A New Family of Time-Limited Nyquist Pulses for OFDM Systems

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***Abstract*— We first provide the motivation for a new family of time-limited Nyquist pulses for Orthogonal Frequency Division Multiplexing (OFDM) systems by multiplying existing Nyquist pulses in the frequency domain. The condition required for the resultant pulses to have a roll-off factor less than or equal to unity is then established. The roll-off factor is found to be a function of two design parameters. Thus, once the roll-off factor is fixed, there is an opportunity to optimize the two design parameters, such that the signal-to-interference ratio (SIR) in the presence of carrier frequency offset is maximized. As an illustration, a new pulse is designed by multiplying the time-limited Better Than Raised Cosine and RC pulses in the frequency domain, followed by optimization for SIR maximization. The obtained pulse is compared with a pulse, which is known to provide the best out-of-band (OOB) power performance among the known pulses for OFDM systems. From the comparison, it is clear that the obtained pulse provides better SIR performance while maintaining the same OOB power performance.**

***Index Terms*— OFDM, pulse shaping, out-of-band power, inter-carrier interference.**

I. Introduction

TRANSMIT pulses for Orthogonal Frequency Division Multiplexing (OFDM) systems have two major requirements. The pulses should produce low Out-of-Band (OOB) power at the transmitter and low Inter Carrier Interference (ICI) at the receiver in the presence of Carrier Frequency Offset (CFO).

Recently, a pulse with an Asymptotic Decay Rate (ADR) of $1/f^5$ has been shown to provide the best OOB performance among the known pulses for OFDM systems [1]. But its ICI performance is inferior even to the RC pulse.

Several time-limited Nyquist pulses that provide low ICI power and, hence high Signal-to-Interference Ratio (SIR), in the presence of CFO have been proposed in [2]–[6]. The ICI performance is known to depend mainly on the height of the first two sidelobes [2].

In view of the above, we can say that the desired characteristic of a time-limited Nyquist pulse for an OFDM system is - ADR should be as high as possible (for low OOB) and height of the first few sidelobes should be as small as possible (for low ICI). Now, if we multiply two time-limited Nyquist pulses in the frequency domain, the height of the sidelobes in the resultant pulse will be lower than that of either of the pulses (assuming the height of the sidelobes in both the pulses is less than unity). Also, the multiplication will result in a higher ADR. This provides a natural motivation for considering frequency domain multiplication of two time-limited Nyquist pulses.

Manuscript received June 20, 2016; accepted July 20, 2016. Date of publication August 3, 2016; date of current version October 7, 2016. The associate editor coordinating the review of this letter and approving it for publication was M. Matthaiou.

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Digital Object Identifier 10.1109/LCOMM.2016.2597841

In this paper, we investigate the result of frequency domain multiplication of two time-limited Nyquist pulses. We first establish a necessary and sufficient condition for the resultant pulse to be Nyquist with roll-off factor less than or equal to unity. The roll-off factor is found to be a function of two parameters which provide an additional dimension for designing an appropriate Nyquist pulse even after the roll-off factor is fixed. This additional dimension can be used for optimizing the resultant Nyquist pulse with respect to any desired criteria such as maximizing the SIR in the presence of CFO. We also consider a specific example - the multiplication of Better Than Raised Cosine (BTRC) and RC pulses and optimize it with respect to the SIR criterion. The resultant pulse is found to have much better SIR performance than [1] while maintaining the same OOB performance.

The paper is organized as follows: In Section II we discuss the multiplication of two time-limited Nyquist pulses, in Section III we establish the necessary condition for the roll-off factor to be less than or equal to unity, in Section IV we discuss pulse design for SIR maximization, in Section V we consider an example while in Section VI we compare its OOB and SIR performance with the RC pulse and the pulse reported in [1]. Section VII concludes the paper.

II. Multiplication of Two Time-Limited Nyquist Pulses

Let us consider the multiplication of two time-limited Nyquist pulses $P_1(f)$ and $P_2(f)$ in the frequency domain to obtain a new pulse. We refer to the new pulse as the M pulse and denote it as $P_{\mathrm{M}}(f)$. The frequency domain representation of the M pulse $P_{\mathrm{M}}(f)$ is given by

$$P_{\mathrm{M}}(f) = P_1(f) \cdot P_2(f) \tag{1}$$

If the ADR's of the multiplying pulses $P_1(f)$ and $P_2(f)$ are $1/f^u$ and $1/f^v$, the ADR of $P_{\mathrm{M}}(f)$ will be $1/f^{(u+v)}$.

Since multiplication in the frequency domain corresponds to convolution in the time domain, the time domain representation of the M pulse $p_{\mathrm{M}}(t)$ can be expressed as

$$p_{\mathrm{M}}(t) = p_1(t) * p_2(t) \tag{2}$$

where $p_{\mathrm{M}}(t)$, $p_1(t)$ and $p_2(t)$ are the time domain representations of the pulses $P_{\mathrm{M}}(f)$, $P_1(f)$ and $P_2(f)$, respectively. The pulses $p_1(t)$ and $p_2(t)$ can be expressed as (for $i = 1, 2$)

$$p_i(t) = \begin{cases} 1, & 0 \le |t| < \dfrac{T_i(1-\alpha_i)}{2} \\ x_i(t), & \dfrac{T_i(1-\alpha_i)}{2} \le |t| < \dfrac{T_i(1+\alpha_i)}{2} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where the roll-off factor $\alpha_i \leq 1$. In the above equation, T_i denotes the duration of $p_i(t)$ when $\alpha_i = 0$ and $1/T_i$ is the minimum subcarrier frequency spacing required when the pulse is to be used in an OFDM system.

Both $p_1(t)$ and $p_2(t)$ are time-limited Nyquist pulses, hence $P_1(f)$ and $P_2(f)$ will have zero crossings at integral values of $\pm 1/T_1$ and $\pm 1/T_2$ respectively. As a result, $P_{\text{M}}(f)$ in (1) will have zero crossings at both $\pm k/T_1$ and $\pm k/T_2$, for integer values of k and hence it will be Nyquist for two different sets of T_i and α_i. Let us denote them as $\alpha_{\text{M}_1}, T_{\text{M}_1}$ and $\alpha_{\text{M}_2}, T_{\text{M}_2}$.

Furthermore, $P_{\text{M}}(f)$ will satisfy the subcarrier orthogonality condition for zero ICI in OFDM systems that is [2]

$$\int p(t) e^{-\frac{j2\pi kt}{T_{\text{M}_i}}} dt = P\left(\frac{k}{T_{\text{M}_i}}\right) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } k = \pm 1, \pm 2, \dots \end{cases}$$

for the subcarrier frequency spacings $1/T_{\text{M}_1}$ and $1/T_{\text{M}_2}$.

The duration of the pulse $p_{\text{M}}(t)$ can be expressed as $T_{\text{M}_1}(1 + \alpha_{\text{M}_1})$ as well as $T_{\text{M}_2}(1 + \alpha_{\text{M}_2})$. Since the duration is also equal to the sum of the duration of the pulses $p_1(t)$ and $p_2(t)$, we can say that

$$T_{\text{M}_1}(1 + \alpha_{\text{M}_1}) = T_{\text{M}_2}(1 + \alpha_{\text{M}_2}) = T_1(1 + \alpha_1) + T_2(1 + \alpha_2). \tag{4}$$

Furthermore, $p_{\text{M}}(t)$ will satisfy the Nyquist for both T_{M_1} and T_{M_2}, i.e.

$$\sum_{k=-\infty}^{\infty} p_{\text{M}}(t - kT_{\text{M}_1}) = A \tag{5}$$

and

$$\sum_{k=-\infty}^{\infty} p_{\text{M}}(t - kT_{\text{M}_2}) = B \tag{6}$$

where A and B are constants.

The pulse $p_{\text{M}}(t)$ is Nyquist but its roll-off factor is not guaranteed to be less than or equal to unity. For time-limited Nyquist pulses used in practical OFDM systems, the roll-off factor is required to be less than or equal to unity. Also, If the roll-off factor is greater than unity, $p_{\text{M}}(t)$ cannot be expressed in the form given in (3).

In the next section, we seek a necessary condition such that $\alpha_{\text{M}} \leq 1$.

III. Derivation of the Condition for $\alpha_{\text{M}} \leq 1$

Without loss of generality, we assume that $T_1 > T_2$. Therefore the first zero crossing of $P_{\text{M}}(f)$ will be at $1/T_1$ and the minimum subcarrier frequency spacing possible in an OFDM system using $p_{\text{M}}(t)$ will be $1/T_1$. Thus, we can take

$$T_{\text{M}_1} = T_1. \tag{7}$$

From (4) and (7), α_{M_1} can be expressed as

$$\alpha_{\text{M}_1} = \alpha_1 + \frac{T_2(1 + \alpha_2)}{T_1} \tag{8}$$

For $0 \leq \alpha_1, \alpha_2 \leq 1$, the range of α_{M_1} can be expressed as

$$\frac{T_2}{T_1} \leq \alpha_{\text{M}_1} \leq 1 + 2\frac{T_2}{T_1}$$

Hence, the values of α_1, α_2, T_1 and T_2 such that the condition

$$0 \leq \alpha_{\text{M}_1} \leq 1 \tag{9}$$

is satisfied can be obtained from (8) as

$$\alpha_1 + \frac{T_2(1 + \alpha_2)}{T_1} \leq 1$$
$$\Rightarrow T_2(1 + \alpha_2) \leq T_1(1 - \alpha_1). \tag{10}$$

Thus, (10) is a necessary condition for the roll-off factor of $p_{\text{M}}(t)$ to be less than or equal to unity. It can be easily shown that the condition in (10) is also a sufficient condition for the roll-off factor of $p_{\text{M}}(t)$ to be less than or equal to unity.

Now let us consider the other scenario in which the subcarrier frequency spacing is taken to be $1/T_2$ i.e. $T_{\text{M}_2} = T_2$.

From (4) the corresponding roll-off factor α_{M_2} will be given by

$$\alpha_{\text{M}_2} = \alpha_2 + \frac{T_1(1 + \alpha_1)}{T_2} \tag{11}$$

and for $0 \leq \alpha_1, \alpha_2 \leq 1$, the range of α_{M_2} is given by

$$\frac{T_1}{T_2} \leq \alpha_{\text{M}_2} \leq 1 + 2\frac{T_1}{T_2} \tag{12}$$

Since $T_1 > T_2$, it is clear that α_{M_2} will always be greater than unity. Hence, this case is not of much interest.

Let us denote the parameters of the desired M pulse as α_{M} and T_{M}. In view of the above, these parameters can be taken as $\alpha_{\text{M}} = \alpha_{\text{M}_1}$ and $T_{\text{M}} = T_{\text{M}_1}$. Using (7) and (8), we obtain

$$T_{\text{M}} = T_1 \tag{13}$$

$$\alpha_{\text{M}} = \alpha_1 + \frac{T_2(1 + \alpha_2)}{T_1} \tag{14}$$

From (13) and (14), it can be observed that for a given T_{M} the roll-off factor α_{M} is a function of the three parameters $\{\alpha_1, \alpha_2, T_2\}$. Hence, once the roll-off factor α_{M} is fixed, there are infinitely many choices for any two of the three parameters α_1, α_2 and T_2 and, in principle, one can optimize their value with respect to some other performance criteria, for eg. SIR.

Next we discuss how to obtain the values of α_1, α_2 and T_2 such that the SIR for a desired T_{M} and α_{M} is maximized.

IV. Pulse Design for SIR Maximization

For a time-limited Nyquist pulse at the transmitter and a rectangular pulse at the receiver, the SIR expression in the presence of a CFO of Δf is given in [2]–[4] and [7]. It is given by

$$\text{SIR} = \frac{|P_{\text{M}}(\Delta f)|^2}{\sum_{k=0, k \neq m}^{N-1} |P_{\text{M}}(\frac{m-k}{T_{\text{M}}} + \Delta f)|^2} \tag{15}$$

It can be seen that the SIR depends on $P_{\text{M}}(f)$. Thus, by optimizing α_1, α_2 and T_2 one can find the $P_{\text{M}}(f)$ which will provide the maximum SIR.

Let us take α_1 and α_2 as the free variables which are to be optimized. Then the problem can be formulated as

$$\begin{aligned} &\underset{\alpha_1, \alpha_2}{\text{argmax}} \ \text{SIR} \\ &\text{subject to } 0 \leq \alpha_1 < \alpha_{\text{M}} \\ &\qquad\qquad 0 \leq \alpha_2 \leq 1. \end{aligned} \tag{16}$$

Algorithm 1 Parameters for SIR Maximization

1: **procedure** –INITIALIZE $\alpha_2 = 0.1$
2: **for** $\alpha_1 = 0.01$ **do**
3: **for** Different values of δf **do**
4: Calculate SIR for each value of δf and sum all
5: **end for**
6: Increment α_1 by 0.01, Repeat steps 3-4 and stop when $\alpha_1 = \alpha_M - 0.1$
7: **end for**
8: Increment the value of α_2 to 0.1, repeat Steps 2-6 and stop when $\alpha_2 = 1$
9: Find α_1, α_2 corresponds to maximum SIR value
10: **end procedure**

The constraint on α_1 is obtained from (14) since all the quantities are positive. Assuming a uniformly distributed CFO error Δf and for a given range of normalized frequency offset $\delta f = \Delta f \cdot T_M$ the optimal values of α_1 and α_2 can be obtained using the exhaustive search given in Algorithm 1.

T_2 can be obtained by substituting the optimal values of α_1 and α_2 in (14) and is given by

$$T_2 = \frac{T_M(\alpha_M - \alpha_1)}{1 + \alpha_2} \tag{17}$$

V. EXAMPLE OF AN M PULSE

In this section we design an M pulse for an ADR of $1/f^5$ which is also the ADR of the NP pulse in [1] which has been recently reported for its superior OOB power performance. We begin by multiplying the frequency domain expressions of the BTRC pulse i.e. $P_1(f)$ and the RC pulse i.e. $P_2(f)$ [2]. Since the ADR of BTRC is $1/f^2$ and the ADR of RC is $1/f^3$, the ADR of the resultant M pulse will be $1/f^5$. Its frequency domain response can be expressed using (1).

The time domain expression of the M pulse can be obtained by convolving the time domain responses of the BTRC and RC pulses. It is easy to derive but since it results in a large, multi-line expression, we have omitted it here.

The values of α_1, α_2 and the corresponding T_2 have been obtained using Algorithm 1 for $T_M = 1$, $\alpha_M = 0.35$ and $\Delta f \cdot T_M$ in the range [−0.3 0.3]. The optimal values of α_1 and α_2 are obtained as 0.16 and 0.4 respectively. Using (13) and (17), we get $T_1 = 1$ and $T_2 = 0.1357$.

Fig. 1 shows the frequency response of the M pulse. We have also plotted the frequency response of the RC pulse and the frequency response of the NP pulse. The roll-off factor for all the pulses has been taken to be $\alpha = 0.35$. From the figure it can be observed that, as expected, the first two sidelobes of the M pulse are smaller than the corresponding sidelobes of the RC and NP pulses. The time domain responses of the M, RC and NP pulses are shown in Fig. 2.

VI. OOB POWER AND SIR PERFORMANCE

For comparing the OOB and SIR performance of different time-limited Nyquist pulses for OFDM systems, their durations (and hence roll-off factors) are kept same. This ensures that all of them provide the same throughput. The durations of all the pulses in this section are $1 + \alpha$ where α denotes the roll-off factor of the pulse.

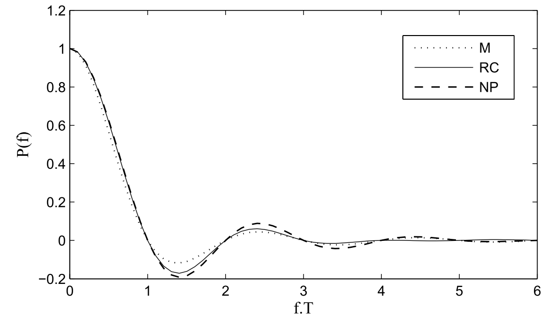

Fig. 1. The frequency spectra P(f) for 0.35 roll-off factor.

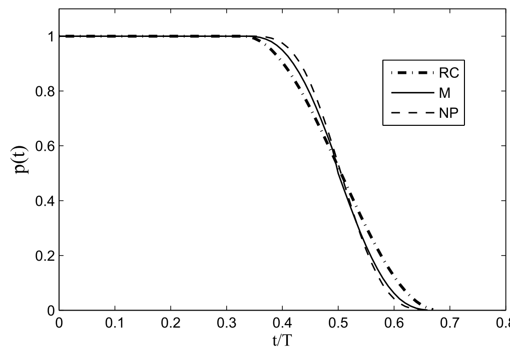

Fig. 2. Time domain response of pulses for 0.35 roll-off factor.

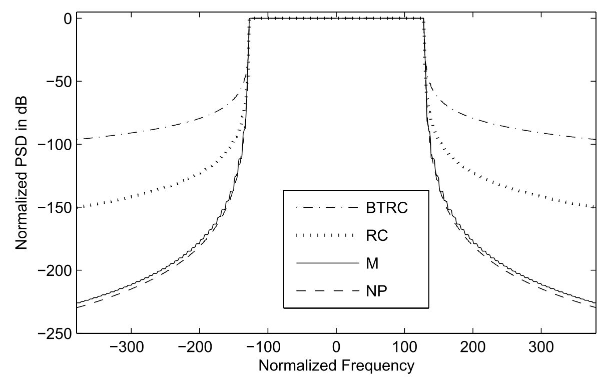

Fig. 3. PSD of M, NP and RC pulses for 0.35 roll-off factor.

The OOB power is evaluated from the power spectral density (PSD) of the transmitted signal. Assuming that the transmitted data symbols are independent and identically distributed, the average PSD $S(f)$ of the baseband OFDM signal having N subcarriers can be expressed as [8]

$$S(f) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \left| P\left(f - \frac{k}{T}\right) \right|^2 \tag{18}$$

Fig. 3 shows the normalized PSD of M, RC, BTRC and NP [1] pulses for $N = 256$, $T = 1$ and $\alpha = 0.35$. It can be seen that the PSD curves of M and NP pulses are overlapping. From the OOB perspective the RC pulse is clearly inferior to both these pulses. Since the ADR of BTRC is $1/f^2$, its OOB performance is significantly inferior to the other

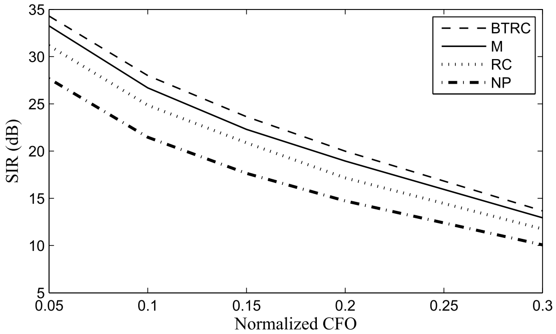

Fig. 4. SIR for 0.35 roll-off factor.

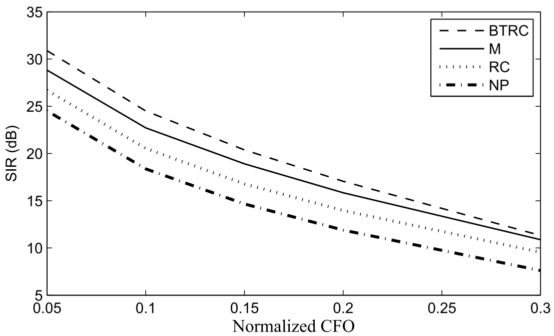

Fig. 5. SIR for 0.2 roll-off factor.

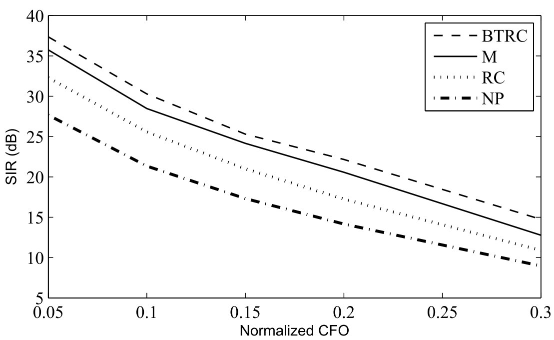

Fig. 6. SIR for 0.5 roll-off factor.

pulses. Further, if for an OFDM system the OOB performance of the RC pulse is acceptable, this superior performance of M and NP pulses can be exploited in another way. Thus, it will allow the choice of much smaller values of α for the M and NP pulses, thereby leading to substantial savings in overhead. In fact, [1] has claimed that the savings in overhead can be as high as 90%.

Now we evaluate the SIR at the receiver in the presence of a CFO of Δf using (15). In Fig. 4 we have plotted the averaged SIR of M, NP, RC and BTRC pulses as a function of the normalized CFO $\Delta f \cdot T$. The number of subcarriers N is 64 and $\alpha = 0.35$ for all the pulses. It can be seen that the M pulse provides about 2dB more SIR in comparison to the RC pulse. Compared to the NP pulse, the M pulse provides 5dB more SIR without any loss in OOB power performance. This is because the first two sidelobes of the M pulse are smaller than the sidelobes of the RC and NP pulses (Fig. 1). Although the BTRC pulse provides about 1 dB more SIR as compared to the M pulse, as pointed out above it is not a good candidate for OFDM systems due to its significantly inferior OOB performance.

The SIR performance of the pulses has also been evaluated for $\alpha = 0.2$ and $\alpha = 0.5$ and shown in Fig. 5 and Fig. 6 respectively. It is clear that the trend in Figs. 4, 5 and 6 is similar and the M pulses retain their superiority over RC and NP pulses even when α is varied.

VII. Conclusion

We have proposed a method to obtain a new family of time-limited Nyquist pulses for OFDM systems by multiplying any two existing time-limited Nyquist pulses in the frequency domain. The new pulses will not only have higher ADRs than their constituent pulses but also lower levels of sidelobes. The necessary and sufficient condition for the roll-off factor of the resulting pulses to be less than or equal to unity has been established. The proposed family provides a way for obtaining better time-limited Nyquist pulses even after the roll-off is fixed. Thus the pulses can be optimized with respect to some performance criterion. As an illustration, a new pulse has been obtained by multiplying BTRC pulse with RC pulse, followed by its optimization so that its SIR in the presence of a uniformly distributed CFO at the receiver is maximized. The obtained pulse has the same OOB power performance as the NP pulse but at the same time provides higher SIR. By considering more pairs of existing pulses, one can obtain several new, and probably useful, pulses.

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