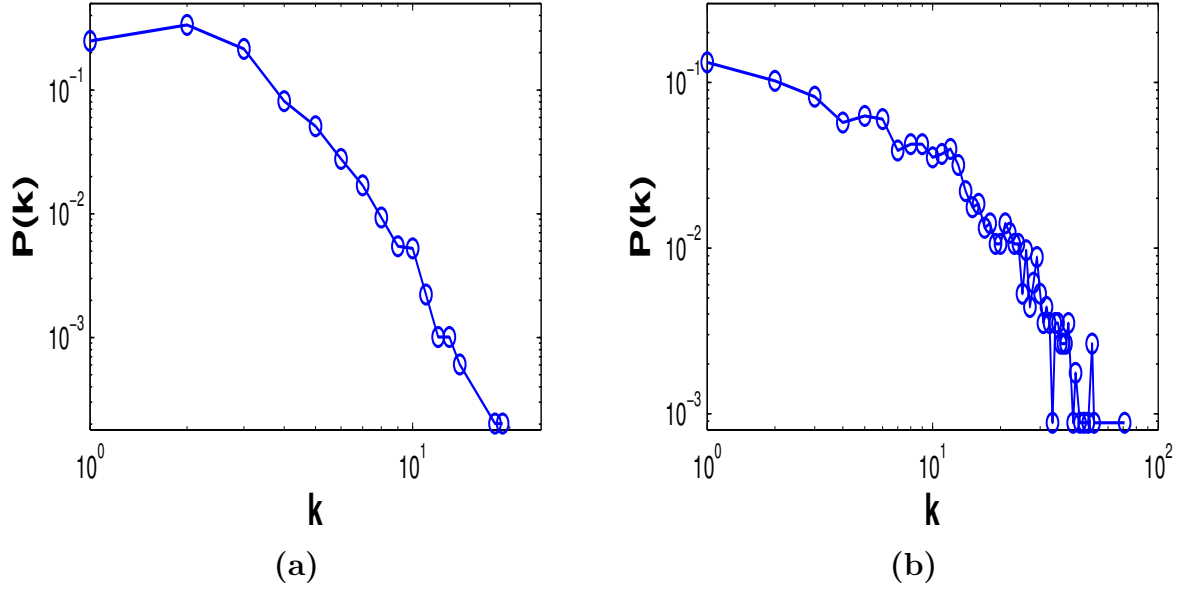

Figure 6.2: The degree distributions of (a) generated scale free network (b) Email network.

6.5 Rumor Spreading Model Used for Comparison

We have used rumor spreading model proposed in chapter 5 for unweighted network. The mean field equations for complex networks while considering non linearly varying number of informed neighbor nodes by a spreader in each time step (not all neighbors

of the node) have been used. Therefore, rate equations for rumor spreading model are,

$$\frac{dI(k,t)}{dt} = -\frac{k\lambda I(k,t)}{\langle k \rangle}\sum_l l^\alpha P(l)S(l,t), \tag{6.5.11}$$

$$\frac{dS(k,t)}{dt} = \frac{k\lambda I(k,t)}{\langle k \rangle}\sum_l l^\alpha P(l)S(l,t) - \frac{k\sigma S(k,t)}{\langle k \rangle}\sum_l [S(l,t) +$$
$$R(l,t)]l^\alpha P(l) - \delta S(k,t), \tag{6.5.12}$$

$$\frac{dR(k,t)}{dt} = \frac{k\sigma S(k,t)}{\langle k \rangle}\sum_l [S(l,t) + R(l,t)]l^\alpha P(l) + \delta S(k,t). \tag{6.5.13}$$

Where, λ, σ and δ are the rumor spreading, stifling and forgetting rates respectively. After solving Equations (1)-(3) for $\delta = 1$, the rumor threshold (below this spreading rate rumor will not spread in the network) is $\lambda_c = \frac{\langle k \rangle}{\langle k^{\alpha+1} \rangle}$

6.6   Simulations and Results

The numerical simulations have been carried out to observe the complete dynamical process with inoculation strategies with spreading ($\lambda = 0.5$), stifling ($\sigma = 0.2$) and spontaneous forgetting ($\delta = 1$) rates. Nodes interact with each other for rumor passing in each time step. After N nodes update their states according to the proposed rumor model, time step is incremented. To reduce the complexity, $\alpha = 1$ is considered. The scale free networks are generated according to the power law, $P(k) = k^{-\gamma}$, where $2 < \gamma \leq 3$ for $N = 5000$ and $\gamma = 2.3$ (Fig. 6.2 (a)). Email network has also been considered for the verification as real world complex network (Fig. 6.2 (b)).The random inoculation is implemented by selecting gN nodes randomly in the network. The targeted inoculation can be done after selection of the fraction of higher degree of nodes. The structural centrality inoculation can be done by getting the diagonal values, l_{ii}^+ of the pseudo inverse matrix L^+, for the corresponding node i. Using l_{ii}^+, we can sort

out the values in an array from low to high and inoculate fraction of the sorted nodes in the array.

(a)   **(b)**
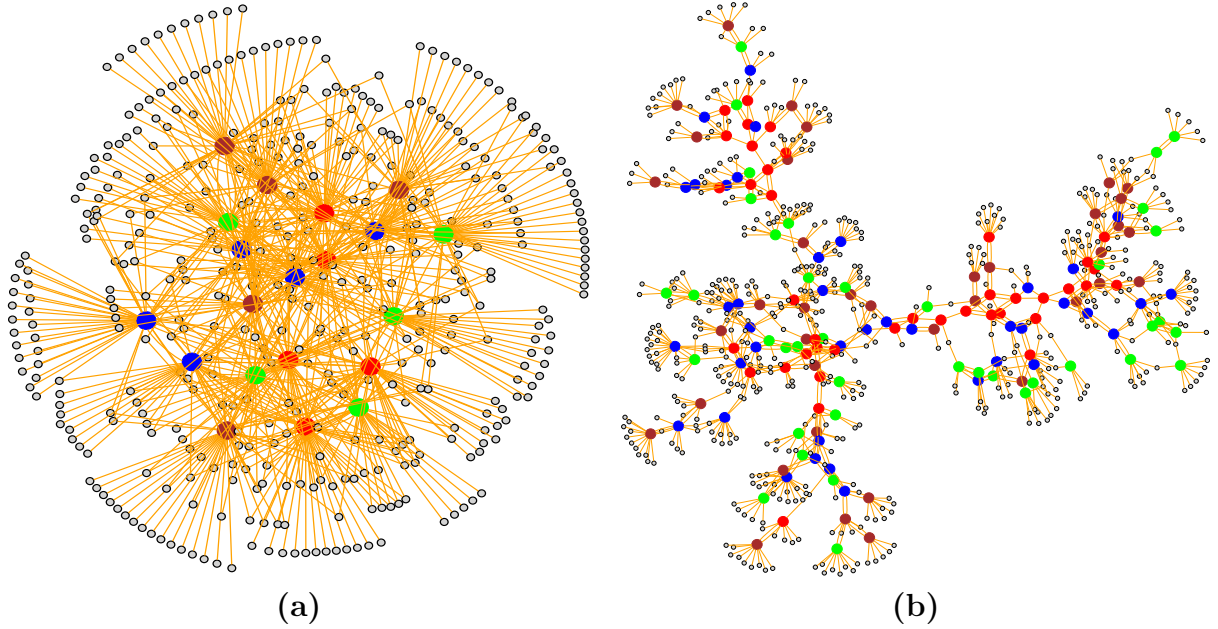
Figure 6.3: The structure of (a) Email network (b) generated scale free network with the different ranking of structural centralities (red $\rightarrow$ blue $\rightarrow$ brown $\rightarrow$ green nodes show higher to low ranking).

The structures of the email and generated scale free network are constructed for some nodes along with the structural centrality (Fig. 6.3). In the degree distribution of email network more number of very high degree nodes are found as compared with the generated scale free network, as shown in Fig. 6.2. Therefore, Fig. 6.3 (a) shows lot of edges around more number of higher degree nodes as compared to the generated scale free networks shown in Fig. 6.3 (b). The most structurally central node represented by red and least by green can be verified from Fig. 6.3. For high structurally central node, less number of hops are required to reach the other nodes, even at less degree. The most structurally centered node provides the well connected path between the two dense nodes shown as a sub-graph.
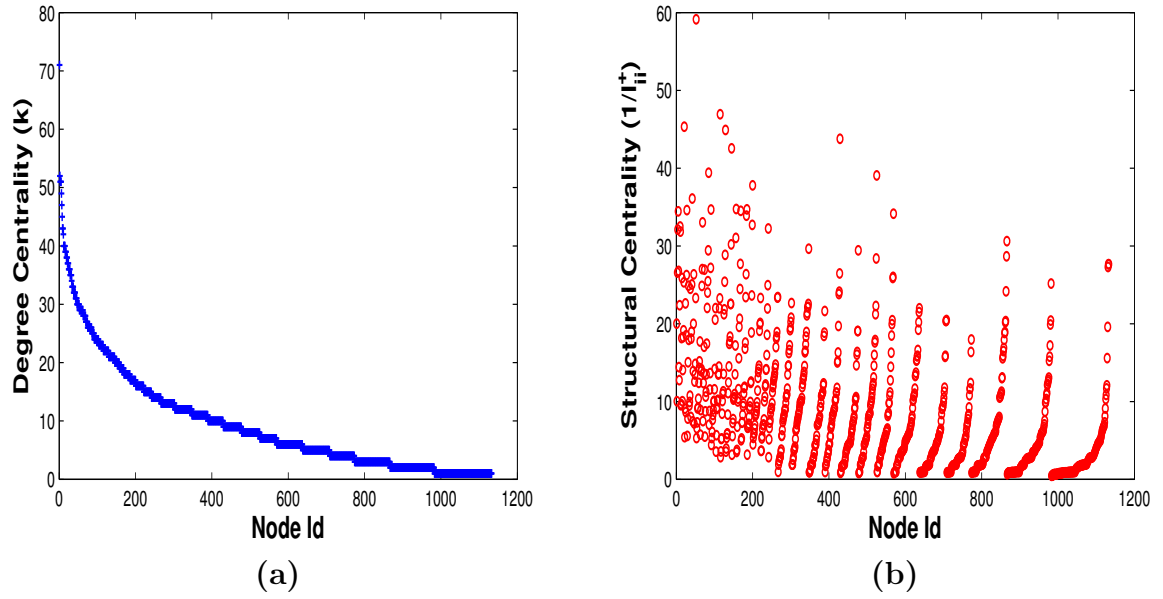
Figure 6.4: Distributions of (a) degree centralities (b) structural centralities with the node ids in email network.

In Fig. 6.4 (a), degree centrality has been mentioned for all the nodes in the decreasing order of degrees and corresponding node's structural centrality is shown in Fig. 6.4 (b) for email networks. It is observed that even with the less degree of nodes, the structural centrality is high, and can affect the network in the case of rumor spreading in comparison with the high degree nodes. Therefore, we observe influential nodes in the structural centrality. Hence, it is required to inoculate these nodes to suppress the rumor in the network.

Using the rumor model from Equations (6.5.11)-(6.5.13), rumor dynamics is studied for random inoculation, and targeted inoculation on the basis of nodal degree and structural centrality. In Fig. 6.5, evolution of size of rumor is plotted against time for email network. Final size of rumor is less in the structural centrality then the targeted inoculation for 10 % inoculation of nodes (Fig. 6.5 (a)). Similar pattern for

rumor evolution with time has been found for 30 % inoculations (Fig. 6.5), but rumor is almost suppressed in this case. Thus, the structural centrality based inoculation suppresses the rumor in the networks more effectively.
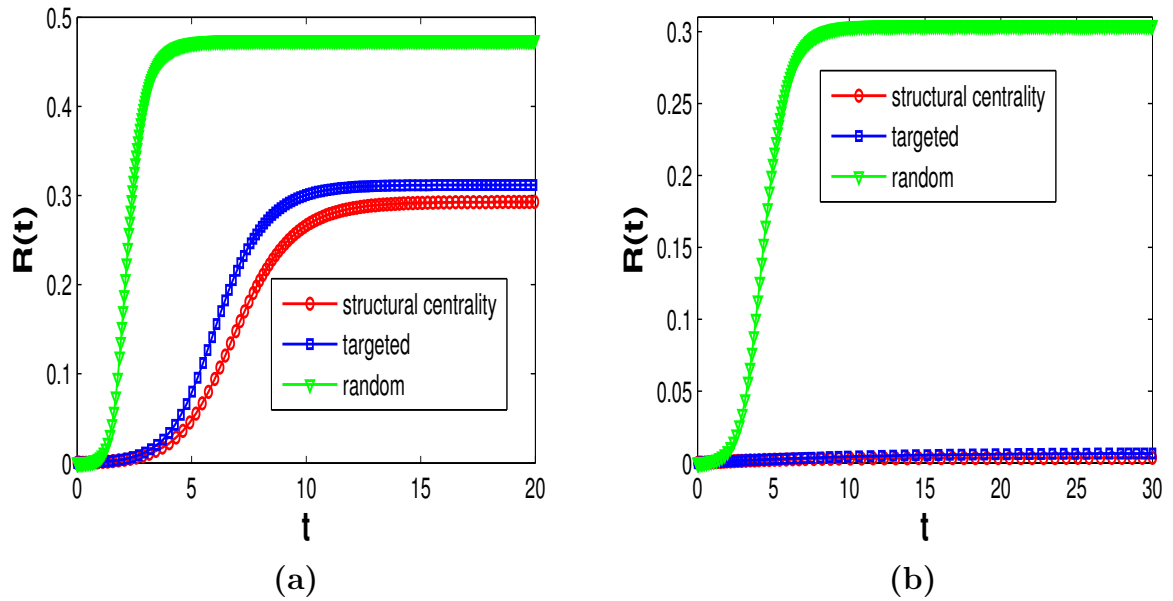
Figure 6.5: Rumor evolution with the time for (a) 10 % inoculations (b) 30 % inoculations for Email network.

Random inoculation is not much effective in both cases i.e., in email network and generated scale free network to suppress the rumor . In the case of generated scale free networks, for very small fraction of time, rumor size has been found to be higher in structural centrality based inoculations initially for 10 % as well as 30 % of inoculations of node, as shown in Fig. 6.6. But later rumor size decreases in the structural centrality based inoculation in comparison with targeted inoculations (the reason is, highest degree is very less in the network but number of high degree nodes are more). Therefore, degree centrality plays important role initially but later structural centrality plays its role.
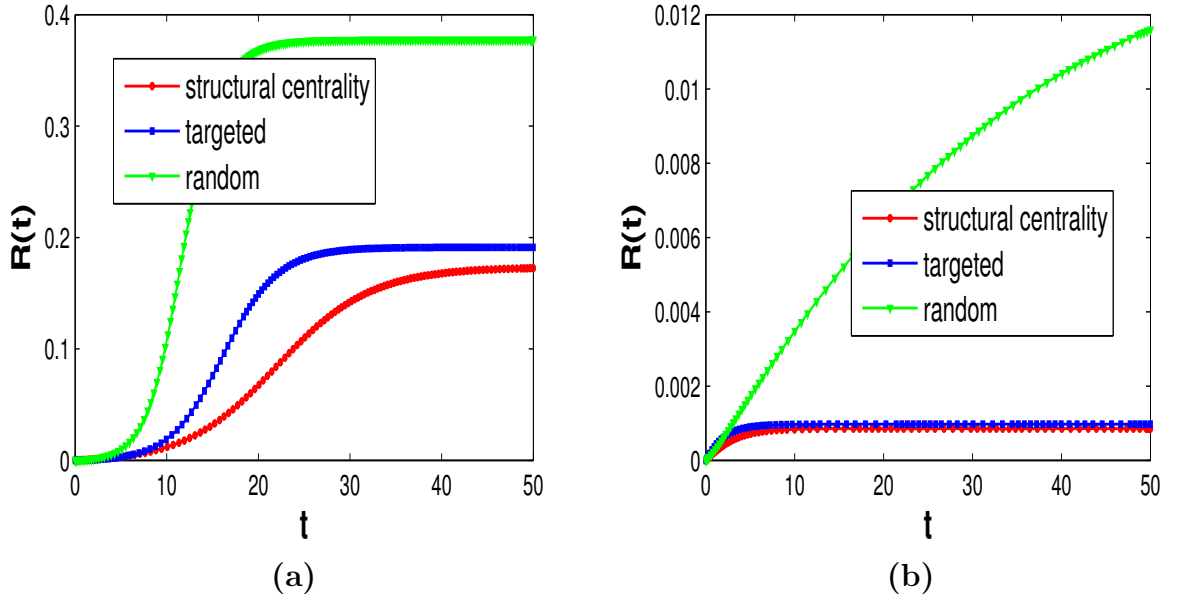
Figure 6.6: Rumor evolution with the time for (a) 10 % inoculations (b) 30 % inoculations for generated scale free network.

6.7   Conclusions

In scale free network we have derived the structural centralities of nodes in the complex networks and ranked it with the help of l_{ii}^{+} values. A node with the high structural centrality needs less number of hops to reach the other node, even at less degree. We have inoculated nodes according to the rank of structural centrality. After this we observed less rumor spreading than the targeted inoculation based on degree centrality and random inoculation. It has also been observed that there are lot of nodes, having low degree but high structural centrality and vice versa.

Chapter 7

Conclusions and Future works

In this work, a new compartment of nodes viz. stifler who rejects the rumor with rate ρ is added to the rumor spreading model. The proposed model supports the small critical inoculation value g_c in random as well as targeted inoculation to control rumor spreading when average degree $\langle k \rangle$ of small world network is small. It is also found that in the targeted inoculation, g_c is smaller than in the case of random inoculation when degree $\langle k \rangle$ is small. When $\langle k \rangle$ is high, mean field approximation fails. In this case, random or targeted inoculation alone will not be effective. Therefore, one should decrease the acceptability of the rumor and apply either random or targeted inoculation method at the same time. After doing this, we got a small value of g_c to control rumor spreading. We have also investigated the rumor diffusion mechanism in scale free networks, with the new compartment of stifling nodes who reject the rumors. After decreasing the rumor acceptability factor, the population who reject the rumor increases. If the degrees of nodes are known, then targeted inoculation strategy is found to be the best for scale free networks. But, if this information about the scale free networks is not available, then neighbor inoculation strategy can be applied which is better than the random inoculation to control the rumor. It has also been observed that decreasing

the rumor acceptability factor makes the inoculation more effective in controlling the rumor in scale free networks.

The rumor spreading model has also been extended to correlated networks. As expected, it is observed that rumor can be stopped more effectively in correlated networks using targeted inoculation. The degree distribution in the reduced network after random inoculation is found to be independent of degree-degree correlation. Thus the performance of random inoculation is independent even if degree-degree correlation is present or not. However, it does not hold true for targeted inoculations. It is interesting to observe that for small values of rumor transmission rate (λ), final size of rumor in uncorrelated networks is larger than in correlated one with or without inoculation. On the other hand, for higher values of transmission rate, rumor size is lower in uncorrelated scale free networks. It is concluded that removal of connections among hub nodes can decrease the rumor spreading.

A modified SIR model has also been proposed by considering standard SIR rumor spreading model with degree dependent tie strength of nodes and nonlinear spread of rumor. The two parameters - nonlinear exponent α and degree dependent tie strength exponent β have been introduced for this purpose. In the modified rumor spreading model, finite rumor spreading threshold has been found for finite scale free networks. Further, the rumor threshold has been found to be fixed for any size of network when $\alpha + \beta + 2 < \gamma$. Random and targeted inoculation schemes have been introduced in the proposed modified model. Rumor threshold in targeted inoculation scheme is found to be highest and without inoculation lowest in the modified model. For random inoculation, the rumor threshold will be in between the two extremes. It has also been observed that for scale free networks targeted inoculation scheme is successful in suppressing the rumor spreading in the network since, one need to inoculate less number

of nodes than in random inoculation. Further, the rumor threshold is found to be more sensitive against α in comparison with β. Finally, it is seen that in real world networks finite rumor threshold can be achieved by considering more realistic parameters (degree dependent tie strength of nodes and nonlinear spread of rumor).

At last, in scale free network we have derived the structural centralities of nodes in the complex networks and ranked it with the help of l_{ii}^{+} values. A node with the high structural centrality needs less number of hops to reach the other node, even at less degree. We have inoculated nodes according to the rank of structural centrality. After this we observed less rumor spreading than the targeted inoculation based on degree centrality and random inoculation. It is also observed that there are lot of nodes, having low degree but high structural centrality and vice versa.

Many more problems worth further investigations have been found during the course of this research work. Some of the problem which can be further pursued are given below:

1. In the present work, the rumor spreading rate has been assumed to be constant. Once take it to be variable entity which may change with progress of rumor spread.

2. Some of the nodes can be considered as trusted, information received from these nodes will be considered reliable information (not rumor).

3. Some new methods can be investigated to find more influential nodes for the real world networks to inoculate them.

4. Complex network has been considered undirected to define the rumor spreading models, in this thesis. This work can be extended for directed networks also.

Appendix A

Chapter 3

A.1

Initial conditions of our model are

$I(0) \approx 1, S(0) \approx 0, R_{acc}(0) = 0, R_{rej}(0) = 0$ and $S(\infty) = 0$. From Equations (3.1.3)
-(3.1.6),

$$\int_0^t \frac{1}{I(\tau)} \, dI(\tau) = -(\lambda + \rho + \eta)\langle k \rangle \int_0^t S(\tau) \, d\tau,$$
$$I(t) = I(0)e^{[-(\lambda+\rho+\eta)\langle k \rangle \int_0^t S(\tau) \, d\tau]}. \tag{A.1.1}$$

From Equation (3.0.1),

$$S(t) + R(t) = 1 - I(t).$$

From Equation (3.1.7), we get,

$$R(0) \approx 0.$$

$$\frac{dR(t)}{dt} = \sigma \langle k \rangle S(t)[1 - I(t)] + [\rho + \eta \langle k \rangle I(t)S(t)] \tag{A.1.2}$$

From Equation (A.1.2),

$$
\begin{aligned}
\frac{dR(t)}{dt} &= \frac{dR_{acc}(t)}{dt} + \frac{dR_{rej}(t)}{dt}, \\
\frac{dR(t)}{dt} &= \sigma\langle k\rangle S(t)(S(t) + R(t)) + \\
&\quad \langle k\rangle I(t)S(t)(\rho + \eta).
\end{aligned}
\tag{A.1.3}
$$

From Equation (3.0.1) and (3.1.7),

$$
\frac{dR(t)}{dt} = \sigma\langle k\rangle S(t)[1 - I(t)] + (\rho + \eta)\langle k\rangle I(t)S(t).
\tag{A.1.4}
$$

After integrating Equation (A.1.4),

$$
\begin{aligned}
\int_0^t dR(\tau) &= \sigma\langle k\rangle \int_0^t S(\tau)\,d\tau + \frac{\sigma - \rho - \eta}{\lambda + \rho + \eta}\int_0^t 1\,dI(\tau), \\
R(t) - R(0) &= \sigma\langle k\rangle \int_0^t S(\tau)\,d\tau + \frac{\sigma - \rho - \eta}{\lambda + \rho + \eta}[I(t) - I(0)].
\end{aligned}
$$

For $t = \infty$,

$$
\begin{aligned}
R(\infty) &= \sigma\langle k\rangle \int_0^t S(\tau)\,d\tau + \frac{\sigma - \rho - \eta}{\lambda + \rho + \eta}[1 - R(\infty) - 1], \\
\langle k\rangle \int_0^t S(\tau)\,d\tau &= \frac{R(\infty)}{\sigma}\left\{\frac{\lambda + \sigma}{\lambda + \rho + \eta}\right\}.
\end{aligned}
\tag{A.1.5}
$$

Put value of Equation (A.1.5) into Equation (A.1.1),

$$
R(\infty) = 1 - e^{-\frac{\lambda + \sigma}{\sigma}R(\infty)}
$$

A.2

Similarly by using initial conditions for g fraction of initial inoculation of the nodes in our model.

$I(0) \approx 1 - g, S(0) \approx 0, R_{acc}(0) = 0, R_{rej}(0) = g$ we will get,

$$R(\infty) = 1 - (1 - g)e^{\frac{\lambda+\sigma}{\sigma}g}e^{-\frac{\lambda+\sigma}{\sigma}R(\infty)}.$$

A.3

From Equations (3.1.3) -(3.1.6),

$$\frac{dI(t)}{dR_{rej}(t)} = \frac{-(\lambda + \rho + \eta)}{\rho}$$

After integrating with t = 0 to ∞,

$$I(\infty) - I(0) = \frac{-(\lambda + \rho + \eta)}{\rho}[R_{rej}(\infty) - R_{rej}(0)].$$

Putting $I(\infty) = 1 - R(\infty)$,

$$R_{rej}(\infty) = \frac{\rho}{\lambda + \rho + \eta}R(\infty).$$

As $R(\infty) = R_{rej}(\infty) + R_{acc}(\infty)$, therefore

$$R_{acc}(\infty) = \frac{(\lambda + \eta)}{\lambda + \rho + \eta}R(\infty).$$

A.4

Let $\frac{\lambda+\sigma}{\sigma} = \beta$,

$$R(\infty) = 1 - (1 - g)e^{\beta(g-R(\infty))}$$

If $(g - R(\infty)) \geq 0$, $R(\infty)$ goes to 0 and for $(g - R(\infty)) < 0$, then $R(\infty)$ goes to 1. Therefore, $R(\infty)$ lies between 0 and 1. For nonzero solution for $R(\infty)$, $(g - R(\infty)) = 0$, thus $R(\infty)$ will be g.

A.5

Form the given rumor equations,

$$\frac{dI(k,t)}{dt} = -k(\lambda + \rho + \eta)I(k,t)\sum_l P(l|k)S(l,t), \tag{A.5.6}$$

$$\frac{dR_{rej}(k,t)}{dt} = \rho k I(k,t)\sum_l P(l|k)S(l,t). \tag{A.5.7}$$

From Eq. (A.5.6),

$$I(k,\infty) = exp\left(\frac{-k(\lambda + \rho + \eta)}{\langle k \rangle}\Theta\right). \tag{A.5.8}$$

From Eqs. (A.5.6) and (A.5.7),

$$\frac{dR_{rej}(k,t)}{dt} = \frac{-\rho}{(\lambda + \rho + \eta)}\frac{dI(k,t)}{dt}. \tag{A.5.9}$$

After integrating both sides,

$$R_{rej}(k,t) = \frac{-\rho}{(\lambda + \rho + \eta)}I(k,t) + C; \ C \text{ is an integrating constant.} \tag{A.5.10}$$

$$C = \frac{\rho}{(\lambda + \rho + \eta)}I(k,0); \ I(k,0) \text{ is the initial fraction of ignorants of degree } k, \text{ is almost 1.}$$

Final size of rumor $R(\infty)$ is,

$$R(\infty) = \sum_l P(l)R(l,\infty), \tag{A.5.11}$$

$$R(\infty) = R_{acc}(\infty) + R_{rej}(\infty). \tag{A.5.12}$$

For $t \to \infty$, Eq. (A.5.10) will be

$$R_{rej}(k, \infty) \;\;=\;\; \frac{\rho}{(\lambda + \rho + \eta)}\left[1 - exp\left(\frac{-k(\lambda + \rho + \eta)}{\langle k \rangle}\Theta\right)\right]. \tag{A.5.13}$$

After multiplying $P(k)$ both sides of Eq. (A.5.14), and summing over all values of k,

$$R_{rej}(\infty) \;\;=\;\; \frac{\rho}{(\lambda + \rho + \eta)}\left[1 - \sum_k P(k)exp\left(\frac{-k(\lambda + \rho + \eta)}{\langle k \rangle}\Theta\right)\right]. \tag{A.5.14}$$

$$R_{rej}(k, \infty) = \frac{\rho}{\lambda + \eta + \delta}R_{acc}(k, \infty). \tag{A.5.15}$$

From Eq. (A.5.12),

$$R_{rej}(\infty) \;\;=\;\; R(\infty) - R_{acc}(\infty).$$

Since $I(\infty) + R_{rej}(\infty) + R_{acc}(\infty) = 1$ or $I(\infty) + R(\infty) = 1$, therefore $R(\infty)$ can be calcultaed using Eq. (A.5.11) as,

$$R(\infty) \;\;=\;\; \sum_l P(l)\left(1 - exp\left(\frac{-(\lambda + \rho + \eta)l\Theta}{\langle k \rangle}\right)\right). \tag{A.5.16}$$

$$\tag{A.5.17}$$

Now, using Eqs. (A.5.16)-(A.5.14),

$$R_{acc}(\infty) \;\;=\;\; = \frac{\lambda + \eta}{\lambda + \eta + \rho}\left[1 - \sum_l P(l)exp\left(\frac{-(\lambda + \eta + \rho)l\Theta}{\langle k \rangle}\right)\right]. \tag{A.5.18}$$

Appendix B

Chapter 5

Rumor threshold (λ_c) for $2 < \gamma < 3$

B.1

$$k_{max} = k_{min} N^{\frac{1}{\gamma - 1}} \tag{B.1.1}$$

$$
\begin{aligned}
\lambda_c &= \frac{\langle k \rangle}{\langle k^2 \rangle} \\
&= \frac{\int_{k_{min}}^{k_{max}} k P(k) dk}{\int_{k_{min}}^{k_{max}} k^2 P(k) dk} \\
&= \frac{\int_{k_{min}}^{k_{max}} k^{1-\gamma} dk}{\int_{k_{min}}^{k_{max}} k^{2-\gamma} dk} \\
&\quad (P(k) \propto k^{-\gamma} \text{ for SF network}) \\
&= \frac{(3-\gamma)}{2-\gamma} \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}
\end{aligned}
\tag{B.1.2, B.1.3}
$$

In eq. (B.1.3), numerator and denominator divided by $k_{max}^{3-\gamma}$,

$$\lambda_c = \frac{(3-\gamma)}{2-\gamma} \frac{1/k_{max} - (k_{min}^{2-\gamma}/k_{max})^{3-\gamma}}{1 - (k_{min}/k_{max})^{\gamma-3}} \qquad (B.1.4)$$

For leading order in k_{max}/k_{min},

$$\lambda_c = \frac{(3-\gamma)}{(\gamma-2)k_{min}}(k_{min}/k_{max})^{\gamma-3} \qquad (B.1.5)$$

B.2

Rumor threshold for $\gamma = 3$ can be calculated from Eq. (B.1.2),

$$\lambda_c = \frac{\int_{k_{min}}^{k_{max}} k^{1-\gamma}dk}{\int_{k_{min}}^{k_{max}} k^{2-\gamma}dk}$$

Put $\gamma = 3$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad (B.2.6)$

$$= \frac{\int_{k_{min}}^{k_{max}} k^{-2}dk}{\int_{k_{min}}^{k_{max}} k^{-1}dk} \qquad (B.2.7)$$

$$= \left[\frac{1}{k_{min}} - \frac{1}{k_{max}}\right] \ln(k_{max}/k_{min}) \qquad (B.2.8)$$

From Eq. (B.1.1),

$$\lambda_c = \left[\frac{1}{k_{min}} - \frac{1}{k_{max}}\right] \ln(N^{1/2})$$

For the leading order,

$$\simeq 2[k_{min} \ln(N)]^{-1} \qquad (B.2.9)$$

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List of Publications

Journal and Conferences:

1. **Anurag Singh** and Y. N. Singh, "Nonlinear spread of rumor and inoculation strategies in the nodes with degree dependent tie stregth in complex networks," *Acta Physica Polonica B*, vol. 44, no. 1, pp. 5–28, Jan 2013.

2. **Anurag Singh** and Y. N. Singh, "Rumor spreading and inoculation of nodes in complex networks," in *Proceedings of the 21st international conference companion on World Wide Web*, ser. WWW '12 Companion. New York, NY, USA: ACM, 2012, pp. 675–678.

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5. **Anurag Singh**, R. Kumar, and Y. N. Singh, "Effects of inoculation based on structural centrality on rumor dynamics in social networks," in *Computing and Combinatorics.* Springer, 2013, pp. 831–840.

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