Assignment-I

1. Age of Cave Painting: The caves near Lascaux, France, contain multiple wall paintings of animals from ancient times. To determine the age, charcoal found in the caves has been investigated using the C-14 dating method. The charcoal gave an average count of 0.97 disintegrations/min/gram of C-14 in a sample. Fresh wood measurements today give 6.68 disintegrations/min/gram. Estimate the age of the charcoal and hence the possible date of the wall paintings. (τ =half-life of C-14 $\approx$ 5730 years).

2. A scrap of paper taken from the Dead Sea Scrolls was found to have a C-14 to C-12 ratio 0.795 times that found in plants living today. Estimate the age of the scroll. (τ =half-life of C-14 $\approx$ 5730 years).

3. Art Forging: In 1945, an art forger van Meegeren, announced that he was the painter of a then-famous painting "Christ and the Disciples at Emmaus" attributed to Vermeer. He wanted to avoid the prosecution for selling a genuine Vermeer to the Nazis during the war. To detect the forgery, the presence of a radioactive isotope of lead, Pb-210, in the pigments of the paint has been used. All paintings contain a small amount of radioactive Pb-210 (half-life 22 years) and an even smaller amount of Ra-226 (half-life 1600 years). In the lead ore, the Pb-210 and Ra-226 are in equilibrium. In the process of paint making, most of the Ra-226 is removed. Thus, the supply of the Pb-210 is cut off, and the Pb-210 decays very rapidly until a new equilibrium is reached with the small amount of Ra-226 present. Radioactive Pb-210 occurs in the following chain
...Radium-226 $\xrightarrow{\text{1600 years}}$ Radon-222 $\xrightarrow{3\frac{4}{5}\text{ days}}$ Polonium-218 $\xrightarrow{\text{3 minutes}}$ Lead-214 $\xrightarrow{\text{27 minutes}}$ Bismuth-214 $\xrightarrow{\text{20 minutes}}$ Polonium-214 $\xrightarrow{\text{less than one second}}$ Lead-210 $\xrightarrow{\text{22 years}}$.....

   Let $y_r(t)$ be the amount of Ra-226 and $y_p(t)$ be the amount of Pb-210 in the ore.
   (i) Assuming that Ra-226 is not replenished in the ore, write the equations (approximate) governing $y_r(t)$ and $y_p(t)$ with $y_r(0) = 1$ and $y_p(0) = 0$.
   (ii) Solve the equations and prove that $y_p(t)/y_r(t)$ approaches a steady value K for a significant time range. This is called radioactive equilibrium.
   (iii) Assume that only 0.01 percent of the Ra-226 remain in the paint. Write a new set of differential equations that models the freshly created mixture in the paint. Solve the system of differential equations to obtain new $y_p(t)$ and new $y_r(t)$. Find the expression for the ratio $Z(t) = y_p(t)/y_r(t)$. Verify that the ratio converges to K as $t \to \infty$. Plot $Z(t)$ over a span of 300 years. What is the value of $Z(t)$ at t = 300? Is it different from K? Explain the relevance of this for detecting an art forgery.

4. Suppose we have an initial quantity U_0 of a radioactive element, uranium, with decay constant m. Suppose the uranium decays to another radioactive element, thorium, with initial quantity T_0 and decay constant n, where $n > m$. If $nT_0 < mU_0$, then show that

initially the amount of thorium increases to an absolute maximum and then decreases toward zero.

5. Consider the differential equation

$$\frac{dc}{dt} + c = f(t), \qquad c(0) = c_0.$$

Solve the above differential equation for the following cases. Also, discuss the long-time behaviour of the solution.

a.

$$f(t) = \begin{cases} 0, & 0 \leq t \leq T \\ M, & t > T \end{cases}$$

b.

$$f(t) = \begin{cases} 0, & 0 \leq t < T_1 \\ M, & T_1 \leq t \leq T_2 \\ 0, & t > T_2, \end{cases}$$

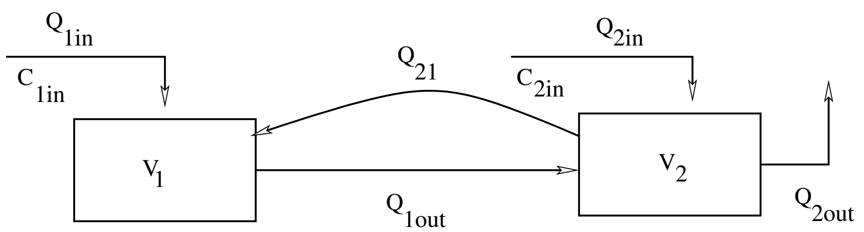
where $L = T_2 - T_1$ and $M\,L = M_0$. Discuss the case when $L \to 0$.

c. $f(t) = M\sin(\omega t)$ for $t \geq 0$

d. $f(t)$ is a periodic function for $t \geq 0$ and is defined by

$$f(t) = \begin{cases} M\sin(\omega t), & 0 \leq t \leq \pi/\omega \\ 0, & \pi/\omega < t \leq 2\pi/\omega, \end{cases} \qquad f(t + 2\pi/\omega) = f(t)$$

6. Consider the hypothetical schematic diagram shown in the figure below. The constant water flow rate are denoted by Q, constant volume of the lake by V and the incoming concentration by C_{in}.



If c_1 and c_2 denote the concentration of lake 1 and lake 2, respectively, then write down the differential equations governing c_1 and c_2. If c_1^0 and c_2^0 are the respective concentration at $t = 0$, then find the equilibrium concentration in the lakes.