ODE: Assignment-4

1. Reduce the following second order differential equation to first order differential equation and hence solve.
   (i) $xy'' + y' = y'^2$ (iii) $yy'' + y'^2 + 1 = 0$ (iii) $y'' - 2y' \coth x = 0$

2. Find the curve $y = y(x)$ for which $y'' = y'$ and the line $y = x$ is tangent at the origin.

3. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
   (i) $y = \cos(ax + b)$ (ii) $y = ax + b/x$ (iii) $y = ae^x + bxe^x$

4.(a) Find the values of m such that $y = e^{mx}$ is a solution of
   (i) $y'' + 3y' + 2y = 0$ (ii) $y'' - 4y' + 4y = 0$ (iii) $y''' - 2y'' - y' + 2y = 0$

(b) Find the values of m such that $y = x^m$ $(x > 0)$ is a solution of
   (i) $x^2y'' - 4xy' + 4y = 0$ (ii) $x^2y'' - 3xy' - 5y = 0$

5. If $p(x), q(x), r(x)$ are continuous functions on an interval $\mathcal{I}$, then show that the set of solutions of the following linear homogeneous equation is a real vector space:
$$y'' + p(x)y' + q(x)y = 0, \qquad x \in \mathcal{I}. \tag{*}$$
Also show that the set of solutions of the linear non-homogeneous equation
$$y''+p(x)y'+q(x)y = r(x), \qquad x \in \mathcal{I} \tag{**}$$
is not a real vector space. Further, suppose $y_1(x), y_2(x)$ are any two solutions of (**). Obtain conditions on the constants a and b so that $ay_1 + by_2$ is also its solution.

6. Are the following functions linearly dependent on the given intervals?

(i)	$\sin 4x$, $\cos 4x$	$(-\infty, \infty)$	(ii)	$\ln x$, $\ln x^3$	$(0, \infty)$
(iii)	$\cos 2x$, $\sin^2 x$	$(0, \infty)$	(iv)	x^3, $x^2\lvert x\rvert$	$[-1, 1]$

7.(a) Show that a solution to (*) with x-axis as tangent at any point in $\mathcal{I}$ must be identically zero on $\mathcal{I}$.

(b) Let $y_1(x), y_2(x)$ be two solutions of (*) with a common zero at any point in $\mathcal{I}$. Show that y_1, y_2 are linearly dependent on $\mathcal{I}$.

(c) Show that $y = x$ and $y = \sin x$ are not a pair solutions of equation (*), where $p(x), q(x)$ are continuous functions on $\mathcal{I} = (-\infty, \infty)$.

8.(a) Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval $\mathcal{I}$. Suppose that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in $\mathcal{I}$. Show that there exists unique $p(x), q(x)$ on $\mathcal{I}$ such that (*) has y_1, y_2 as fundamental solutions.

(b) Construct equations of the form (*) from the following pairs of solutions:
   (i) e^{-x}, xe^{-x} (ii) $e^{-x}\sin 2x$, $e^{-x}\cos 2x$ (iii) x, $x^2 + 1$

9. Let $y_1(x), y_2(x)$ are two linearly independent solutions of (*). Show that
   (i) between consecutive zeros of y_1, there exists a unique zero of y_2;
   (ii) $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions of (*) <u>iff</u> $\alpha\delta \neq \beta\gamma$.