ODE: Assignment-7

1. Locate and classify the singular points in the following:
   (i) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$ (ii) $(3x+1)xy'' - xy' + 2y = 0$

2. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
   (a) $9x^2y'' + (9x^2+2)y = 0$ (b) $x^2(x^2-1)y'' - x(1+x^2)y' + (1+x^2)y = 0$
   (c) $xy'' + (1-2x)y' + (x-1)y = 0$ (d) $x(x-1)y'' + 2(2x-1)y' + 2y = 0$

3. Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

4. Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

5. Find a solution bounded near $x = 0$ of the following ODE
   $$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0$$

6. Using recurrence relations, show that
   (i) $J_0''(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

7. Show that
   (i) $\int x^4 J_1(x)\,dx = (4x^3 - 16x)J_1(x) - (x^4 - 8x^2)J_0(x) + C$
   (ii) $\int J_5(x)\,dx = -2J_4(x) - 2J_2(x) - J_0(x) + C$

8. Express
   (i) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
   (iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

9. Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

10. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

11. Show that any nontrivial solution of $u'' + q(x)u = 0,\ q(x) < 0$ has at most one zero.

12. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

13. Let y_ν be a nontrivial solution of Bessel's equation of order ν on the positive x-axis. Show that (i) if $0 \le \nu < 1/2$, then every interval of length π contains at least one zero of $y_\nu(x)$; (ii) if $\nu = 1/2$, then the distance between successive zeros of y_ν is exactly π; and (iii) if $\nu > 1/2$, then every interval of length π contains at most one zero of $y_\nu(x)$.

14. Show that the Bessel functions J_ν ($\nu \ge 0$) satisfy
    $$\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)\,dx = \frac{1}{2}J_{\nu+1}^2(\lambda_n)\delta_{mn},$$
    where λ_i are the positive zeros of J_ν.

15. Find the eigen values and eigen functions of the following Strum-Liouville problems:
    (i) $y'' + \lambda y = 0, \quad y(0) = y'(1) + y(1) = 0$
    (ii) $(e^{2x}y')' + (\lambda+1)e^{2x}y = 0, \quad y(0) = y(\pi) = 0.$ [Substitute $y = e^{-x}u$]

16. If $p(x), q(x), r(x)$ are all greater than zero on (a, b), then prove that the eigen values of the Strum-Liouville problem, $(p(x)y')' - q(x)y + \lambda r(x)y = 0$, are positive with any of the boundary conditions: (i) $p(a) = p(b) = 0$, (ii) $y(a) - ky'(a) = y(b) + my'(b) = 0, k, m > 0$, (iii) $p(a) = p(b)$ with $y(b) = y(a),\ y'(b) = y'(a)$.

17. Consider the Strum-Liouville problem

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

with $p(x) > 0$ on $[a, b]$ and $y(a) \neq y(b),\ y'(a) \neq y'(b)$. Show that every eigen function is unique except for a constant factor.