ODE: Assignment-6

1. The equation $y'' + y' - xy = 0$ has a power series solution of the form $y = \sum_{n=0} a_n x^n$.

   (i) Find the power series solutions $y_1(x)$ and $y_2(x)$ such that $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

2. Consider the equation $(1 + x^2)y'' - 4xy' + 6y = 0$.

   (i) Find its general solution in the form $y = a_0 y_1(x) + a_1 y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

3.(i) Show that the fundamental system of solutions of Legendre equation

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

consists of $y_1(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$ and $y_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$, where $a_0 = a_1 = 1$ and

$$a_{2n+2} = -\frac{(p - 2n)(p + 2n + 1)}{(2n + 1)(2n + 2)} a_{2n}, \qquad n = 0, 1, 2, \cdots$$

$$a_{2n+1} = -\frac{(p - 2n + 1)(p + 2n)}{2n(2n + 1)} a_{2n-1}, \qquad n = 1, 2, 3, \cdots .$$

(ii) Verify that

$$y_1(x) = P_0(x) = 1,\ y_2(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x} \quad \text{for } p = 0$$

$$y_2(x) = P_1(x) = x,\ y_1(x) = 1 - \frac{x}{2} \ln \frac{1 - x}{1 + x} \quad \text{for } p = 1.$$

(iii) The expression, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n]$, is called the Rodrigues' formula for Legendre polynomial P_n of degree n. Assuming this, find P_1, P_2, P_3.

4. Using Rodrigues' formula for $P_n(x)$, show that

   (i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

   (iii) $\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n + 1} \delta_{mn}$ (iv) $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ if $m < n$

5. Suppose $m > n$. Show that $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ if $m - n$ is odd. What happens if $m - n$ is even?

6. The function on the left side of

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

is called the generating function of the Legendre polynomial P_n. Using this show that

(i) $(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$ (ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(iii) $P_{n+1}'(x) - xP_n'(x) = (n + 1)P_n(x)$; (iv) $P_n(1) = 1,\ P_n(-1) = (-1)^n$

(v) $P_0(0) = 1,\ P_{2n+1}(0) = 0,\ P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2^n n!}, \quad n \geq 1$

7. Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:

(i) $f(x) = x^3 + x + 1$ (ii) $f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x < 0 \\ x & \text{if} \quad 0 \le x \le 1 \end{cases}$ (first three nonzero terms)