ODE: Assignment-3

1. Solve the following linear first order differential equations using the method of variation of parameters:
   (i) $xy' - 2y = x^4$ (ii) $y' + (\cos x)y = \sin x \cos x$
   [Method: To solve $y' + P(x)y = R(x)$, first find the general solution $y_h(x) = cy_1(x)$ of the homogeneous equation $y' + P(x)y = 0$. Assume that $y_p(x) = u_1(x)y_1(x)$ is a (particular) solution of the original equation. Find u_1 and the general solution of the original equation is given by $y(x) = y_h(x) + y_p(x)$]

2. Solve $y' + y = x + 1$, $y' + y = \cos 2x$. Hence solve $y' + y = 1 + x/2 - \cos^2 x$

3. Show that the set of solutions of the homogeneous linear equation, $y' + P(x)y = 0$ on an interval $I = [a, b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

4. Let $f(x, y)$ be continuous on the closed rectangle $R : |x - x_0| \leq a$, $|y - y_0| \leq b$.
   (i) Show that y is a solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ <u>iff</u>
   $$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\, dt.$$
   (ii) Let $|f(x, y)| \leq M$ and $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)\, dt$, with $y_0(x) = y_0$. Show by the method of induction that $|y_n(x) - y_0| \leq b$ for $|x - x_0| \leq h$, where $h = \min\{a, b/M\}$.

5. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
   (i) $y' = 2\sqrt{x}$, $y(0) = 1$ (ii) $y' + xy = x$, $y(0) = 0$ (iii) $y' = 2\sqrt{y}/3$, $y(0) = 0$

6. Apply (i) Euler method and (ii) improved Euler method to compute $y(x)$ at $x = 0.2, 0.4, 0.6, 0.8, 1.0$ for the initial value problem: $y' = xy + xy^2$, $y(0) = 1$. Compare the errors in each point with the exact solution.

7. Solve $y' = (y - x)^{2/3} + 1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where (x_0, y_0) lies on the line $y = x$.

8. Discuss the existence and uniqueness of the solution of the inital value problem
   $$(x^2 - 2x)y' = 2(x - 1)y, \qquad y(x_0) = y_0.$$